

INFINITESIMAL BENDING OF CURVES ON THE RULED SURFACES

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ABSTRACT

In this paper we study infinitesimal bending of curves that lie on the ruled surfaces in Euclidean 3-dimensional space. We obtain an infinitesimal bending field under whose effect all bent curves remain on the same ruled surface as the initial curve. Specially, we consider infinitesimal bending of the curves which belong to the cylinder as well as to the hyperbolic paraboloid and find corresponding infinitesimal bending fields. We examine the variation of the curvature of a curve under infinitesimal bending on the hyperbolic paraboloid. Some examples are visualized using program packet *Mathematica*.

Keywords: Infinitesimal bending, Curve, Variation, Ruled surface, Cylinder, Hyperbolic paraboloid.

INTRODUCTION

Infinitesimal bending of curves and surfaces is a special part of the surface bending theory which presents one of the main consisting parts of the global differential geometry. The problems of infinitesimal bending have physical applications (in the study of elasticity, for example) and have a long history.

Historically, the first result of the surface bending theory belongs to Cauchy. Later, bending theory was developed thanks to the works of leading mathematicians of the considered area like Blaschke, Cohn-Vossen, A. D. Alexandrov, A. V. Pogorelov, I. N. Vekua, V. T. Fomenko, I. Kh. Sabitov, I. I. Karatopraklieva, V. A. Alexandrov and many others.

Infinitesimal bending is determined by the stationary of arc length with appropriate precision. A concept of infinitesimal bending dealt first with infinitesimal bending of surfaces and then with the same problem in the theory of curves and manifolds.

Infinitesimal bending of surfaces and manifolds was widely studied in (Aleksandrov, 1936; Alexandrov, 2010; Efimov, 1948; Hinterleitner et al., 2008; Ivanova-Karatopraklieva & Sabitov, 1995; Kon-Fossen, 1959; Najdanović, 2014; Vekua, 1959; Velimirović, 2009). Infinitesimal bending of curves was considered in (Efimov, 1948; Najdanovic, 2015; Najdanovic & Velimirovic, 2017a,b; Rancic et al., 2009; Velimirović, 2001a,b, 2009; Velimirović et al., 2010; Yano et al., 1946).

First we shall give some basic facts, definitions and theorems according to (Velimirović, 2001a & Velimirović, 2009).

Let us consider continuous regular curve $C \subset \mathcal{R}^3$, given with the equation

$$C : \mathbf{r} = \mathbf{r}(u), \quad u \in I \subseteq \mathcal{R} \quad (1)$$

included in a family of the curves

$$C_\epsilon : \tilde{\mathbf{r}}(u, \epsilon) = \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon \mathbf{z}(u), \quad u \in I, \quad \epsilon \in (-1, 1), \quad (2)$$

where u is a real parameter and we get C for $\epsilon = 0$ ($C = C_0$).

Definition 1. Family of curves C_ϵ is **infinitesimal bending of a curve** C if

$$ds_\epsilon^2 - ds^2 = o(\epsilon),$$

where $\mathbf{z} = \mathbf{z}(u)$, $\mathbf{z} \in C^1$ is **infinitesimal bending field** of the curve C .

Theorem 2. (Efimov, 1948) *Necessary and sufficient condition for $\mathbf{z}(u)$ to be an infinitesimal bending field of a curve C is to be*

$$d\mathbf{r} \cdot d\mathbf{z} = 0, \quad (3)$$

where \cdot stands for the scalar product in \mathcal{R}^3 . □

The next theorem is related to determination of the infinitesimal bending field of a curve C .

Theorem 3. (Velimirović, 2001a) *Infinitesimal bending field for the curve C (1) is*

$$\mathbf{z}(u) = \int [p(u)\mathbf{n}_1(u) + q(u)\mathbf{n}_2(u)] du, \quad (4)$$

where $p(u)$ and $q(u)$, are arbitrary integrable functions and vectors $\mathbf{n}_1(u)$ and $\mathbf{n}_2(u)$ are respectively unit principal normal and binormal vector fields of the curve C . □

As

$$\mathbf{n}_1 = \frac{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})\ddot{\mathbf{r}} - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\dot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|\|\ddot{\mathbf{r}}\|}, \quad \mathbf{n}_2 = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{\|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\|},$$

infinitesimal bending field can be written in the form

$$\mathbf{z}(t) = \int [p(t)\frac{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})\ddot{\mathbf{r}} - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\dot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|\|\ddot{\mathbf{r}}\|} + q(t)\frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{\|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\|}] dt$$

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where $p(t), q(t)$ are arbitrary integrable functions, or in the form

$$\mathbf{z}(t) = \int [P_1(t)\dot{\mathbf{r}} + P_2(t)\ddot{\mathbf{r}} + Q(t)(\dot{\mathbf{r}} \times \ddot{\mathbf{r}})]dt \quad (5)$$

where $P_i(t)$, $i = 1, 2$, $Q(t)$ are arbitrary integrable functions, too.

An interesting problem is the infinitesimal bending of a plane curve which stays in the plane after bending. This problem was considered in the paper (Velimirović, 2001a). It was found corresponding infinitesimal bending field, i. e. it was proved the next theorem.

Theorem 4. *Infinitesimal bending field that plane curve*

$$K : \rho = \rho(\theta) \quad (6)$$

under infinitesimal bending includes in a family of planes curves

$$K_\epsilon : \rho_\epsilon = \rho_\epsilon(\theta), \quad \epsilon \in (-1, 1), \quad (7)$$

is

$$\mathbf{z}(\theta) = \rho(\theta) \sin \theta \mathbf{i} - \rho(\theta) \cos \theta \mathbf{j}, \quad (8)$$

where \mathbf{i} and \mathbf{j} are unit vectors in the direction of Cartesian axes. ■

Also, in the same paper it was proved that the area of the region determined by a plane curve being infinitesimally bent staying plane is stationary.

An interesting question considered in the paper (Velimirović et al., 2010) is about infinitesimal bending of a spherical curve but so that all bent curves are on the same sphere. It was proved that there isn't infinitesimal bending of a spherical curve belonging to the sphere.

In this paper we confront the question: Is it possible to infinitesimally bend a curve C which lies on the ruled surface S , but so that all bent curves of the family C_ϵ stay on S ? The answer is affirmative and in the sequel we shall give an explicit formula for such an infinitesimal bending field.

DETERMINATION OF INFINITESIMAL BENDING FIELD

Theorem 5. *Let a ruled surface S be given by*

$$S : \mathbf{r} = \mathbf{r}(u, v) = \rho(u) + v \mathbf{e}(u) \quad (u \in \mathcal{I}, v \in \mathcal{R}, \|\mathbf{e}(u)\| = 1), \quad (9)$$

with a directrix $\rho = \rho(u)$ and generatrices in the direction of the vector $\mathbf{e}(u)$, and let a curve

$$C : \mathbf{r} = \mathbf{r}(t) = \mathbf{r}(u(t), v(t)) \quad (10)$$

be on the surface S . Then infinitesimal bending field which given curve leaves on the surface S is

$$\mathbf{z}(t) = c \mathbf{e}(u(t)) e^{-\int \frac{u^2(\rho_u \cdot \mathbf{e}_u + v\|\mathbf{e}_u\|^2)}{u\rho_u \cdot \mathbf{e} + v} dt}, \quad (11)$$

where $u\rho_u \cdot \mathbf{e} + v \neq 0$ and c is a constant.

Proof. Let

$$C : \mathbf{r} = \mathbf{r}(t) = \rho(u(t)) + v(t) \mathbf{e}(u(t))$$

be a curve on the surface S and

$$C_\epsilon : \mathbf{r}_\epsilon(t) = \rho(u(t)) + v(t) \mathbf{e}(u(t)) + \epsilon \mathbf{z}(t)$$

be an infinitesimal bending of the curve C determined with the field \mathbf{z} . As the family of the curves C_ϵ , $\epsilon \in (-1, 1)$ belongs to the surface S , the field \mathbf{z} must be in the form

$$\mathbf{z}(t) = \mathbf{e}(u(t)) z_1(t), \quad (12)$$

where $z_1(t)$ is a real continuous differentiable function.

Having in mind that \mathbf{z} is an infinitesimal bending field, the condition (3) must be satisfied, i. e.

$$\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0, \quad (13)$$

i. e.

$$(\rho_u \dot{u} + \dot{v} \mathbf{e} + v \mathbf{e}_u \dot{u}) \cdot (\mathbf{e}_u \dot{u} z_1 + \mathbf{e} \dot{z}_1) = 0. \quad (14)$$

As it is valid $\|\mathbf{e}\| = 1$, we conclude that must be satisfied $\mathbf{e} \cdot \dot{\mathbf{e}} = 0$ ($\mathbf{e} \perp \dot{\mathbf{e}} \Leftrightarrow \dot{\mathbf{e}} \cdot \mathbf{e} + \mathbf{e} \cdot \dot{\mathbf{e}} = 0 \Leftrightarrow (\mathbf{e} \cdot \dot{\mathbf{e}}) = 0 \Leftrightarrow (\|\mathbf{e}\|^2) = 0 \Leftrightarrow \|\mathbf{e}\| = \text{const}$). Using this fact, we obtain homogenous linear differential equation

$$z_1 \dot{u}^2 (\rho_u \cdot \mathbf{e}_u + v\|\mathbf{e}_u\|^2) + \dot{z}_1 (u\rho_u \cdot \mathbf{e} + v) = 0, \quad (15)$$

whose solution is

$$z_1(t) = c e^{-\int \frac{u^2(\rho_u \cdot \mathbf{e}_u + v\|\mathbf{e}_u\|^2)}{u\rho_u \cdot \mathbf{e} + v} dt}, \quad (16)$$

c is a constant and $u\rho_u \cdot \mathbf{e} + v \neq 0$. Putting (16) into (12), we obtain (11). ■

Let us note that if the directrix $C : \rho = \rho(u)$ is at the same time the striction line of the ruled surface S , then it is valid

$$\dot{\mathbf{e}} \cdot \dot{\rho} = 0$$

(see Gray (1998)), therefore $\mathbf{e}(u)$ is the field of the infinitesimal bending of curve C which that curve includes in a family of curves on the ruled surface S ,

$$C_\epsilon : \tilde{\rho}(u, \epsilon) = \rho_\epsilon(u) = \rho(u) + \epsilon \mathbf{e}(u).$$

In addition, each curve of the family C_ϵ is "parallel" to the curve C , i. e. the cut of the each generatrix between C and C_ϵ is of the same length. Indeed,

$$\|\tilde{\rho}(u_1, \epsilon) - \rho(u_1)\| = \|\epsilon \mathbf{e}(u_1)\| = \epsilon = \|\epsilon \mathbf{e}(u_2)\| = \|\tilde{\rho}(u_2, \epsilon) - \rho(u_2)\|,$$

due to $\|\mathbf{e}(u_1)\| = \|\mathbf{e}(u_2)\| = 1$. Note that for the directrix of the ruled surface can be taken every curve on the surface which is cut or touched by the generatrices, and therefore, the directrix can be the striction line, if there is one.

The ruled surfaces are not the only surfaces on which it is possible to infinitesimally bend curves. We will show that in the following example.

Example 6. Let S be the rotation surface in \mathcal{R}^3 of class C^∞ , given as the graph of the function

$$F(x, y) = \begin{cases} e^{\frac{-1}{(r^2-1)^2}}, & r > 1 \\ 0, & 0 \leq r \leq 1 \end{cases} \quad x^2 + y^2 = r^2.$$

Note that S is not a ruled surface (in fact it is not ruled in any open neighborhood of any point of the unit circle $(\cos \theta, \sin \theta, 0)$). Consider the curve $C \subset S$ given by

$$C : \mathbf{r}(t) = (t, o, f(t)),$$

where

$$f(t) = \begin{cases} e^{\frac{-1}{(t^2-1)^2}}, & |t| > 1 \\ 0, & |t| \leq 1 \end{cases}$$

Let

$$g(t) = \begin{cases} 0, & |t| \geq 1 \\ e^{\frac{-1}{(t^2-1)^2}}, & |t| < 1 \end{cases}$$

Then the family of curves given by

$$C_\epsilon : \mathbf{r}_\epsilon = (t, \epsilon g(t), f(t))$$

belongs to the surface S and presents infinitesimal bending of the curve C determined with the field $\mathbf{z} = (0, g(t), 0)$.

INFINITESIMAL BENDING OF CURVES ON THE CYLINDER

A cylinder is an example of the ruled surfaces. Let us find an infinitesimal bending field of a curve on the cylinder, that leaves the given curve on the cylinder after bending.

Theorem 7. Let the cylinder be given by the equation $S : x^2 + y^2 = a^2$. Let $C : \mathbf{r}(t) : (\alpha, \beta) \rightarrow \mathcal{R}^3$ be a regular continuous curve on the cylinder S and $\mathbf{z}(t)$ be a vector field of class C^1 which given curve includes in the family of the curves $C_\epsilon : \mathbf{r}_\epsilon = \mathbf{r}(t) + \epsilon \mathbf{z}(t)$, $\epsilon \in (-1, 1)$, on the cylinder S , under infinitesimal bending.

a) If the curve C is in the plane $z = \text{const}$, then infinitesimal bending field is $\mathbf{z}(t) = z_3(t)\mathbf{k}$, where $\mathbf{k} = (0, 0, 1)$ and $z_3(t)$ is an arbitrary real function of class C^1 .

b) Otherwise, infinitesimal bending field is a constant vector $\mathbf{z} = c\mathbf{k}$, where c is a real constant. Bending is rigid and reduces to the translation along z -axis.

Proof. The parametric equation of the cylinder of height h and of semi-axis a is

$$\mathbf{r}(u, v) = (a \cos u, a \sin u, v), \quad u \in [0, 2\pi], \quad v \in [0, h]. \quad (17)$$

Let

$$\mathbf{r}(t) = \mathbf{r}(u(t), v(t)) = (a \cos u(t), a \sin u(t), v(t)), \quad t \in (\alpha, \beta) \quad (18)$$

be the curve which lies on the cylinder (17). Infinitesimal bending will be

$$\begin{aligned} \mathbf{r}_\epsilon(t) &= \mathbf{r}(t) + \epsilon \mathbf{z}(t) \\ &= (a \cos u(t) + \epsilon z_1(t), a \sin u(t) + \epsilon z_2(t), v(t) + \epsilon z_3(t)), \end{aligned} \quad (19)$$

where $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ and $z_1(t), z_2(t), z_3(t)$ are real continuous differentiable functions. As the curves \mathbf{r}_ϵ must be on the surface (17), it must be valid

$$(a \cos u(t) + \epsilon z_1(t))^2 + (a \sin u(t) + \epsilon z_2(t))^2 = a^2,$$

i.e. after recombination

$$2a(\cos u(t)z_1(t) + \sin u(t)z_2(t)) + \epsilon(z_1^2(t) + z_2^2(t)) = 0,$$

for each $\epsilon \in (-1, 1) \setminus \{0\}$. Therefore, it must be valid

$$\cos u(t)z_1(t) + \sin u(t)z_2(t) = 0 \quad (20)$$

and

$$z_1^2(t) + z_2^2(t) = 0. \quad (21)$$

From (21) we have $z_1(t) = z_2(t) = 0$, so that $\mathbf{z}(t) = (0, 0, z_3(t))$. As \mathbf{z} is infinitesimal bending field, it holds (13). Therefore,

$$(-a \sin u(t)\dot{u}(t), a \cos u(t)\dot{u}(t), \dot{v}(t)) \cdot (0, 0, \dot{z}_3(t)) = 0,$$

i.e.

$$\dot{v}(t)\dot{z}_3(t) = 0. \quad (22)$$

From here it is $\dot{v}(t) = 0$ or $\dot{z}_3(t) = 0$. We distinguish two cases.

- When $\dot{v}(t) = 0$, we have $v(t) = \text{const}$, i.e. the curve $\mathbf{r}(t)$ is in the plane $z = \text{const}$. Infinitesimal bending field in this case is $\mathbf{z}(t) = (0, 0, z_3(t)) = z_3(t)\mathbf{k}$.
- When $\dot{z}_3(t) = 0$, i.e. $z_3(t) = \text{const}$, we obtain infinitesimal bending field of an arbitrary curve on the cylinder in the form $\mathbf{z} = (0, 0, c) = c\mathbf{k}$, c is a real constant. ■

Example 8. Let us consider some curves on the cylinder $S : \mathbf{r}(u, v) = (3 \cos u, 3 \sin u, v)$ and their infinitesimal bending. These examples are visualized using the software package *Mathematica* (Gray, 1998).

For the circle $C_1 : \mathbf{r}(t) = (3 \cos t, 3 \sin t, 0)$, which is in the plane $z = 0$, we can take infinitesimal bending field $\mathbf{z}(t) = t\mathbf{k}$. On the figure Fig. (1) we can see the curve C_1 and bent curves for $\epsilon = 0.1, 0.3, 0.5, 0.9$. Obviously, the curve C_1 clefs under that infinitesimal bending and is included in a family of helices on the cylinder.

For the same circle $C_1 : \mathbf{r}(t) = (3 \cos t, 3 \sin t, 0)$, we can take $\mathbf{z}(t) = t(2\pi - t)\mathbf{k}$. As it is valid $\mathbf{r}_\epsilon(t = 0) = \mathbf{r}_\epsilon(t = 2\pi)$, the curve remains closed (Fig. (2)).

If we take the curve $C_2 : \mathbf{r}(t) = (3 \cos t^3, 3 \sin t^3, t^3)$ and infinitesimal bending field $\mathbf{z} = 4\mathbf{k}$, we get only rigid motion, i.e. translation along the z -axis (Fig. (3)).

INFINITESIMAL BENDING OF CURVES ON THE HYPERBOLIC PARABOLOID

A hyperbolic paraboloid is an example of doubly ruled surfaces. In the sequel we will find infinitesimal bending field of an arbitrary curve belonging to this surface.

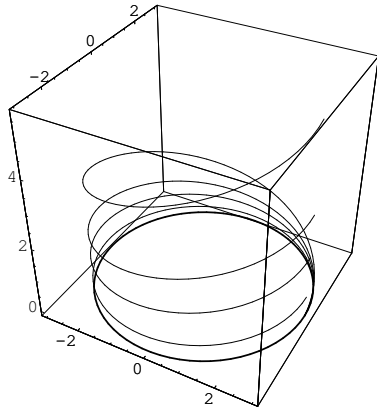


Figure 1. Infinitesimal bending of the circle C_1 on the cylinder.

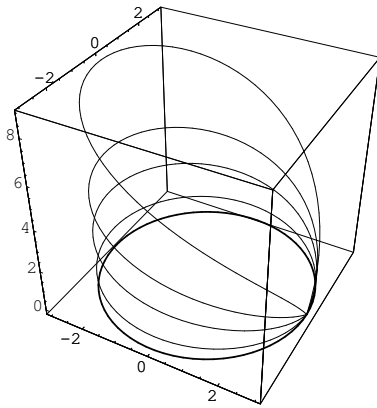


Figure 2. Infinitesimal bending of the circle C_1 on the cylinder.

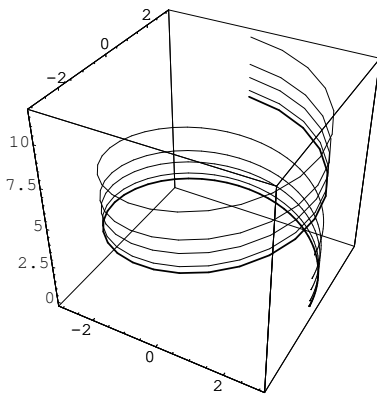


Figure 3. Trivial infinitesimal bending of the circle C_2 on the cylinder.

Theorem 9. Let a hyperbolic paraboloid be given by the equation $S : \mathbf{r}(u, v) = (u, v, uv)$ and continuous regular curve on it by $C : \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$. Let $\mathbf{z}(t)$ be a vector field of class C^1 which given curve under infinitesimal bending includes in the family of the curves $C_\epsilon : \mathbf{r}_\epsilon = \mathbf{r}(t) + \epsilon \mathbf{z}(t)$, $\epsilon \in (-1, 1)$, on the hyperbolic paraboloid S . Then the equations

$$\mathbf{z}(t) = ce^{-\int \frac{(uv)'\dot{u}}{\dot{v}+u(uv)'} dt} (0, 1, u(t)), \quad (23)$$

$\dot{v} + u(uv)' \neq 0$, and

$$\mathbf{z}(t) = ce^{-\int \frac{(uv)'\dot{v}}{\dot{u}+v(uv)'} dt} (1, 0, v(t)), \quad (24)$$

$\dot{u} + v(uv)' \neq 0$, determine the field $\mathbf{z}(t)$. c is an arbitrary constant.

Proof. Let

$$\begin{aligned} \mathbf{r}_\epsilon(t) &= \mathbf{r}(t) + \epsilon \mathbf{z}(t) \\ &= (u(t) + \epsilon z_1(t), v(t) + \epsilon z_2(t), u(t)v(t) + \epsilon z_3(t)) \end{aligned} \quad (25)$$

be an infinitesimal bending of C determined by the field $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$, z_1, z_2, z_3 are real continuous differentiable functions. As the curves (25) are on the surface S , it must be satisfied the next condition

$$(u(t) + \epsilon z_1(t))(v(t) + \epsilon z_2(t)) = u(t)v(t) + \epsilon z_3(t), \quad \forall \epsilon \in (-1, 1).$$

By dividing with $\epsilon \neq 0$ we obtain

$$z_1(t)v(t) + z_2(t)u(t) - z_3(t) + \epsilon z_1(t)z_2(t) = 0. \quad (26)$$

Since the condition (26) must be valid for each $\epsilon \in (-1, 1) \setminus \{0\}$, it must be

$$z_1(t)v(t) + z_2(t)u(t) - z_3(t) = 0 \quad (27)$$

and

$$z_1(t)z_2(t) = 0. \quad (28)$$

From (28) we get $z_1(t) = 0$ or $z_2(t) = 0$. We distinguish two cases.

– Let be valid $z_1(t) = 0$. The equation (27) reduces to $z_2(t)u(t) = z_3(t)$. From here we have

$$\mathbf{z}(t) = (0, z_2(t), u(t)z_2(t)). \quad (29)$$

Since $\mathbf{z}(t)$ is an infinitesimal bending field, it holds (13). By substituting $\dot{\mathbf{r}} = (\dot{u}, \dot{v}, \dot{u}v + u\dot{v})$ and $\dot{\mathbf{z}} = (0, \dot{z}_2, \dot{u}z_2 + uz_2)$, in the equation (13) we obtain

$$(\dot{v} + (uv)'\dot{u})z_2 + (uv)'\dot{u}z_2 = 0. \quad (30)$$

Solving the homogenous linear differential equation (30) by z_2 , we obtain

$$z_2(t) = ce^{-\int \frac{(uv)'\dot{u}}{\dot{v}+u(uv)'} dt}, \dot{v} + u(uv)' \neq 0. \quad (31)$$

From here and from (29) we get (23).

– Analogically we solve the case $z_2(t) = 0$ and get (24).

It is easy to show that the fields (23) and (24) present infinitesimal bending fields of the curve C which that curve leaves on the hyperbolic paraboloid S , after bending. ■

Example 10. Let the curve $C_3 : \mathbf{r}(t) = (t, t, t^2)$, $t \in (a, b) \subseteq \mathcal{R}$, be given on the surface $S : \mathbf{r}(u, v) = (u, v, uv)$. According to Theorem 9, after the necessary calculations, we obtain that infinitesimal bending fields of the curve C_3 which given curve leaves on the surface S , for $c = 1$, have the next form

$$\mathbf{z}_1(t) = \left(0, \frac{1}{\sqrt{1+2t^2}}, \frac{t}{\sqrt{1+2t^2}}\right)$$

and

$$\mathbf{z}_2(t) = \left(\frac{1}{\sqrt{1+2t^2}}, 0, \frac{t}{\sqrt{1+2t^2}} \right).$$

The resulting deformations are shown in Fig. (4). We can see the curve C_3 and bent curves $C_{3\epsilon}$ for $\epsilon = \pm 0.25, \pm 0.5, \pm 0.75, \pm 1$ under the field \mathbf{z}_1 .

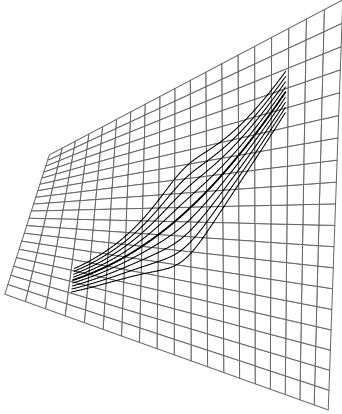


Figure 4. Infinitesimal bending of the curve C_3 on the hyperbolic paraboloid.

Example 11. For the curve $C_4 : \mathbf{r}(t) = (t, t^2, t^3)$ the corresponding infinitesimal bending fields are

$$\mathbf{z}_1(t) = \left(0, \frac{1}{\sqrt{2+3t^2}}, \frac{t}{\sqrt{2+3t^2}} \right)$$

and

$$\mathbf{z}_2(t) = \left(\frac{1}{\sqrt{1+3t^4}}, 0, \frac{t^2}{\sqrt{1+3t^4}} \right).$$

Graphical presentation of the family of bent curves $C_{4\epsilon}$ under the field \mathbf{z}_2 is given in Fig. (5).

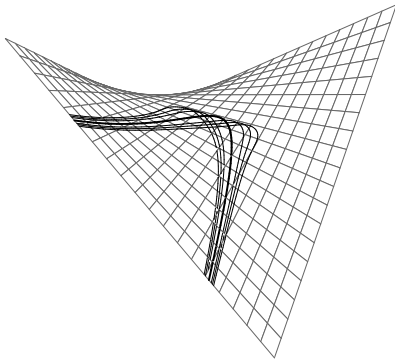


Figure 5. Infinitesimal bending of the curve C_4 on the hyperbolic paraboloid.

VARIATION OF THE CURVATURE OF CURVES ON THE HYPERBOLIC PARABOLOID

Under infinitesimal bending, geometric magnitudes describing a curve are changing and this change is determined by the variation. We define the variation according to (Vekua (1959)).

Definition 12. Let $\mathcal{A} = \mathcal{A}(u)$ be the magnitude that characterizes a geometric property on the curve C and $\mathcal{A}_\epsilon = \mathcal{A}_\epsilon(u)$ the corresponding magnitude on the curve C_ϵ being infinitesimal bending of the curve C ,

$$\Delta \mathcal{A} = \mathcal{A}_\epsilon - \mathcal{A} = \epsilon \delta \mathcal{A} + \epsilon^2 \delta^2 \mathcal{A} + \dots + \epsilon^n \delta^n \mathcal{A} + \dots \quad (32)$$

Coefficients $\delta \mathcal{A}, \delta^2 \mathcal{A}, \dots, \delta^n \mathcal{A}, \dots$ are **the first, the second, ..., the n th variation** of the geometric magnitude \mathcal{A} , respectively under infinitesimal bending C_ϵ of the curve C .

Obviously, for the first variation is effective

$$\delta \mathcal{A} = \frac{d}{d\epsilon} \mathcal{A}_\epsilon(u) \Big|_{\epsilon=0}, \quad (33)$$

i. e.

$$\delta \mathcal{A} = \lim_{\epsilon \rightarrow 0} \frac{\Delta \mathcal{A}}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{A}_\epsilon(u) - \mathcal{A}(u)}{\epsilon}. \quad (34)$$

Infinitesimal bending is a kind of deformation under which the coefficients of the first fundamental form don't get the variations of the first order, i. e. these variations are zero. The magnitudes expressed by the coefficients of the first fundamental form and derivatives of these coefficients also have no variation of the first order (for example Cristoffel's symbols, area of a region on the surface and other). However, coefficients of the second fundamental form have, generally speaking, variations different from zero.

Let us examine the variation of the curvature K of a curve

$$C : \mathbf{r}(t) = \mathbf{r}(u(t), v(t)) \quad (35)$$

on the hyperbolic paraboloid

$$S : \mathbf{r}(u, v) = (u, v, uv). \quad (36)$$

The curve C infinitesimally bends on S under infinitesimal bending field (23). The curvature of the curves

$$C_\epsilon : \mathbf{r}_\epsilon = \mathbf{r}(t) + \epsilon \mathbf{z}(t) \quad (37)$$

is given by the formula

$$K_\epsilon = \frac{\|\mathbf{r}_\epsilon \times \dot{\mathbf{r}}_\epsilon\|}{\|\mathbf{r}_\epsilon\|^3}. \quad (38)$$

By direct calculation we get

$$\|\mathbf{r}_\epsilon \times \dot{\mathbf{r}}_\epsilon\|^2 = \|\mathbf{r} \times \dot{\mathbf{r}}\|^2 + 2\epsilon P_1(t) + \epsilon^2 P_2(t) + \epsilon^3 P_3(t) + \epsilon^4 P_4(t),$$

where

$$\begin{aligned} P_1(t) &= (\dot{v}(uv)'' - \dot{v}(uv)')A(t) + (\dot{u}(uv)'' - \dot{u}(uv)')C(t) \\ &\quad + (\dot{u}\dot{v} - \dot{u}\dot{v})D(t); \\ P_2(t) &= 2(\dot{v}(uv)'' - \dot{v}(uv)')B(t) + A^2(t) + C^2(t) + D^2(t); \\ P_3(t) &= 2A(t)B(t); \quad P_4(t) = B^2(t); \end{aligned} \quad (39)$$

$$A(t) = (\ddot{u}\dot{v} - \dot{u}\ddot{v})z_2 + (4\dot{u}\dot{v} + \ddot{u}\dot{v})z_2' - \dot{u}\dot{v}z_2'';$$

$$B(t) = \ddot{u}z_2z_2' + 2\dot{u}z_2'' - \dot{u}z_2z_2'';$$

$$C(t) = (2\dot{u}^2 - u\ddot{u})z_2 + u\dot{u}z_2'; \quad D(t) = \dot{u}z_2' - \dot{u}z_2'';$$

The function $z_2(t)$ is determined by Eq. (31). Also,

$$\|\dot{\mathbf{r}}_\epsilon\|^2 = \|\dot{\mathbf{r}}\|^2 + \epsilon^2[(\dot{u}z_2 + uz_2')^2 + z_2'^2]. \quad (40)$$

It is valid

$$\lim_{\epsilon \rightarrow 0} \frac{K_\epsilon^2 - K^2}{\epsilon} = \frac{d}{d\epsilon} K_\epsilon^2 \Big|_{\epsilon=0} = \frac{2P_1(t)}{\|\dot{\mathbf{r}}\|^6}, \quad (41)$$

and also,

$$\lim_{\epsilon \rightarrow 0} \frac{K_\epsilon^2 - K^2}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{K_\epsilon - K}{\epsilon} \cdot (K_\epsilon + K) = \delta K \cdot 2K, \quad (42)$$

where δK is the variation of the curvature K of curve (35). By comparing (41) and (42), using $K \neq 0$ we obtain

$$\delta K = \frac{P_1(t)}{K\|\dot{\mathbf{r}}\|^6}. \quad (43)$$

Let us verify the case $K = 0$. From (41) and (42) we conclude $P_1(t) = 0$. Now it is

$$\delta K = \lim_{\epsilon \rightarrow 0} \frac{K_\epsilon}{\epsilon} = \frac{\sqrt{\epsilon^2 P_2(t) + \epsilon^3 P_3(t) + \epsilon^4 P_4(t)}}{\epsilon [\sqrt{\|\dot{\mathbf{r}}\|^2 + \epsilon^2((\dot{u}z_2 + uz_2')^2 + z_2'^2)}]^3}$$

i. e.

$$\delta K = \frac{\sqrt{P_2(t)}}{\|\dot{\mathbf{r}}\|^3}. \quad (44)$$

Therefore, it holds

Theorem 13. *The first variation of the curvature K of curve (35) on the hyperbolic paraboloid (36) under infinitesimal bending (23) doesn't have to be zero and it is given by Eq. (43) for $K \neq 0$, i. e. (44), for $K = 0$, where P_1 and P_2 are the functions given in (39).*

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