

CURVES ON RULED SURFACES UNDER INFINITESIMAL BENDING

MARIJA NAJDANOVIĆ^{1,*}, MIROSLAV MAKSIMOVIĆ¹, LJUBICA VELIMIROVIĆ²

¹Faculty of Sciences and Mathematics, University of Priština in Kosovska Mitrovica, Serbia

²Faculty of Sciences and Mathematics, University of Niš, Niš, Serbia

ABSTRACT

Infinitesimal bending of curves lying with a given precision on ruled surfaces in 3-dimensional Euclidean space is studied. In particular, the bending of curves on the cylinder, the hyperbolic paraboloid and the helicoid are considered and appropriate bending fields are found. Some examples are graphically presented.

Keywords: Infinitesimal bending, Curve, Ruled surface, Cylinder, Hyperbolic paraboloid, Helicoid.

INTRODUCTION

Infinitesimal bending is a kind of deformations of geometric objects under which the arc length is stationary with appropriate precision which is described by the following equation

$$ds_\epsilon^2 - ds^2 = o(\epsilon), \quad \epsilon > 0, \quad \epsilon \rightarrow 0.$$

It means that the difference of the squares of the line elements of deformed and initial object is an infinitesimal of the order higher than the first with respect to the infinitesimal parameter ϵ . Many other geometric magnitudes stay invariant in the sense that they don't get the variations of the first order (for example, the coefficients of the first fundamental form, Cristoffel's symbols, Gaussian curvature etc.). Many papers are devoted to the infinitesimal bending of curves, surfaces and manifolds (see (Aleksandrov, 1936; Efimov, 1948; Kon-Fossen, 1959; Vekua, 1959; Ivanova-Karatopraklieva & Sabitov, 1995; Velimirović, 2001a,b; Hinterleitner et al., 2008; Rančić et al., 2009; Alexandrov, 2010; Najdanović, 2015; Najdanović & Velimirović, 2017; Kauffman et al., 2019; Najdanović et al., 2019; Rančić et al., 2019; Rýparová & Mikeš, 2019; Belova et al., 2021; Maksimović et al., 2021).

In (Najdanović & Velimirović, 2018) the authors studied the infinitesimal bending of curves that lie on ruled surfaces in Euclidean 3-dimensional space. It was proven that it is possible infinitesimally bend such a curve so that all bent curves remain on the same surface as the initial curve. Corresponding infinitesimal bending field under whose effect all bent curves remain on the same ruled surface was obtained.

The connection between ruled surfaces and infinitesimal bending of curves is also considered in (Gözütok et al., 2020). Some interesting papers on ruled surfaces are (Li & Pei, 2016; Li et al., 2021).

In this paper we observe a curve on a ruled surface and set the condition that all bent curves remain on the initial surface with a given precision. More precisely, let

$$C : \mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t)), \quad t \in J \subseteq \mathbb{R},$$

be the curve on the surface S given by the implicit equation

$$S : F(x, y, z) = 0.$$

So, it is valid

$$F(x(t), y(t), z(t)) = 0.$$

Suppose that

$$C_\epsilon : \mathbf{r}_\epsilon(t) = (x_\epsilon(t), y_\epsilon(t), z_\epsilon(t))$$

is an infinitesimal bending of the curve C and we get C for $\epsilon = 0$. The problem we pose is to determinate an infinitesimal bending of C so that all bent curves C_ϵ are on the surface S with a given precision, ie. so that the following condition is valid:

$$F(x_\epsilon(t), y_\epsilon(t), z_\epsilon(t)) = o(\epsilon). \quad (1)$$

Below we are going to consider infinitesimal bending of curves on a cylinder, on a hyperbolic paraboloid and on a helicoid. Some examples are graphically presented using program packet *Mathematica*.

INFINITESIMAL BENDING OF CURVES

At the beginning we are giving basic definitions and theorems regarding infinitesimal bending of curves according to (Efimov, 1948; Vekua, 1959; Velimirović, 2001a).

Let a regular curve C be given in the vector form

$$C : \mathbf{r} = \mathbf{r}(t), \quad t \in J \subseteq \mathbb{R} \quad (2)$$

included in the family of the curves

$$C_\epsilon : \mathbf{r}_\epsilon(t) = \mathbf{r}(t) + \epsilon \mathbf{z}(t), \quad t \in J, \quad (3)$$

* Corresponding author: marija.najdanovic@pr.ac.rs

where $\epsilon > 0$, $\epsilon \rightarrow 0$ is an infinitesimal parameter and we get C for $\epsilon = 0$ ($C = C_0$).

Definition 1. A family of curves C_ϵ given by (3) is called **infinitesimal deformation** of the curve C given by (2). The field $\mathbf{z} = \mathbf{z}(t)$, $\mathbf{z} \in C^1$, is **infinitesimal deformation field** of C .

Definition 2. An infinitesimal deformation C_ϵ given by (3) is called **infinitesimal bending** of the curve C given by (2) if

$$ds_\epsilon^2 - ds^2 = o(\epsilon),$$

where the field $\mathbf{z} = \mathbf{z}(t)$, $\mathbf{z} \in C^1$, is **infinitesimal bending field** of C .

Theorem 3. (Efimov, 1948) *Necessary and sufficient condition for $\mathbf{z}(t)$ to be an infinitesimal bending field of the curve C is to be*

$$d\mathbf{r} \cdot d\mathbf{z} = 0, \quad (4)$$

where \cdot stands for the scalar product in \mathbb{R}^3 . \square

Definition 4. An infinitesimal bending field is **trivial** if it can be given in the form

$$\mathbf{z} = \mathbf{a} \times \mathbf{r} + \mathbf{b},$$

where \mathbf{a} and \mathbf{b} are constant vectors.

According to Vekua (1959) we have the next theorem.

Theorem 5. *Under infinitesimal bending of curves each line element undertakes a nonnegative addition, which is the infinitesimal value of at least the second order with respect to ϵ , i. e.*

$$ds_\epsilon - ds = o(\epsilon) \geq 0.$$

\square

Infinitesimal bending field of a curve C is determined in the following theorem.

Theorem 6. (Velimirović, 2001a) *Infinitesimal bending field for the curve C given by (2) is*

$$\mathbf{z}(t) = \int [p(t)\mathbf{n}_1(t) + q(t)\mathbf{n}_2(t)] dt, \quad (5)$$

where $p(t)$ and $q(t)$ are arbitrary integrable functions and vectors $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ are respectively unit principal normal and binormal vector fields of the curve C . \square

INFINITESIMAL BENDING OF CURVES ON CYLINDER

Let be given a cylinder by the implicit equation

$$S : F(x, y, z) \equiv x^2 + y^2 - a^2 = 0, \quad (6)$$

$a > 0$, or by the vector parametric equation

$$S : \mathbf{r}(u, v) = (a \cos u, a \sin u, v), \quad u \in [0, 2\pi], \quad v \in [0, h]. \quad (7)$$

Let

$$C : \mathbf{r}(t) = \mathbf{r}(u(t), v(t)) = (a \cos u(t), a \sin u(t), v(t)), \quad t \in J, \quad (8)$$

be the curve on the cylinder S . Suppose that

$$C_\epsilon : \mathbf{r}_\epsilon(t) = (a \cos u(t) + \epsilon z_1(t), a \sin u(t) + \epsilon z_2(t), v(t) + \epsilon z_3(t)) \quad (9)$$

is an infinitesimal bending of C , where $z_1(t)$, $z_2(t)$, $z_3(t)$ are real continuous differentiable functions. In order to stay on the cylinder S with a given precision, it is necessary to apply the condition (1), so we have

$$(a \cos u(t) + \epsilon z_1(t))^2 + (a \sin u(t) + \epsilon z_2(t))^2 - a^2 = o(\epsilon).$$

From the last equation we obtain the condition

$$\cos u(t)z_1(t) + \sin u(t)z_2(t) = 0 \quad (10)$$

which allows the bent curves to stay on the cylinder S with a given precision. For $\cos u(t) \neq 0$ we can express z_1 as a function of z_2 :

$$z_1(t) = -\tan u(t)z_2(t). \quad (11)$$

Therefore, we are looking for the infinitesimal bending field in the following form

$$\mathbf{z}(t) = (-\tan u(t)z_2(t), z_2(t), z_3(t)). \quad (12)$$

In order for the field (12) to be an infinitesimal bending field, it is necessary that the condition $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$ is valid. Since $\dot{\mathbf{r}}(t) = (-a \sin u(t) \dot{u}(t), a \cos u(t) \dot{u}(t), \dot{v}(t))$ and $\dot{\mathbf{z}}(t) = (-\frac{\dot{u}(t)}{\cos^2 u(t)} z_2(t) - \tan u(t) \dot{z}_2(t), \dot{z}_2(t), \dot{z}_3(t))$, we obtain

$$a \dot{u}(t)^2 \frac{\sin u(t)}{\cos^2 u(t)} z_2(t) + \frac{a \dot{u}(t)}{\cos u(t)} \dot{z}_2(t) + \dot{v}(t) \dot{z}_3(t) = 0. \quad (13)$$

This is the relationship between z_2 and z_3 . Let us choose arbitrarily z_3 and solve the linear differential equation by z_2 . The solution is

$$z_2(t) = e^{-\int \dot{u}(t) \tan u(t) dt} \left[c - \frac{1}{a} \int \frac{\dot{v}(t) \cos u(t)}{\dot{u}(t)} \dot{z}_3(t) e^{\int \dot{u}(t) \tan u(t) dt} dt \right], \quad (14)$$

$u(t) \neq \text{const}$, $\cos u(t) \neq 0$ and c is a constant. If $\dot{v}(t) = 0 \Rightarrow v(t) = \text{const}$, the curve C is a circle on the cylinder. In that case we choose z_3 arbitrarily and determine z_2 from

$$z_2(t) = ce^{-\int \dot{u}(t) \tan u(t) dt}.$$

If $u(t) = \text{const}$, then the equation (13) reduces to $\dot{v}(t) \dot{z}_3(t) = 0$ wherefrom we have $\dot{z}_3(t) = 0 \Rightarrow z_3(t) = \text{const}$ and we choose z_2 arbitrarily.

Based on the previous considerations, the following theorems hold.

Theorem 7. The field $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ whose components z_1 and z_2 satisfy the condition (10) includes the curve (8) under infinitesimal deformation into the family of deformed curves on the cylinder (6) with a given precision. \square

Theorem 8. The field $\mathbf{z}(t)$ given by (12) where $z_3(t)$ is arbitrary real continuous differentiable function, and $z_2(t)$ is given in (14), is infinitesimal bending field of the curve (8) so that all bent curves are on the cylinder (6) with a given precision. \square

Example 9. Let be $u(t) = t, v(t) = 0$. Then the curve C is a circle $\mathbf{r}(t) = (a \cos t, a \sin t, 0)$. We have

$$z_2(t) = ce^{-\int \tan t dt} = \bar{c} \cos t,$$

$$z_1(t) = -c \tan t \cos t = -\bar{c} \sin t,$$

z_3 -arbitrarily, c, \bar{c} are constants. So, the infinitesimal bending field is $\mathbf{z}(t) = (-\bar{c} \sin t, \bar{c} \cos t, z_3(t))$. By a simple check, we conclude that the conditions $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$ and $(a \cos t - \bar{c} \sin t)^2 + (a \sin t + \bar{c} \cos t)^2 - a^2 = \bar{c}^2 = o(\epsilon)$ are satisfied. An illustration of the infinitesimal bending is shown in Figures 1 and 2.

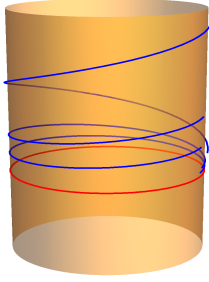


Figure 1. Circle (red) on the cylinder and infinitesimally bent curves (blue), for $\bar{c} = 1, z_3(t) = t + 1$ and $\epsilon = 0.05, 0.1, 0.25$.

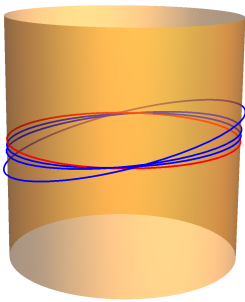


Figure 2. Circle (red) on the cylinder and infinitesimally bent curves (blue), for $\bar{c} = 1, z_3(t) = \cos t$ and $\epsilon = 0.05, 0.1, 0.25$.

INFINITESIMAL BENDING OF CURVES ON HYPERBOLIC PARABOLOID

Let the curve $C : \mathbf{r} = \mathbf{r}(t)$ lie on the hyperbolic paraboloid

$$S : F(x, y, z) \equiv xy - z = 0 \quad (15)$$

with the vector parametric equation

$$S : \mathbf{r}(u, v) = (u, v, uv).$$

Of course, the following equation holds

$$C : \mathbf{r} = \mathbf{r}(t) = \mathbf{r}(u(t), v(t)) = (u(t), v(t), u(t)v(t)). \quad (16)$$

Let $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ be an infinitesimal bending field under which all bent curves are on the surface (15) with a given precision. As the infinitesimal bending is in the following form

$$\mathbf{r}_\epsilon(t) = \mathbf{r}(t) + \epsilon \mathbf{z}(t) \equiv (x_\epsilon(t), y_\epsilon(t), z_\epsilon(t)),$$

the condition (1) reduces to

$$(u(t) + \epsilon z_1(t))(v(t) + \epsilon z_2(t)) - u(t)v(t) - \epsilon z_3(t) = o(\epsilon)$$

ie.

$$z_1(t)v(t) + z_2(t)u(t) - z_3(t) = 0. \quad (17)$$

This is the necessary and sufficient condition which allows that all bent curves are approximately on S . From Eq. (17) we obtain

$$z_3(t) = z_1(t)v(t) + z_2(t)u(t). \quad (18)$$

Therefore,

$$\mathbf{z}(t) = (z_1(t), z_2(t), z_1(t)v(t) + z_2(t)u(t)) \quad (19)$$

is the required field. It is also necessary to apply the condition $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$. Since $\dot{\mathbf{r}} = (\dot{u}, \dot{v}, \dot{u}v + u\dot{v})$, and $\dot{\mathbf{z}} = (\dot{z}_1, \dot{z}_2, \dot{z}_1v + z_1\dot{v} + \dot{z}_2u + z_2\dot{u})$, we obtain

$$[\dot{v} + (uv)'\dot{u}]z_2 + (uv)'\dot{u}z_2 + \varphi = 0, \quad (20)$$

where

$$\varphi = [\dot{u} + (uv)'\dot{v}]z_1 + (uv)'\dot{v}z_1.$$

If we arbitrarily choose z_1 , we obtain the function z_2 by solving the linear differential equation (20), and z_3 from Eq. (18). The equation (20) reduces to

$$\dot{z}_2 + \frac{(uv)'\dot{u}}{\dot{v} + (uv)'\dot{u}}z_2 = -\frac{\varphi}{\dot{v} + (uv)'\dot{u}}, \quad \dot{v} + (uv)'\dot{u} \neq 0$$

whose solution is

$$z_2 = e^{-\int \frac{(uv)'\dot{u}}{\dot{v} + (uv)'\dot{u}} dt} \left[c - \int \frac{\varphi}{\dot{v} + (uv)'\dot{u}} e^{\int \frac{(uv)'\dot{u}}{\dot{v} + (uv)'\dot{u}} dt} dt \right], \quad (21)$$

c is a constant.

In this way we have proved the following theorems.

Theorem 10. The field $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ whose components z_1, z_2 and z_3 satisfy the condition (17) includes the curve (16) under infinitesimal deformation into the family of deformed curves on the hyperbolic paraboloid (15) with a given precision. \square

Theorem 11. The field $\mathbf{z}(t)$ given by (19) where $z_1(t)$ is arbitrary real continuous differentiable function, and $z_2(t)$ is given in (21), is infinitesimal bending field of the curve (16) so that all bent curves are on the hyperbolic paraboloid (15) with a given precision. \square

Example 12. Let the curve $C : \mathbf{r}(t) = (t, t, t^2)$, $t \in J \subseteq \mathbb{R}$, be given on the hyperbolic paraboloid $S : \mathbf{r}(u, v) = (u, v, uv)$. Let us find the field \mathbf{z} according to the previous theorem.

Let be $z_1(t) = t$. Since $u(t) = t$, $v(t) = t$, $\dot{u} = 1$, $\dot{v} = 1$, $\dot{z}_1 = 1$, we obtain $\varphi = 1 + 4t^2$. Also,

$$\begin{aligned} z_2 &= e^{-\int \frac{2t}{1+2t^2} dt} \left[c - \int \frac{1+4t^2}{1+2t^2} e^{\int \frac{2t}{1+2t^2} dt} dt \right] \\ &= \frac{c}{\sqrt{1+2t^2}} - t. \end{aligned}$$

Next, we have

$$z_3 = z_1 v + z_2 u = \frac{ct}{\sqrt{1+2t^2}}.$$

Therefore, the corresponding infinitesimal bending field is

$$\mathbf{z}(t) = \left(t, \frac{c}{\sqrt{1+2t^2}} - t, \frac{ct}{\sqrt{1+2t^2}} \right), \quad (22)$$

so

$$\mathbf{r}_\epsilon = \left(t + \epsilon t, t + \epsilon \left(\frac{c}{\sqrt{1+2t^2}} - t \right), t^2 + \epsilon \frac{ct}{\sqrt{1+2t^2}} \right).$$

It is easy to check that the following is true: $\dot{\mathbf{r}} = (1, 1, 2t)$, $\dot{\mathbf{z}} = \left(1, \frac{-2ct}{\sqrt{(1+2t^2)^3}} - 1, \frac{c}{\sqrt{(1+2t^2)^3}} \right)$, $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$,

$$\begin{aligned} (t + \epsilon t) \left(t + \epsilon \left(\frac{c}{\sqrt{1+2t^2}} - t \right) \right) - t^2 - \epsilon \frac{ct}{\sqrt{1+2t^2}} \\ = \epsilon^2 t \left(\frac{c}{\sqrt{1+2t^2}} - t \right) = o(\epsilon). \end{aligned}$$

Figure 3 shows infinitesimal bending of curve C under field (22).

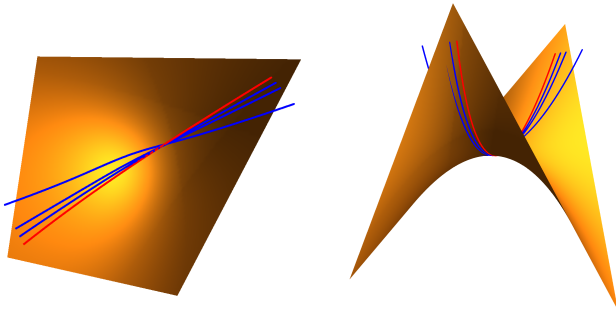


Figure 3. Curve C (red) on the hyperbolic paraboloid and infinitesimally bent curves (blue), for $c = 1$ and $\epsilon = 0.05, 0.1, 0.25$.

INFINITESIMAL BENDING OF CURVES ON HELICOID

Let $C : \mathbf{r} = \mathbf{r}(t)$ be the curve on the helicoid

$$S : F(x, y, z) \equiv \frac{y}{x} - \tan \frac{z}{c} = 0, \quad (23)$$

c -constant, with the vector parametric equation

$$\mathbf{r}(u, v) = (u \cos v, u \sin v, cv).$$

Then the equation of the curve C has the form

$$C : \mathbf{r}(t) = \mathbf{r}(u(t), v(t)) = (u(t) \cos v(t), u(t) \sin v(t), cv(t)). \quad (24)$$

Let $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ be an infinitesimal bending field which given curve leaves on the helicoid S , where $z_1(t)$, $z_2(t)$, $z_3(t)$ are real continuous differentiable functions. Then will be

$$\mathbf{r}_\epsilon(t) = (u(t) \cos v(t) + \epsilon z_1(t), u(t) \sin v(t) + \epsilon z_2(t), cv(t) + \epsilon z_3(t))$$

the corresponding infinitesimal bending and the condition

$$\frac{u(t) \sin v(t) + \epsilon z_2(t)}{u(t) \cos v(t) + \epsilon z_1(t)} - \tan \frac{cv(t) + \epsilon z_3(t)}{c} = o(\epsilon)$$

must be satisfied. From the last equation we have

$$\frac{u(t) \sin v(t) + \epsilon z_2(t)}{u(t) \cos v(t) + \epsilon z_1(t)} - \frac{\tan v(t) + \tan(\epsilon \frac{z_3(t)}{c})}{1 - \tan v(t) \tan(\epsilon \frac{z_3(t)}{c})} = o(\epsilon),$$

or, after using Maclaurin series for $y = \tan x$:

$$\frac{u(t) \sin v(t) + \epsilon z_2(t)}{u(t) \cos v(t) + \epsilon z_1(t)} - \frac{\tan v(t) + \epsilon \frac{z_3(t)}{c} + o(\epsilon^2)}{1 - \tan v(t)(\epsilon \frac{z_3(t)}{c} + o(\epsilon^2))} = o(\epsilon).$$

From here we obtain

$$z_2(t) - u(t) \frac{\sin^2 v(t)}{\cos v(t)} \frac{z_3(t)}{c} - z_1(t) \tan v(t) - u(t) \cos v(t) \frac{z_3(t)}{c} = 0,$$

ie.

$$z_2(t) = z_1(t) \tan v(t) + \frac{z_3(t)u(t)}{c \cos v(t)}. \quad (25)$$

Thus the field $\mathbf{z}(t)$ has the following form

$$\mathbf{z}(t) = \left(z_1(t), z_1(t) \tan v(t) + \frac{z_3(t)u(t)}{c \cos v(t)}, z_3(t) \right), \quad (26)$$

$\cos v(t) \neq 0$, and the functions $z_1(t)$ and $z_2(t)$ are obtained from the condition

$$\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0,$$

where

$$\begin{aligned} \dot{\mathbf{r}} &= (\dot{u}(t) \cos v(t) - u(t)\dot{v}(t) \sin v(t), \dot{u}(t) \sin v(t) \\ &\quad + u(t)\dot{v}(t) \cos v(t), c\dot{v}(t)), \\ \dot{\mathbf{z}} &= \left(\dot{z}_1(t), \dot{z}_1(t) \tan v(t) + \frac{z_1(t)\dot{v}(t)}{\cos^2 v(t)} \right. \\ &\quad \left. + \frac{(\dot{z}_3(t)u(t) + z_3(t)\dot{u}(t)) \cos v(t) + z_3(t)u(t)\dot{v}(t) \sin v(t)}{c \cos^2 v(t)}, \dot{z}_3(t) \right). \end{aligned}$$

For simplicity, let us consider a helix which is obtained for $u = 1$ and $v = t$. Therefore, the vector parametric equation of a helix is

$$\mathbf{r}(t) = (\cos t, \sin t, ct). \quad (27)$$

The field \mathbf{z} reduces to

$$\mathbf{z} = \left(z_1(t), z_1(t) \tan t + \frac{z_3(t)}{c \cos t}, z_3(t) \right), \quad (28)$$

$\cos t \neq 0$. Since $\dot{\mathbf{r}} = (-\sin t, \cos t, c)$, $\dot{\mathbf{z}} = (\dot{z}_1(t), \dot{z}_1(t) \tan t + \frac{z_1(t)}{\cos^2 t} + \frac{\dot{z}_3(t) \cos t + z_3(t) \sin t}{c \cos^2 t}, \dot{z}_3(t))$, using the condition $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$ we get the equation

$$\dot{z}_3 + \frac{1}{1+c^2} z_3(t) \tan t = -\frac{c}{1+c^2} \frac{z_1(t)}{\cos t}. \quad (29)$$

This is the relationship between z_1 and z_3 . We choose one of these functions arbitrarily, and we get the other by solving the equation (29). Thus, if z_1 is arbitrary real continuous differentiable function, we obtain

$$z_3(t) = (\cos t)^{\frac{1}{1+c^2}} \left[c_1 - \frac{c}{1+c^2} \int \frac{z_1(t)}{(\cos t)^{\frac{2+c^2}{1+c^2}}} dt \right], \quad (30)$$

where c_1 is a constant.

Theorem 13. *The field $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ whose components z_1 , z_2 and z_3 satisfy the condition (25) includes the curve (24) under infinitesimal deformation into the family of deformed curves on the helicoid (23) with a given precision. \square*

Theorem 14. *The field $\mathbf{z}(t)$ given by (28) where $z_1(t)$ is arbitrary real continuous differentiable function, and $z_3(t)$ is given in (30), is infinitesimal bending field of the helix (27) so that all bent curves are on the helicoid (23) with a given precision. \square*

Example 15. Let be $z_1(t) = (1 + 2c^2) \sin t \cos^2 t$. Then we obtain after necessary integration (for $c_1 = 0$) $z_3(t) = c \cos^2 t$, and $z_2(t) = (1 + 2c^2) \sin^2 t \cos t + \cos t$.

The corresponding infinitesimal bending field which given helix leaves at the helicoid with a given precision is

$$\mathbf{z}(t) = ((1 + 2c^2) \sin t \cos^2 t, (1 + 2c^2) \sin^2 t \cos t + \cos t, c \cos^2 t). \quad (31)$$

By a simple check, we conclude that this is an infinitesimal bending field and the condition (1) is valid.

Infinitesimal bending of a helix on the helicoid under bending field (31) is shown in Figure 4.

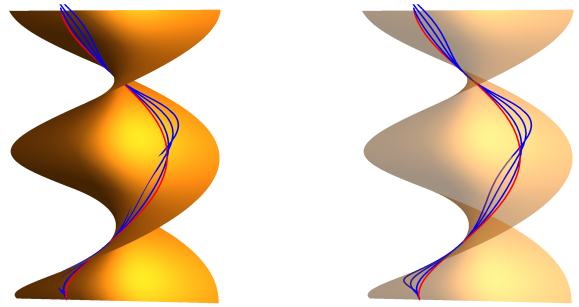


Figure 4. Helix (red) on the helicoid and infinitesimally bent curves (blue), for $\epsilon = 0.05, 0.15, 0.25$.

CONCLUSION

In this paper, infinitesimal bending of curves that are approximately on the cylinder, the hyperbolic paraboloid and the helicoid, respectively, are examined. The corresponding infinitesimal bending fields are obtained and some examples with graphical illustration are considered. This is a continuation of the research published in the paper (Najdanović & Velimirović, 2018).

ACKNOWLEDGEMENT

This work is supported by the Serbian Ministry of Education, Science and Technological Development under the research grant 451-03-68/2020-14/200123 (the first and the second author) and 451-03-68/2020-14/200124 (the third author).

REFERENCES

- Aleksandrov, A. D. 1936, O beskonechno malyh izgibaniyah neregulyarnyh poverhnosti, *Matem. sbornik*, 1(43), pp. 307-321.
- Alexandrov, V. 2010, New manifestations of the Darboux's rotation and translation fields of a surface, *New Zealand Journal of Mathematics*, 40, pp. 59-65.
- Belova, O., Mikeš, J., & Sherkuziyev, M. 2021, An Analytical Inflexibility of Surfaces Attached Along a Curve to a Surface Regarding a Point and Plane, *Results Math*, 76(56). doi:10.1007/s00025-021-01362-0
- Efimov, N. 1948, Kachestvennye voprosy teorii deformacii poverhnosti, *UMN*. 3. 2, pp. 47-158.
- Gözütok, U., Çoban, H. A., & Sağıroğlu, Y. 2020, Ruled surfaces obtained by bending of curves, *Turk J Math*, 44, pp. 300-306. doi:10.3906/mat-1908-21
- Hinterleitner, I., Mikeš, J., & Stránská, J. 2008, Infinitesimal Fplanar transformations, *Russian Mathematics*, 52(4), pp. 13-18. doi:10.3103/s1066369x08040026
- Ivanova-Karatopraklieva, I. & Sabitov, I. K. 1995, Bending of surfaces. Part II, *Journal of Mathematical Sciences*, 74(3), pp. 997-1043. doi:10.1007/bf02362831
- Kauffman, L. H., Velimirović, Lj. S., Najdanović, M. S., & Rančić, S. R. 2019, Infinitesimal bending of knots and energy change.,

- Journal of Knot Theory and Its Ramifications, 28(11), 1940009. doi:10.1142/S0218216519400091
- Kon-Fossen, S. E. 1959, *Nekotorye voprosy differ. geometrii v celom* (Moskva: Fizmatgiz)
- Li, Y. & Pei, D. 2016, Evolutes of dual spherical curves for ruled surfaces, *Mathematical Methods in the Applied Sciences* 39(11), pp. 3005-3015. doi:10.1002/mma.3748
- Li, Y., Wang, Z., & Zhao, T. 2021, Geometric Algebra of Singular Ruled Surfaces, *Advances in Applied Clifford Algebras*, 31(2), pp. 1-19. doi:10.1007/s00006-020-01097-1
- Maksimović, M., Velimirović, Lj., & Najdanović, M. 2021, Infinitesimal bending of DNA helices, *Turk J Math*, 45, pp. 520-528. doi:10.3906/mat-2003-106
- Najdanović, M. 2015, Infinitesimal bending influence on the Willmore energy of curves, *Filomat*, 29(10), pp. 2411-2419. doi:10.2298/fil1510411n
- Najdanović, M. S., Rančić, S. R., Kauffman, L. H., & Velimirović, Lj. S. 2019, The total curvature of knots under second-order infinitesimal bending, *Journal of Knot Theory and Its Ramifications*, 28(1), 1950005. doi:10.1142/S0218216519500056
- Najdanović, M. S. & Velimirović, Lj. S. 2017, On the Willmore energy of curves under second order infinitesimal bending, *Miskolc Mathematical Notes*, 17(2), pp. 979-987. doi:10.18514/mmn.2017.2133
- Najdanović, M. S. & Velimirović, Lj. S. 2018, Infinitesimal bending of curves on the ruled surfaces, *University thought - Publication in Natural Sciences*, 8(1), pp. 46-51. doi:10.5937/univtho8-17403
- Rančić, S., Najdanović, M., & Velimirović, Lj. 2019, Total normalcy of knots, *Filomat*, 33(4), pp. 1259-1266. doi:10.2298/FIL1904259R
- Rančić, S., Velimirović, Lj., & Zlatanović, M. 2009, Curvebend graphical tool for presentation of infinitesimal bending of curves, *Filomat*, 23(2), pp. 108-116. doi:10.2298/fil0902108r
- Rýparová, L. & Mikeš, J. 2019, Infinitesimal Rotary Transformation, *Filomat*, 33(4), pp. 1153-1157. doi.org/10.2298/FIL1904153R
- Vekua, I. 1959, *Obobschennye analiticheskie funkcii* (Moskva)
- Velimirović, Lj. 2001a, Change of geometric magnitudes under infinitesimal bending, *Facta universitatis - series: Mechanics, Automatic Control and Robotics*, 3(11), pp. 135-148.
- Velimirović, Lj. 2001b, Infinitesimal bending of curves, *Matematički bilten Skopje, Makedonija*, 25(LI), pp. 25-36.