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EXISTENCE OF INVARIANT POINTS AND APPLICATIONS TO SIMULTANEOUS APPROXIMATION

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ABSTRACT

For the set of ε-simultaneous approximation and ε-simultaneous coapproximation, we derive certain Brosowski-Meinardus type invariant point results in this paper. As a consequence, some results on ε-approximation, ε-coapproximation, best approximation, and best coapproximation are also deduced.

Keywords: ε-simultaneous approximatively compact set, Starshaped set, Best approximation, Best simultaneous approximation, ε-simultaneous approximation.

INTRODUCTION AND PRELIMINARIES

The study of best approximation theory plays an important role in nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality, complementarity problems, and so forth. The idea of applying fixed point theorems to approximation theory was initiated in normed linear spaces by Meinardus (1963). Later, Brosowski (1969) generalized the result of Meinardus and proved a nice result on invariant approximation. Thereafter, various generalizations of Brosowski's results appeared in the literature.

Singh (1979a) observed that the linearity of the operator $\mathscr{T}$ and convexity of the set $P_{\mathcal{G}}(\mathfrak{x})$ can be relaxed and proved an interesting result. Later, Singh (1979b) demonstrated that previous result of Singh (1979a) remains valid if $\mathscr{T}$ is assumed to be nonexpansive only on the set $P_{\mathcal{G}}(\mathfrak{x}) \cup \{\mathfrak{x}\}$. Thenceforth, many results have been obtained in this direction by many researchers (see Chandok (2019); Chandok & Narang (2011a,b, 2012a,b, 2013); Khan & Akbar (2009a,b); Mukherjee & Som (1985); Narang & Chandok (2009a,b,c); Rao & Mariadoss (1983) and references cited therein).

In this article, we obtain some similar types of results on $\mathscr{T}$-invariant points for the set of ε-simultaneous approximation and ε-simultaneous coapproximation for a Hardy-Roger type contraction mapping defined on a Takahashi space $(\mathscr{X}, d, W)$. For such class of mappings, we also deduce some results on $\mathscr{T}$-invariant points for the set of ε-approximation, ε-coapproximation, best approximation and best coapproximation.

Definition 1. Let $(\mathscr{X}, d)$ be a metric space, $\emptyset \neq \mathcal{G} \subset \mathscr{X}$, $\mathcal{F}$ a nonempty bounded subset of $\mathscr{X}$. For $\mathfrak{x} \in \mathscr{X}$, assume that

$$d_{\mathcal{F}}(\mathfrak{x}) = \{\sup d(y, \mathfrak{x}) : y \in \mathcal{F}\},$$

$$D(\mathcal{F}, \mathcal{G}) = \{\inf d_{\mathcal{F}}(\mathfrak{x}) : \mathfrak{x} \in \mathcal{G}\},$$

and

$$P_{\mathcal{G}}(\mathcal{F}) = \{g_0 \in \mathcal{G} : d_{\mathcal{F}}(g_0) = D(\mathcal{F}, \mathcal{G})\}.$$

An element $g_0 \in P_{\mathcal{G}}(\mathcal{F})$ is said to be a **best simultaneous approximation** of $\mathcal{F}$ with respect to $\mathcal{G}$ (see Chandok & Narang (2011a)).

For $\varepsilon > 0$, we define

$$\begin{aligned} P_{\mathcal{G}(\varepsilon)}(\mathcal{F}) &= \{g_0 \in \mathcal{G} : d_{\mathcal{F}}(g_0) \leq D(\mathcal{F}, \mathcal{G}) + \varepsilon\} \\ &= \{g_0 \in \mathcal{G} : \sup_{y \in \mathcal{F}} d(y, g_0) \leq \inf_{g \in \mathcal{G}} \sup_{y \in \mathcal{F}} d(y, g) + \varepsilon\}. \end{aligned}$$

An element $g_0 \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is said to be a ε-simultaneous approximation of $\mathcal{F}$ with respect to $\mathcal{G}$ (see Chandok & Narang (2011a)).

It can be easily seen that for $\varepsilon > 0$, the set $P_{\mathcal{G}}(\varepsilon)(\mathcal{F})$ is always a nonempty bounded set and is closed if $\mathcal{G}$ is closed.

In case $\mathcal{F} = \{p\}$, $p \in \mathscr{X}$, then elements of $P_{\mathcal{G}}(p)$ are called **best approximations** to p in $\mathcal{G}$ and of $P_{\mathcal{G}(\varepsilon)}(p)$ are called ε-**approximation** to p in $\mathcal{G}$.

For $\varepsilon > 0$, we define

$$R_{\mathcal{G}(\varepsilon)}(\mathcal{F}) = \{g_0 \in \mathcal{G} : \sup_{g \in \mathcal{G}} d(g_0, g) + \varepsilon \leq \inf_{g \in \mathcal{G}} \sup_{y \in \mathcal{F}} d(y, g)\}.$$

An element $g_0 \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is said to be a ε-simultaneous coapproximation of $\mathcal{F}$ with respect to $\mathcal{G}$ (see Chandok & Narang (2011a)).

In case $\mathcal{F} = \{p\}$, $p \in \mathscr{X}$, then elements of $R_{\mathcal{G}}(p)$ are called **best coapproximations** to p in $\mathcal{G}$ and of $R_{\mathcal{G}(\varepsilon)}(p)$ are called ε-**coapproximation** to p in $\mathcal{G}$.

Let $\mathscr{T}$ be a self mapping defined on a subset $\mathcal{G}$ of a metric space $\mathscr{X}$. A best approximant $\mathfrak{y}$ in $\mathcal{G}$ to an element $\mathfrak{x}_0$ in $\mathscr{X}$ with $\mathscr{T}\mathfrak{x}_0 = \mathfrak{x}_0$ is an invariant approximation in $\mathscr{X}$ to $\mathfrak{x}_0$ if $\mathscr{T}\mathfrak{y} = \mathfrak{y}$.

Example 2. Let $\mathscr{X} = \mathbb{R}$ with usual metric and $\mathcal{G} = [0, 1] \subset \mathscr{X}$. Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ as

$$\mathscr{T}\mathfrak{x} = \begin{cases} \mathfrak{x}, \mathfrak{x} < 2 \\ \frac{\mathfrak{x}+2}{2}, \mathfrak{x} \geq 2. \end{cases}$$

Clearly, $\mathscr{T}(\mathcal{G}) = \mathcal{G}$ and $\mathscr{T}(2) = 2$. Also, $P_{\mathcal{G}}(2) = \{1\}$. Hence $\mathscr{T}$ has a fixed point in $\mathscr{X}$ which is a best approximation to 2 in $\mathcal{G}$. Thus, 2 is an invariant approximation.

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Definition 3. A sequence $\{y_n\}$ in $\mathcal{G}$ is called a ε-**minimizing sequence** for $\mathcal{F}$, if

$$\limsup_{\mathfrak{x}\in\mathcal{F}} d(\mathfrak{x}, y_n) \leq D(\mathcal{F}, \mathcal{G}) + \varepsilon.$$

The set $\mathcal{G}$ is said to be ε-**simultaneous approximatively compact with respect to** $\mathcal{F}$ (see Chandok & Narang (2011a)) if for every $\mathfrak{x} \in \mathcal{F}$, each ε-minimizing sequence $\{y_n\}$ in $\mathcal{G}$ has a subsequence $\{y_{n_i}\}$ converging to an element of $\mathcal{G}$.

Inspired by the work of Takahashi (1970) and Guay et al. (1982), we have the following definition.

Definition 4. Let $\mathscr{X}$ be a nonempty set, d be a metric on $\mathscr{X}$ and $W : \mathscr{X} \times \mathscr{X} \times [0, 1] \to \mathscr{X}$ be a continuous mapping satisfying, for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{u} \in \mathscr{X}$ and $\lambda \in [0, 1]$,

1. $d(\mathfrak{u}, W(\mathfrak{x}, \mathfrak{y}, \lambda)) \leq \lambda d(\mathfrak{u}, \mathfrak{x}) + (1 - \lambda)d(\mathfrak{u}, \mathfrak{y})$,
2. $d(W(\mathfrak{x}, \mathfrak{u}, \lambda), W(\mathfrak{y}, \mathfrak{u}, \lambda)) \leq d(\mathfrak{x}, \mathfrak{y})$.

Then the triple $(\mathscr{X}, d, W)$ is called a **Takahashi space**.

A normed linear space and each of its convex subset are simple examples of Takahashi spaces with W given by $W(\mathfrak{x}, \mathfrak{y}, \lambda) = \lambda\mathfrak{x} + (1 - \lambda)\mathfrak{y}$ for $\mathfrak{x}, \mathfrak{y} \in \mathscr{X}$ and $0 \leq \lambda \leq 1$. For definition of convex set, q-starshaped set and starshaped set see Chandok & Narang (2011a) and references cited therein.

Definition 5. Let $\mathcal{G}$ be a nonempty subset of a metric space $(\mathscr{X}, d)$ and $\mathscr{T} : \mathcal{G} \to \mathcal{G}$ be a self map. Then $\mathscr{T}$ is said to be **asymptotically regular** (see, Browder & Petryshyn (1966)) if for all $\mathfrak{x} \in \mathcal{G}$, $d(\mathscr{T}^n(\mathfrak{x}), \mathscr{T}^{n+1}(\mathfrak{x})) \to 0$ as $n \to \infty$.

Definition 6. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfies **condition (A)** (see Mukherjee & Verma (1989)) if

$$d(\mathscr{T}^n\mathfrak{x}, \mathfrak{y}) \leq d(\mathfrak{x}, \mathfrak{y}),$$

for all $\mathfrak{x}, \mathfrak{y} \in \mathscr{X}$ and for some positive integer n.

MAIN RESULTS

Inspired by the work of Hardy-Roger, we define the following contraction:

Definition 7. Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a **HR-type contraction** if there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + 2\gamma < 1$, $\alpha + \gamma \neq 1$ such that for all $\mathfrak{x}, \mathfrak{y} \in \mathscr{X}$, we have

$$\begin{aligned} d(\mathscr{T}\mathfrak{x}, \mathscr{T}\mathfrak{y}) \quad \leq \quad & \alpha\frac{d(\mathfrak{x}, \mathscr{T}\mathfrak{x})d(\mathfrak{y}, \mathscr{T}\mathfrak{y})}{1 + d(\mathfrak{x}, \mathfrak{y})} + \beta(d(\mathfrak{x}, \mathfrak{y})) + \\ & \gamma(d(\mathfrak{x}, \mathscr{T}\mathfrak{x}) + d(\mathfrak{y}, \mathscr{T}\mathfrak{y})). \qquad (1) \end{aligned}$$

Remark 8. On a metric space, every HR-type contraction has at most one fixed point. Indeed, let $\mathfrak{x}$ and $\mathfrak{y}$ be two distinct fixed points of $\mathscr{T}$, which is a HR-type contraction. Then

$$\begin{aligned} d(\mathfrak{x}, \mathfrak{y}) = d(\mathscr{T}\mathfrak{x}, \mathscr{T}\mathfrak{y}) \quad \leq \quad & \alpha\frac{d(\mathfrak{x}, \mathscr{T}\mathfrak{x})d(\mathfrak{y}, \mathscr{T}\mathfrak{y})}{1 + d(\mathfrak{x}, \mathfrak{y})} + \beta(d(\mathfrak{x}, \mathfrak{y})) + \\ & \gamma(d(\mathfrak{x}, \mathscr{T}\mathfrak{x}) + d(\mathfrak{y}, \mathscr{T}\mathfrak{y})) \\ = \quad & \beta(d(\mathfrak{x}, \mathfrak{y})), \end{aligned}$$

which is a contradiction as $0 \leq \beta < 1$ and $d(\mathfrak{x}, \mathfrak{y}) > 0$.

The following result will be needed in the sequel.

Proposition 9. *Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a HR-type contraction on a metric space $(\mathscr{X}, d)$. Then for all $\mathfrak{x} \in \mathscr{X}$, the sequence $\{d(\mathscr{T}^n\mathfrak{x}, \mathscr{T}^{n+1}\mathfrak{x})\}$ is decreasing and $\mathscr{T}$ is asymptotically regular.*

Proof. Let $\mathfrak{x}_0$ be an arbitrary point in $\mathscr{X}$ and $\{\mathfrak{x}_n\}$ be sequence in $\mathscr{X}$ such that $\mathfrak{x}_{n+1} = \mathscr{T}\mathfrak{x}_n = \mathscr{T}^n\mathfrak{x}_0$, for every $n \geq 0$. Using (1), we have

$$\begin{aligned} d(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+1}) \quad = \quad & d(\mathscr{T}\mathfrak{x}_{n+1}, \mathscr{T}\mathfrak{x}_n) \\ \leq \quad & \alpha\frac{d(\mathfrak{x}_{n+1}, \mathscr{T}\mathfrak{x}_{n+1})d(\mathfrak{x}_n, \mathscr{T}\mathfrak{x}_n)}{1 + d(\mathfrak{x}_{n+1}, \mathfrak{x}_n)} + \beta(d(\mathfrak{x}_{n+1}, \mathfrak{x}_n)) + \\ & \gamma(d(\mathfrak{x}_{n+1}, \mathscr{T}\mathfrak{x}_{n+1}) + d(\mathfrak{x}_n, \mathscr{T}\mathfrak{x}_n)) \\ = \quad & \alpha\frac{d(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2})d(\mathfrak{x}_n, \mathfrak{x}_{n+1})}{1 + d(\mathfrak{x}_{n+1}, \mathfrak{x}_n)} + \beta(d(\mathfrak{x}_{n+1}, \mathfrak{x}_n)) + \\ & \gamma(d(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}) + d(\mathfrak{x}_n, \mathfrak{x}_{n+1})) \\ \leq \quad & (\alpha + \gamma)d(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}) + (\beta + \gamma)d(\mathfrak{x}_{n+1}, \mathfrak{x}_n). \end{aligned}$$

This implies

$$d(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+1}) \leq \frac{\beta + \gamma}{1 - \alpha - \gamma}d(\mathfrak{x}_{n+1}, \mathfrak{x}_n). \qquad (2)$$

Since $L = \frac{\beta+\gamma}{1-\alpha-\gamma} < 1$, the sequence $\{d(\mathscr{T}^n\mathfrak{x}_0, \mathscr{T}^{n+1}\mathfrak{x}_0)\}$ is a decreasing sequence. Using mathematical induction, we have

$$d(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+1}) \leq (L)^{n+1}\, d(\mathfrak{x}_1, \mathfrak{x}_0). \qquad (3)$$

Taking the limit $n \to \infty$, we have $d(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+1}) \to 0$, that is, $d(\mathscr{T}^n\mathfrak{x}_0, \mathscr{T}^{n+1}\mathfrak{x}_0) \to 0$. Hence the result.

Using the above proposition, we prove the following:

Theorem 10. *Every HR-type contraction on a complete metric space has unique fixed point.*

Proof. Using Proposition , the sequence $\{d(\mathscr{T}^n\mathfrak{x}_0, \mathscr{T}^{n+1}\mathfrak{x}_0)\}$ is decreasing and $d(\mathscr{T}^n\mathfrak{x}_0, \mathscr{T}^{n+1}\mathfrak{x}_0) \to 0$ as $n \to \infty$ for all $\mathfrak{x}_0 \in \mathscr{X}$. We claim that $\{\mathfrak{x}_n\}$ is a Cauchy sequence. For $m > n$, and $L = \dfrac{\beta + \gamma}{1 - \alpha - \gamma} < 1$ we have

$$\begin{aligned} d(\mathfrak{x}_n, \mathfrak{x}_m) &\leq d(\mathfrak{x}_n, \mathfrak{x}_{n+1}) + d(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}) + \ldots + d(\mathfrak{x}_{m-1}, \mathfrak{x}_m) \\ &\leq (L^n + L^{n+1} + \ldots + L^{m-1})d(\mathfrak{x}_0, \mathfrak{x}_1) \\ &\leq \frac{L^n(1 - L^{m-n})}{1 - L}d(\mathfrak{x}_0, \mathfrak{x}_1). \end{aligned}$$

Therefore, $d(\mathfrak{x}_m, \mathfrak{x}_n) \to 0$, when $m, n \to \infty$. Thus $\{\mathfrak{x}_n\}$ is a Cauchy sequence in a complete metric space $\mathscr{X}$ and so there exists $u \in \mathscr{X}$ such that $\lim_{n\to\infty} \mathfrak{x}_n = u$.

Now, we'll show that the point u is a fixed point of $\mathscr{T}$. On the contrary, suppose that $\mathscr{T}u \neq u$, then $d(u, \mathscr{T}u) > 0$. Consider

$$\begin{aligned} d(\mathfrak{x}_{n+1}, \mathscr{T}u) = d(\mathscr{T}\mathfrak{x}_n, \mathscr{T}u) \leq & \alpha\frac{d(\mathfrak{x}_n, \mathscr{T}\mathfrak{x}_n)d(u, \mathscr{T}u)}{1 + d(\mathfrak{x}_n, u)} + \beta(d(\mathfrak{x}_n, u)) + \\ & \gamma(d(\mathfrak{x}_n, \mathscr{T}\mathfrak{x}_n) + d(u, \mathscr{T}u)) \\ = & \alpha\frac{d(\mathfrak{x}_n, \mathfrak{x}_{n+1})d(u, \mathscr{T}u)}{1 + d(\mathfrak{x}_n, u)} + \beta(d(\mathfrak{x}_n, u)) + \\ & \gamma(d(\mathfrak{x}_n, \mathfrak{x}_{n+1}) + d(u, \mathscr{T}u)). \end{aligned}$$

Taking $n \to \infty$, we have $d(u, \mathscr{T}u) \leq \gamma d(u, \mathscr{T}u)$, it implies that $d(u, \mathscr{T}u) = 0$. Hence u is a fixed point of $\mathscr{T}$. Using Remark , we obtain that $\mathscr{T}$ has unique fixed point.

Example 11. Let $\mathscr{X} = [0,1]$ and d be the usual metric on $\mathscr{X}$. Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ as $\mathscr{T}\mathfrak{x} = \begin{cases} \frac{\mathfrak{x}}{10}, & \mathfrak{x} \in [0, \frac{1}{2}] \\ \frac{\mathfrak{x}}{5} - \frac{1}{20}, & \mathfrak{x} \in (\frac{1}{2}, 1]. \end{cases}$

Suppose $\alpha = \frac{1}{8}, \beta = \frac{1}{4}, \gamma = \frac{1}{8} \in [0,1)$ with $\alpha + \beta + 2\gamma = \frac{5}{8} < 1$. We may check that

$$d(\mathscr{T}\mathfrak{x}, \mathscr{T}y) \leq \frac{1}{8}\frac{d(\mathfrak{x}, \mathscr{T}\mathfrak{x})d(y, \mathscr{T}y)}{1 + d(\mathfrak{x}, y)} + \frac{1}{4}(d(\mathfrak{x}, y)) + \frac{1}{8}(d(\mathfrak{x}, \mathscr{T}\mathfrak{x}) + d(y, \mathscr{T}y)),$$

for all $\mathfrak{x}, y \in \mathscr{X}$. Thus using Theorem , $\mathscr{T}$ has unique fixed point. Notice that $0 \in \mathscr{X}$ is the fixed point of $\mathscr{T}$.

Theorem 12. *Let $(\mathscr{X}, d, W)$ be a complete Takahashi space, G be a nonempty subset of $\mathscr{X}$ and $\mathcal{F}$ a nonempty bounded subset of $\mathscr{X}$. Suppose that $\mathscr{T}_n$ is a self map on $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ such that $\mathfrak{x}_{n+1} = \mathscr{T}_n\mathfrak{x} = W(\mathscr{T}^n\mathfrak{x}, q, \lambda_n)$, where $\lambda_n \in (0,1)$ and satisfying the following for some positive integer n,*

$$\begin{aligned} d(\mathscr{T}^n\mathfrak{x}, \mathscr{T}^n y) \quad \leq \quad & \alpha\left(\frac{dist(\mathfrak{x}, [\mathscr{T}^n\mathfrak{x}, q])dist(y, [\mathscr{T}^n y, q])}{1 + d(\mathfrak{x}, y)}\right) + \\ & \beta(d(\mathfrak{x}, y)) + \\ & \gamma(dist(\mathfrak{x}, [\mathscr{T}^n\mathfrak{x}, q]) + dist(y, [\mathscr{T}^n y, q])), \end{aligned} \tag{4}$$

for all $\mathfrak{x}, y, q \in \mathscr{X}$, where $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + 2\gamma < 1$, $\alpha + \gamma \neq 1$. If $\mathscr{T}$ is continuous and $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is compact, and q-starshaped, then it contains a $\mathscr{T}$-invariant point.

Proof. Define $\mathscr{T}_n : P_{\mathcal{G}(\varepsilon)}(\mathcal{F}) \to P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ as $\mathscr{T}_n z = W(\mathscr{T}^n z, q, \lambda_n)$, $z \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ where $\{\lambda_n\}$ is a sequence in $(0,1)$ such that $\lambda_n \to 1$. Consider

$$\begin{aligned} d(\mathscr{T}_n z, \mathscr{T}_n y) &= d(W(\mathscr{T}^n z, q, \lambda_n), W(\mathscr{T}^n y, q, \lambda_n)) \\ &\leq \lambda_n d(\mathscr{T}^n z, \mathscr{T}^n y) \\ &\leq \lambda_n\Big[\alpha\Big(\frac{d(z, [\mathscr{T}^n z, q])d(y, [\mathscr{T}^n y, q])}{1 + d(z, y)}\Big) + \beta(d(z, y)) + \\ &\gamma(d(z, [\mathscr{T}^n z, q]) + d(y, [\mathscr{T}^n y, q]))\Big] \\ &\leq \lambda_n\Big[\alpha\Big(\frac{d(z, \mathscr{T}_n z)d(y, \mathscr{T}_n y)}{1 + d(z, y)}\Big) + \beta(d(z, y)) + \\ &\gamma(d(z, \mathscr{T}_n z) + d(y, \mathscr{T}_n y))\Big], \end{aligned}$$

where $\lambda_n(\alpha + \beta + 2\gamma) < 1, z, y \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Therefore by Theorem , each $\mathscr{T}_n$ has a unique fixed point z_n in $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Since $\{\mathscr{T}^n z_n\}$ is a sequence in the compact set $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$, there exists a subsequence $\{\mathscr{T}^{n_i} z_{n_i}\}$ of $\{\mathscr{T}^n z_n\}$ such that $\{\mathscr{T}^{n_i} z_{n_i}\} \to z \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Moreover,

$$z_{n_i} = \mathscr{T}_{n_i} z_{n_i} = W[\mathscr{T}^{n_i} z_{n_i}, q, \lambda_{n_i}] \to z.$$

As $\mathscr{T}$ is continuous, $\mathscr{T}^{n_i} z_{n_i} \to \mathscr{T}^{n_i} z$. By the uniqueness of the limit, we have $\lim_{n\to\infty} \mathscr{T}^{n_i} z = z$ and so $\lim_{n\to\infty} \mathscr{T}^{n_i+1} z = \mathscr{T}z$.

Now, we show that $d(z, \mathscr{T}z) = 0$. Consider

$$d(z, \mathscr{T}z) \leq d(z, \mathscr{T}^{n_i} z) + d(\mathscr{T}^{n_i} z, \mathscr{T}^{n_i+1} z) + d(\mathscr{T}^{n_i+1} z, \mathscr{T}z).$$

Letting $n \to \infty$, in the above inequality, and using $\mathscr{T}$ is asymptotically regular, we have $d(z, \mathscr{T}z) \to 0$. Therefore $\mathscr{T}z = z$. i.e. z is $\mathscr{T}$-invariant.

Using Proposition 2.1 of Chandok & Narang (2011a), we have the following result.

Corollary 13. *Let $(\mathscr{X}, d, W)$ be a complete Takahashi space, G be a nonempty subset of $\mathscr{X}$ and $\mathcal{F}$ a nonempty bounded subset of $\mathscr{X}$. Suppose that $\mathscr{T}_n$ is a self map on $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ such that $\mathfrak{x}_{n+1} = \mathscr{T}_n\mathfrak{x} = W(\mathscr{T}^n\mathfrak{x}, q, \lambda_n)$, where $\lambda_n \in (0,1)$ and satisfying the inequality* (4). *If $\mathscr{T}$ is continuous, $\mathcal{G}$ is ε-simultaneous approximatively compact with respect to $\mathcal{F}$ and $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is starshaped, then it contains a $\mathscr{T}$-invariant point.*

For $\mathcal{F} = \{\mathfrak{x}\}$ and $\varepsilon = 0$, we have the following result on the set of best approximation.

Corollary 14. *Let $(\mathscr{X}, d, W)$ be a complete Takahashi space, G be a nonempty subset of $\mathscr{X}$. Suppose that $\mathscr{T}_n$ is a self map on $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ such that $\mathfrak{x}_{n+1} = \mathscr{T}_n\mathfrak{x} = W(\mathscr{T}^n\mathfrak{x}, q, \lambda_n)$, where $\lambda_n \in (0,1)$ and satisfying the inequality* (4). *If $\mathscr{T}$ is continuous, $\mathcal{G}$ is approximatively compact, $\mathscr{T}$-invariant subset of $\mathscr{X}$ and $\mathfrak{x}$ a $\mathscr{T}$-invariant point and $P_{\mathcal{G}}(\mathfrak{x})$ is starshaped, then $P_{\mathcal{G}}(\mathfrak{x})$ contains a $\mathscr{T}$-invariant point.*

We now prove a result for $\mathscr{T}$-invariant points from the set of ε-simultaneous coapproximations.

Theorem 15. *Let $(\mathscr{X}, d, W)$ be a complete Takahashi space, G be a nonempty subset of $\mathscr{X}$ and $\mathcal{F}$ a nonempty bounded subset of $\mathscr{X}$. Suppose that $\mathscr{T}_n$ is a self map on $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ such that $\mathfrak{x}_{n+1} = \mathscr{T}_n\mathfrak{x} = W(\mathscr{T}^n\mathfrak{x}, q, \lambda_n)$, where $\lambda_n \in (0,1)$ and satisfying the inequality* (4). *Assume that $\mathscr{T}$ is continuous and satisfying condition (A). If $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is compact and q-starshaped, then $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ contains a $\mathscr{T}$-invariant point.*

Proof. Let $g_0 \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Consider

$$d(\mathscr{T}^n g_0, g) + \varepsilon \leq d(g_0, g) + \varepsilon \leq \inf_{g \in \mathcal{G}} sup_{y \in \mathcal{F}} d(y, g),$$

and so $\mathscr{T}^n g_0 \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ i.e. $\mathscr{T}^n : R_{\mathcal{G}(\varepsilon)}(\mathcal{F}) \to R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Since $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is q-starshaped, $W(z, q, \lambda) \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ for all $z \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$, $\lambda \in [0,1]$. Let $\{\lambda_n\}$, $0 \leq \lambda_n < 1$, be a sequence of real numbers such that $\lambda_n \to 1$ as $n \to \infty$. Define $\mathscr{T}_n$ as $\mathscr{T}_n(z) = W(\mathscr{T}^n z, q, \lambda_n)$, $z \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Since $\mathscr{T}$ is a self mapping on $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ and $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is starshaped, each $\mathscr{T}_n$ is a well defined and maps $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ into $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Moreover,

$$\begin{aligned}
d(\mathcal{T}_n y, \mathcal{T}_n z) &= d(W(\mathcal{T}^n y, q, \lambda_n), W(\mathcal{T}^n z, q, \lambda_n)) \\
&\leq \lambda_n d(\mathcal{T}^n y, \mathcal{T}^n z) \\
&\leq \lambda_n \Big[\alpha \Big(\frac{d(y, [\mathcal{T}^n y, q]) d(z, [\mathcal{T}^n z, q])}{1 + d(y, z)} \Big) + \beta(d(y, z)) + \\
&\gamma(d(y, [\mathcal{T}^n y, q]) + d(z, [\mathcal{T}^n z, q])) \Big] \\
&\leq \lambda_n \Big[\alpha \Big(\frac{d(y, \mathcal{T}^n y) d(z, \mathcal{T}^n z)}{1 + d(y, z)} \Big) + \beta(d(y, z)) + \\
&\gamma(d(y, \mathcal{T}^n y) d(z, \mathcal{T}^n z)) \Big],
\end{aligned}$$

where $\lambda_n[\alpha + \beta] < 1$. So by Theorem each $\mathcal{T}_n$ has a unique fixed point $u_n \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ i.e. $\mathcal{T}_n u_n = u_n$ for each n. Since $\{\mathcal{T}^n u_n\}$ is a sequence in the compact set $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$, there exists a subsequence $\{\mathcal{T}^{n_i} u_{n_i}\}$ of $\{\mathcal{T}^n u_n\}$ such that $\{\mathcal{T}^{n_i} u_{n_i}\} \to u \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Moreover,

$$u_{n_i} = \mathcal{T}_{n_i} u_{n_i} = W[\mathcal{T}^{n_i} u_{n_i}, q, \lambda_{n_i}] \to u.$$

As $\mathcal{T}$ is continuous, $\mathcal{T}^{n_i} u_{n_i} \to \mathcal{T}^{n_i} u$. By the uniqueness of the limit, we have $\lim_{n\to\infty} \mathcal{T}^{n_i} u = u$ and so $\lim_{n\to\infty} \mathcal{T}^{n_i+1} u = \mathcal{T} u$.

Now, we show that $d(u, \mathcal{T} u) = 0$. Since $\mathcal{T}$ is asymptotically regular, we have

$$d(u, \mathcal{T} u) \leq d(u, \mathcal{T}^{n_i} u) + d(\mathcal{T}^{n_i} u, \mathcal{T}^{n_i+1} u) + d(\mathcal{T}^{n_i+1} u, \mathcal{T} u) \to 0.$$

Therefore $\mathcal{T} u = u$. i.e. u is $\mathcal{T}$-invariant.

Remark 16.

1. By taking $\mathcal{F} = \{\mathfrak{x}_1, \mathfrak{x}_2\}$, $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathcal{X}$, the set $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ (respectively, $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$) is the set of ε-simultaneous approximation (respectively, ε-simultaneous coapproximation) to the pair of points $\mathfrak{x}_1, \mathfrak{x}_2$ and so we can obtain the results for such pair of points $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ (respectively, $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$).
2. By taking $\mathcal{F} = \{\mathfrak{x}\}$, $\mathfrak{x} \in \mathcal{X}$, the set $P_{\mathcal{G}(\varepsilon)}(\mathfrak{x})$ (respectively, $R_{\mathcal{G}(\varepsilon)}(\mathfrak{x})$) is the set of ε-approximation(respectively, ε-coapproximation) to point $\mathfrak{x}$ and so we can obtain the results on the set of ε-approximation (respectively, ε-coapproximation).
3. By taking $\mathcal{F} = \{\mathfrak{x}\}$ and $\varepsilon = 0$, we can obtain the results on the set of best approximation (respectively, best coapproximation).

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