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GENERALIZED α-MIN SPECIAL TYPE CONTRACTION RESULTS ON 2-MENGER SPACES

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ABSTRACT

The goal of this paper is to present some novel probabilistic α-minimum contraction results on probabilistic 2-metric spaces. Our findings are based on probabilistic 2-metric spaces, which are the probabilistic generalisations of 2-metric spaces. An illustrative example backs up our findings.

Keywords: 2-Menger spaces, Cauchy sequence, Fixed point, ϕ-function, Altering distance function.

INTRODUCTION

Fixed point theory and its applications are important areas of study in mathematics. Metric fixed point theory is used in differential calculus, integral calculus, optimization problems, matrix equations, and a variety of other disciplines of study. Probabilistic 2-metric spaces (2-PMS), which are the probabilistic generalisation of 2-metric spaces, are studied in this paper. Zeng (1987) pioneered these spaces in which distribution function plays the role of metric. 2-Menger spaces are probabilistic 2-metric spaces in which the triangle inequality is hypothesised using the t-norm. Khan et al. (1984) developed the innovative notion of altering distance function in 1984. The "altering distance function" is a control function that changes the distance between two points in metric space. In Choudhury & Das (2008), the concept of changing the distance function has recently been expanded to the context of Menger spaces. This control function is referred to as the ϕ-function, and it is extremely useful for proving fixed point conclusions in Menger spaces. This approach is also applicable to many other situations in this area, such as coincidence point problems. Some recent works using ϕ-function are mentioned in Bhandari (2017a); Bhandari & Choudhury (2017); Bhandari (2017b); Choudhury et al. (2012); Dutta et al. (2009). In recent research works, probabilistic metric (PM) spaces have an important role. Many authors have established various types of results on this popular directions. Some generalized works in this line may be referred as Kutbi et al. (2015); Mihet (2009).

Main features of this paper are following:
1. A new probabilistic $\alpha-$ min special type contraction result.
2. For such contraction, unique fixed point is obtained.
3. Here we use a control function.
4. An illustrative example validates our theorem.

PRELIMINARIES

Some important definitions and mathematical preliminaries are discussed in this section. These are helpful to prove our main results.

Definition 1. A distribution function (see Hadzic & Pap (2001); Schweizer & Sklar (1983)) is a mapping $\Gamma : \mathbb{R} \to \mathbb{R}^+$ if it is non-decreasing and left continuous with $\inf_{\eta\in\mathbb{R}} \Gamma(\eta) = 0$ and $\sup_{\eta\in\mathbb{R}} \Gamma(\eta) = 1$, where $\mathbb{R}$ is the set of reals and $\mathbb{R}^+$ is the set of non-negative reals respectively.

This function has an very important role in our present discussion.

Definition 2. A probabilistic metric space (briefly, PM-space) Hadzic & Pap (2001); Schweizer & Sklar (1983) is an ordered pair (S, Γ), where S is a non-empty set and Γ is a mapping from $S \times S$ into the set of all distribution functions. The function $\Gamma_{\kappa,\mu}$ is assumed to satisfy the following conditions for all $\kappa, \mu, \nu \in S$,

(i) $\Gamma_{\kappa,\mu}(0) = 0$,
(ii) $\Gamma_{\kappa,\mu}(\eta) = 1$ for all $\eta > 0$ if and only if $\kappa = \mu$,
(iii) $\Gamma_{\kappa,\mu}(\eta) = \Gamma_{\mu,\kappa}(\eta)$ for all $\eta > 0$,
(iv) if $\Gamma_{\kappa,\mu}(\eta_1) = 1$ and $\Gamma_{\mu,\nu}(\eta_2) = 1$ then $\Gamma_{\kappa,\nu}(\eta_1 + \eta_2) = 1$ for all $\eta_1, \eta_2 > 0$.

The theory on these spaces have been discussed vastly in the book of Schweizer and Sklar Schweizer & Sklar (1983).

Example 3. Let $S = [0, 7]$ and $\Gamma_{\kappa,\mu}(\eta) = \frac{\eta}{\eta+|\kappa-\mu|}$, then (S, Γ) is a PM space.

Shi et al. (2003) introduced the following definition of n-th order t-norm. It is a function which is used to construct our main results.

Definition 4. A mapping $T : \Pi_{i=1}^{n}[0, 1] \to [0, 1]$ is called a n-th order t-norm if the following conditions are satisfied:

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(i) $T(0,0,\dots,0) = 0$, $T(a,1,1,\dots,1) = a$ for all $a \in [0,1]$,
(ii) $T(a_1, a_2, a_3, \dots, a_n) = T(a_2, a_1, a_3, \dots, a_n) = T(a_2, a_3, a_1, \dots, a_n) = \dots = T(a_2, a_3, a_4, \dots, a_n, a_1)$,
(iii) $a_i \geq b_i$, $i = 1,2,3,\dots,n$, implies $T(a_1, a_2, a_3, \dots, a_n) \geq T(b_1, b_2, b_3, \dots, b_n)$,
(iv) $T(T(a_1, a_2, a_3, \dots, a_n), b_2, b_3, \dots, b_n)$
$$= T(a_1, T(a_2, a_3, \dots, a_n, b_2), b_3, \dots, b_n)$$
$$= T(a_1, a_2, T(a_3, a_4, \dots, a_n, b_2, b_3), b_4, \dots, b_n)$$
$$\vdots$$
$$= T(a_1, a_2, \dots, a_{n-1}, T(a_n, b_2, b_3, \dots, b_n)).$$

When $n = 2$, we have a binary t-norm, which is commonly known as t-norm.

In this paper we use third order minimum t-norm. The following are the examples of different types of third order t-norms:

(i) The minimum t-norm, $\Delta = T_m$, defined by $T_m(a,b,c) = \min\{a,b,c\}$.
(ii) The product t-norm, $\Delta = T_p$, defined by $T_p(a,b,c) = a.b.c$.
(iii) The Lukasiewicz t-norm, $\Delta = T_L$, defined by $T_L(a,b,c) = \max\{a+b+c-1, 0\}$.

Menger spaces (see Hadzic & Pap (2001); Schweizer & Sklar (1983)) are the particular types of probabilistic metric spaces. The definition is given below.

Definition 5. A Menger space is a triplet (S, Γ, Δ), where S is a non empty set, Γ is a function defined on $S \times S$ to the set of all distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $\Gamma_{\kappa,\mu}(0) = 0$ for all $\kappa, \mu \in S$,
(ii) $\Gamma_{\kappa,\mu}(s) = 1$ for all $s > 0$ if and only if $\kappa = \mu$,
(iii) $\Gamma_{\kappa,\mu}(s) = \Gamma_{\mu,\kappa}(s)$ for all $\kappa, \mu \in S$, $s > 0$ and
(iv) $\Gamma_{\kappa,\mu}(u+v) \geq \Delta(\Gamma_{\kappa,\nu}(u), \Gamma_{\nu,\mu}(v))$ for all $u, v \geq 0$ and $\kappa, \mu, \nu \in S$.

A metric space becomes a Menger probabilistic metric space if we write $\Gamma_{\kappa,\mu}(\eta) = H(\eta - d(\kappa,\mu))$ where H is the Heavyside function given by

$$H(\eta) = \begin{cases} 1 \text{ if } & \eta > 0, \\ 0 \text{ if } & \eta \leq 0. \end{cases}$$

In 1963, S. Gähler (see Gähler (1963, 1965)) introduced the concept of 2-metric spaces. In metric spaces we consider a real valued function d on $S \times S$ but here we consider the real valued function d on $S \times S \times S$.

Definition 6. Let S be a non empty set. A real valued function d on $S \times S \times S$ is said to be a 2-metric on S if for all $\kappa, \mu, \nu, w \in S$,

(i) given distinct elements $\kappa, \mu \in S$, there exists an element ν of S such that
$$d(\kappa, \mu, \nu) \neq 0,$$
(ii) $d(\kappa,\mu,\nu) = 0$ when at least two of κ, μ, ν are equal,
(iii) $d(\kappa,\mu,\nu) = d(\kappa,\nu,\mu) = d(\mu,\nu,\kappa)$,
(iv) $d(\kappa,\mu,\nu) \leq d(\kappa,\mu,w) + d(\kappa,w,\nu) + d(w,\mu,\nu)$.

When d is a 2-metric on S, the ordered pair (S,d) is called a 2-metric space.

Example 7. If we consider three vertices κ, μ, ν of a triangle, then area of triangle may be taken as $d(\kappa,\mu,\nu)$. Then the metric function d satisfies all the conditions of 2-metric.

A probabilistic 2-metric space is a probabilistic generalization of 2-metric space. In 1987, Zeng Zeng (1987) introduced the concept of probabilistic 2-metric spaces.

Definition 8. A probabilistic 2-metric space is an order pair (S, Γ) where S is an arbitrary set and Γ is a mapping from $S \times S \times S$ into the set of all distribution functions such that the following conditions are satisfied:

(i) $\Gamma_{\kappa,\mu,\nu}(\eta) = 0$ for $\eta \leq 0$ and for all $\kappa, \mu, \nu \in S$,
(ii) $\Gamma_{\kappa,\mu,\nu}(\eta) = 1$ for all $\eta > 0$ if and only if at least two of κ, μ, ν are equal,
(iii) for distinct points $\kappa, \mu \in S$, there exists a point $\nu \in S$ such that $\Gamma_{\kappa,\mu,\nu}(\eta) \neq 1 \quad$ for $\eta > 0$,
(iv) $\Gamma_{\kappa,\mu,\nu}(\eta) = \Gamma_{\kappa,\nu,\mu}(\eta) = \Gamma_{\nu,\mu,\kappa}(\eta)$ for all $\kappa, \mu, \nu \in S$ and $\eta > 0$,
(v) $\Gamma_{\kappa,\mu,w}(\eta_1) = 1$, $\Gamma_{\kappa,w,\nu}(\eta_2) = 1$ and $\Gamma_{w,\mu,\nu}(\eta_3) = 1$ then $\Gamma_{\kappa,\mu,\nu}(\eta_1+\eta_2+\eta_3) = 1$, for all $\kappa, \mu, \nu, w \in S$ and $\eta_1, \eta_2, \eta_3 > 0$.

Example 9. Let $\Gamma_{\kappa,\mu,\nu}(\eta) = \begin{cases} \frac{\eta}{\eta + \min|\kappa-\mu|,|\kappa-\nu|,|\mu-\nu|} & \text{if} \quad \eta > 0, \\ 0, & \text{if} \quad \eta \leq 0, \end{cases}$
for all $(\kappa, \mu, \nu) \in S^3$. Then (S, Γ) is a probabilistic 2-metric spaces.

In Menger spaces, we use a function Γ which is defined on $S \times S$ to the set of all distribution functions but in case of 2-Menger spaces (see Shih-sen & Nan-Jing (1989)) we use the function Γ which is defined on $S \times S \times S$ to the set of all distribution functions.

Definition 10. Let S be a nonempty set. A triplet (S, Γ, Δ) is said to be a 2-Menger space if Γ is a mapping from $S \times S \times S$ into the set of all distribution functions satisfying the following conditions:

(i) $\Gamma_{\kappa,\mu,\nu}(0) = 0$,
(ii) $\Gamma_{\kappa,\mu,\nu}(\eta) = 1$ for all $\eta > 0$ if and only if at least two of $\kappa, \mu, \nu \in S$ are equal,
(iii) for distinct points $\kappa, \mu \in S$ there exists a point $\nu \in S$ such that $\Gamma_{\kappa,\mu,\nu}(\eta) \neq 1 \quad$ for $\eta > 0$,
(iv) $\Gamma_{\kappa,\mu,\nu}(\eta) = \Gamma_{\kappa,\nu,\mu}(\eta) = \Gamma_{\nu,\mu,\kappa}(\eta)$, for all $\kappa, \mu, \nu \in S$ and $\eta > 0$,
(v) $\Gamma_{\kappa,\mu,\nu}(\eta) \geq \Delta(\Gamma_{\kappa,\mu,w}(\eta_1), \Gamma_{\kappa,w,\nu}(\eta_2), \Gamma_{w,\mu,\nu}(\eta_3))$

where $\eta_1, \eta_2, \eta_3 > 0$, $\eta_1 + \eta_2 + \eta_3 = \eta$, $\kappa, \mu, \nu, w \in S$ and Δ is the 3rd order t norm.

Definition 11. Hadzic (1994) A sequence $\{\kappa_n\}$ in a 2-Menger space (S, Γ, Δ) is said to be converge to a limit κ if given $\epsilon > 0, 0 < \lambda < 1$ there exists a positive integer $N_{\epsilon,\lambda}$ such that
$$\Gamma_{\kappa_n,\kappa,a}(\epsilon) \geq 1 - \lambda \tag{1.1}$$
for all $n > N_{\epsilon,\lambda}$ and for every $a \in S$.

Definition 12. Hadzic (1994) A sequence $\{\kappa_n\}$ in a 2-Menger space (S,Γ,Δ) is said to be a Cauchy sequence in S if given $\epsilon > 0, 0 < \lambda < 1$ there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$\Gamma_{\kappa_n,\kappa_m,a}(\epsilon) \geq 1 - \lambda \tag{1.2}$$

for all $m, n > N_{\epsilon,\lambda}$ and for every $a \in S$.

In our main theorem we have used a complete 2-Menger spaces. Completeness property of spaces have an important role in our results.

Definition 13. Hadzic (1994) A 2-Menger space (S,Γ,Δ) is said to be complete if every Cauchy sequence is convergent in S.

We use the following control function Φ which Choudhury et al. presented in Choudhury & Das (2008).

Definition 14. A function $\phi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(\eta) = 0$ if and only if $\eta = 0$,
(ii) $\phi(\eta)$ is strictly monotone increasing and $\phi(\eta) \to \infty$ as $\eta \to \infty$,
(iii) ϕ is left continuous in $(0,\infty)$,
(iv) ϕ is continuous at 0.

Example 15. $\phi(\eta) = \eta^2$, $\phi(\eta) = \sqrt{\eta}$, $\phi(\eta) = \eta$ are some examples of Φ-function.

In numerous research works, many authors Choudhury & Bhandari (2014); Choudhury et al. (2015); Choudhury & Bhandari (2016) use this exciting property.

MAIN RESULTS

Motivated by Dutta et al. (2009); Gopal et al. (2014), we begin this section by introducing the concept of $\alpha - \min$ special type contraction and α-admissible mappings in 2-Menger spaces.

Definition 16. Let (S,Γ,Δ) be a 2-Menger space and $h : S \to S$ be a mapping. We say that h is an $\alpha - \min$ special type contraction mapping if there exits function $\alpha : S \times S \times (0,\infty) \to \mathbb{R}^+$ satisfying the following inequality

$$\alpha(\kappa,\mu,\eta)\left(\frac{1}{\Gamma_{h\kappa,h\mu,a}(\phi(\eta))} - 1\right) \leq \min\left(\frac{1}{\Gamma_{\kappa,\mu,a}(\phi(\frac{\eta}{c}))} - 1, \frac{1}{\Gamma_{\kappa,h\kappa,a}(\phi(\frac{\eta}{c}))} - 1, \frac{1}{\Gamma_{\mu,h\mu,a}(\phi(\frac{\eta}{c}))} - 1\right) \tag{1}$$

for all $\kappa,\mu,a \in S$, $\eta > 0$, where $0 < c < 1$, $\phi \in \Phi$.

Definition 17. Let (S,Γ,Δ) be a 2- Menger space, $h : S \to S$ be a given mapping and $\alpha : S \times S \times (0,\infty) \to \mathbb{R}^+$ be a function, we say that h is α-admissible if for all $\kappa,\mu,a \in S$, and $\eta > 0$, we have

$$\alpha(\kappa,\mu,\eta) \geq 1 \Rightarrow \alpha(h\kappa,h\mu,\eta) \geq 1.$$

Theorem 18. *Let (S,Γ,Δ) be a complete 2-Menger space, Δ is a minimum t-norm and $h : S \to S$ be an $\alpha - \min$ special type contraction mapping satisfying the following conditions:*

(i) h is α-admissible,
(ii) there exists $\kappa_0 \in S$ such that $\alpha(\kappa_0, h\kappa_0, \eta) \geq 1$, for all $\eta > 0$,
(iii) if $\{\kappa_n\}$ is a sequence in S such that $\alpha(\kappa_n,\kappa_{n+1},\eta) \geq 1$ for all $n \in \mathbb{N}$ and for all $\eta > 0$.

Then h has a unique fixed point, that is, there exists a point $\kappa \in S$ such that $h\kappa = \kappa$.

Proof. Let $\kappa_0 \in S$ be such that $\alpha(\kappa_0, h\kappa_0, \eta) \geq 1$ for all $\eta > 0$. We consider a sequence $\{\kappa_n\}$ in S so that $\kappa_{n+1} = h\kappa_n$, for all $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Clearly, $\kappa_{n+1} \neq \kappa_n$ for all $n \in N$, otherwise h has trivially a fixed point.
As h is α-admissible, we get $\alpha(\kappa_0, h\kappa_0, \eta) = \alpha(\kappa_0,\kappa_1,\eta) \geq 1$ implies $\alpha(h\kappa_0, h\kappa_1, \eta) = \alpha(\kappa_1,\kappa_2,\eta) \geq 1$. Also, by induction, we get

$$\alpha(\kappa_n,\kappa_{n+1},\eta) \geq 1, \text{ for all } n \in \mathbb{N} \text{ and for all } \eta > 0.$$

From the properties of function ϕ, we can find $\eta > 0$ such that $\Gamma_{\kappa_0,\kappa_1,a}(\phi(\eta)) > 0$, for all $a \in S$.

Now, using (1) for all $a \in S$, $\eta > 0$ and $c \in (0,1)$, we get

$$\begin{aligned}
\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta))} - 1 &= \frac{1}{\Gamma_{h\kappa_n,h\kappa_{n-1},a}(\phi(\eta))} - 1 \\
&\leq \alpha(\kappa_n,\kappa_{n-1},\eta)\frac{1}{\Gamma_{h\kappa_n,h\kappa_{n-1},a}(\phi(\eta))} - 1 \\
&\leq \min\Big(\frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))} - 1, \frac{1}{\Gamma_{\kappa_n,h\kappa_n,a}(\phi(\frac{\eta}{c}))} - 1, \\
&\qquad \frac{1}{\Gamma_{\kappa_{n-1},h\kappa_{n-1},a}(\phi(\frac{\eta}{c}))} - 1\Big) \\
&= \min\Big(\frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))} - 1, \frac{1}{\Gamma_{\kappa_n,\kappa_{n+1},a}(\phi(\frac{\eta}{c}))} - 1, \\
&\qquad \frac{1}{\Gamma_{\kappa_{n-1},\kappa_n,a}(\phi(\frac{\eta}{c}))} - 1\Big) \\
&= \min\Big(\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{\eta}{c}))} - 1, \frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))} - 1\Big).
\end{aligned} \tag{2}$$

The above inequality holds since $\alpha(\kappa_n,\kappa_{n-1},\eta) \geq 1$.

We now claim that for all $a \in S$, $\eta > 0$, $n \geq 1$ and $c \in (0,1)$,

$$\min\left(\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{\eta}{c}))} - 1, \frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))} - 1\right) = \frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))} - 1, \tag{3}$$

holds.

If possible, let for some $s > 0$,

$$\min\left(\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c}))} - 1, \frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{s}{c}))} - 1\right) = \frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c}))} - 1,$$

then using (2), we get

$$\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(s))} - 1 \leq \frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c}))} - 1,$$

that is,

$$\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(s)) \geq \Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c})), \tag{4}$$

which is impossible for all $c \in (0,1)$ (For $\phi(\eta)$ is strictly monotone increasing, $\phi(\frac{s}{c}) > \phi(s)$, that is, $\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c})) \geq \Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(s))$, by the monotone property of Γ). Then, for all $\eta > 0$ and $a \in S$, we get

$$\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta))} - 1 \leq \frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))} - 1,$$

that is,

$$\begin{aligned}\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta)) &\geq \Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))\\ &\geq \Gamma_{\kappa_{n-1},\kappa_{n-2},a}(\phi(\frac{\eta}{c^2}))\\ &\vdots\\ &\geq \Gamma_{\kappa_1,\kappa_0,a}(\phi(\frac{\eta}{c^n})).\end{aligned}$$

Hence

$$\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta)) \geq \Gamma_{\kappa_1,\kappa_0,a}(\phi(\frac{\eta}{c^n})). \tag{5}$$

Now, taking limit $n \to \infty$ on both sides of (5), for all $\eta > 0$ and $a \in S$, we obtain

$$\lim_{n\to\infty} \Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta)) = 1. \tag{6}$$

Now, we have to prove that $\{\kappa_n\}$ is a Cauchy sequence. On the contrary, there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{\kappa_{m(\ell)}\}$ and $\{\kappa_{n(\ell)}\}$ of $\{\kappa_n\}$ with $m(\ell) > n(\ell) > \ell$ such that

$$\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\epsilon) < 1 - \lambda. \tag{7}$$

We take $m(\ell)$ corresponding to $n(\ell)$ to be the smallest integer satisfying (7), so that

$$\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)},a}(\epsilon) \geq 1 - \lambda. \tag{8}$$

If $\epsilon_1 < \epsilon$ then we have

$$\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\epsilon_1) \leq \Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\epsilon).$$

So, it is feasible to construct $\{\kappa_{m(\ell)}\}$ and $\{\kappa_{n(\ell)}\}$ with $m(\ell) > n(\ell) > \ell$ and satisfying (7), (8) whenever ϵ is replaced by a smaller positive value. By the continuity of ϕ at 0 and strictly monotone increasing property with $\phi(0) = 0$, it is possible to find $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.
Then, by the above condition, it is possible to get an increasing sequence of integers $\{m(\ell)\}$ and $\{n(\ell)\}$ with $m(\ell) > n(\ell) > \ell$ such that

$$\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\phi(\epsilon_2)) < 1 - \lambda, \tag{9}$$

and

$$\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)},a}(\phi(\epsilon_2)) \geq 1 - \lambda. \tag{10}$$

Now, from (9), we get

$$1 - \lambda > \Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\phi(\epsilon_2)),$$

that is,

$$\frac{1}{1-\lambda} < \frac{1}{\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\phi(\epsilon_2))},$$

that is,

$$\frac{1}{1-\lambda} - 1 < \frac{1}{\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\phi(\epsilon_2))} - 1.$$

using the inequality (1), we get

$$\begin{aligned}\frac{\lambda}{1-\lambda} &< \frac{1}{\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\phi(\epsilon_2))} - 1\\ &\leq \alpha(\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},\eta)(\frac{1}{\Gamma_{h\kappa_{m(\ell)-1},h\kappa_{n(\ell)-1},a}(\phi(\epsilon_2))} - 1)\\ &\leq \min(\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))} - 1, \frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{m(\ell)},a}(\phi(\frac{\epsilon_2}{c}))} - 1,\\ &\quad \frac{1}{\Gamma_{\kappa_{n(\ell)-1},\kappa_{n(\ell)},a}(\phi(\frac{\epsilon_2}{c}))} - 1).\end{aligned} \tag{11}$$

Now, we can choose $\beta_1, \beta_2 > 0$ such that

$$\begin{aligned}&\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))\\ &\geq \Delta(\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},\kappa_{n(\ell)}}(\beta_1), \Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)},a}(\phi(\epsilon_2)), \Gamma_{\kappa_{n(\ell)},\kappa_{n(\ell)-1},a}(\beta_2)),\end{aligned} \tag{12}$$

holds, where $\phi(\frac{\epsilon_2}{c}) = \beta_1 + \beta_2 + \phi(\epsilon_2)$.
Now, using (6) and (10), we have

$$\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},\kappa_{n(\ell)}}(\beta_1) \geq 1 - \lambda, \tag{13}$$

$$\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)},a}(\phi(\epsilon_2)) \geq 1 - \lambda \tag{14}$$

and

$$\Gamma_{\kappa_{n(\ell)},\kappa_{n(\ell)-1},a}(\beta_2) \geq 1 - \lambda. \tag{15}$$

Since Δ is a min t-norm, using (13), (14), and (15) in (12), we have $\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c})) \geq \Delta(1-\lambda, 1-\lambda, 1-\lambda) = 1-\lambda$, $\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))} \leq \frac{1}{1-\lambda}$, that is,

$$\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))} - 1 \leq \frac{1}{1-\lambda} - 1 = \frac{\lambda}{1-\lambda}. \tag{16}$$

Again, using (6), we have $\Gamma_{\kappa_{m(k)-1},\kappa_{m(\ell)},a}(\phi(\frac{\epsilon_2}{c})) \geq 1-\lambda$, $\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{m(\ell)},a}(\phi(\frac{\epsilon_2}{c}))} \leq \frac{1}{1-\lambda}$, that is,

$$\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{m(\ell)},a}(\phi(\frac{\epsilon_2}{c}))} - 1 \leq \frac{1}{1-\lambda} - 1 = \frac{\lambda}{1-\lambda}. \tag{17}$$

Again, $\frac{1}{\Gamma_{\kappa_{n(\ell)-1},\kappa_{n(\ell)},a}(\phi(\frac{\epsilon_2}{c}))} \leq \frac{1}{1-\lambda}$, $\frac{1}{\Gamma_{\kappa_{n(\ell)-1},\kappa_{n(\ell)},a}(\phi(\frac{\epsilon_2}{c}))} \leq \frac{1}{1-\lambda}$, that is,

$$\frac{1}{\Gamma_{\kappa_{n(\ell)-1},\kappa_{n(\ell)},a}(\phi(\frac{\epsilon_2}{c}))} - 1 \leq \frac{1}{1-\lambda} - 1 = \frac{\lambda}{1-\lambda}. \tag{18}$$

Now, using (16), (17) and (18) in (11), we have

$$\begin{aligned}\frac{\lambda}{1-\lambda} &< \min(\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))} - 1,\\ &\quad \frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{m(\ell)},a}(\phi(\frac{\epsilon_2}{c}))} - 1, \frac{1}{\Gamma_{\kappa_{n(\ell)-1},\kappa_{n(\ell)},a}(\phi(\frac{\epsilon_2}{c}))} - 1)\\ &\leq \min(\frac{\lambda}{1-\lambda}, \frac{\lambda}{1-\lambda}, \frac{\lambda}{1-\lambda})\\ &= \frac{\lambda}{1-\lambda},\end{aligned}$$

which is a contradiction. Hence $\{\kappa_n\}$ is a Cauchy sequence.
Since (S, Γ, Δ) be a complete 2-Menger space, $\kappa_n \to u$ as $n \to \infty$, for some $u \in S$. Moreover, we get

$$\Gamma_{hu,u,a}(\epsilon) \geq \Delta(\Gamma_{hu,u,\kappa_{n+1}}(\frac{\epsilon}{3}), \Gamma_{hu,\kappa_{n+1},a}(\frac{\epsilon}{3}), \Gamma_{\kappa_{n+1},u,a}(\frac{\epsilon}{3})). \quad (19)$$

Next, using the properties of function ϕ, we can find $\eta_2 > 0$ such that $\phi(\eta_2) < \frac{\epsilon}{3}$. Since $\kappa_n \to u$ as $n \to \infty$, there exists $n_0 \in N$ such that, for all $n > n_0$ (sufficiently large), we have

$$\frac{1}{\Gamma_{\kappa_{n+1},hu,a}(\frac{\epsilon}{3})} - 1 \leq \frac{1}{\Gamma_{h\kappa_n,hu,a}(\phi(\eta_2))} - 1 \leq \alpha(\kappa_n, u, \eta)(\frac{1}{\Gamma_{h\kappa_n,hu,a}(\phi(\eta_2))} - 1)$$
$$\leq \min(\frac{1}{\Gamma_{\kappa_n,u,a}(\phi(\frac{\eta_2}{c}))} - 1, \frac{1}{\Gamma_{\kappa_n,h\kappa_n,a}(\phi(\frac{\eta_2}{c}))} - 1, \frac{1}{\Gamma_{u,hu,a}(\phi(\frac{\eta_2}{c}))} - 1)$$
$$= \min(\frac{1}{\Gamma_{\kappa_n,u,a}(\phi(\frac{\eta_2}{c}))} - 1, \frac{1}{\Gamma_{\kappa_n,\kappa_{n+1},a}(\phi(\frac{\eta_2}{c}))} - 1, \frac{1}{\Gamma_{u,hu,a}(\phi(\frac{\eta_2}{c}))} - 1).$$

Taking limit $n \to \infty$ on both sides, we have $\frac{1}{\Gamma_{u,hu,a}(\phi(\eta_2))} - 1 \leq \min(0, 0, \frac{1}{\Gamma_{u,hu,a}(\phi(\frac{\eta_2}{c}))} - 1) = 0$. Hence $\frac{1}{\Gamma_{u,hu,a}(\phi(\eta_2))} \leq 1, \Gamma_{u,hu,a}(\phi(\eta_2)) \geq 1$. Hence $hu = u$. So, it is proved that h has a fixed point.

Now, we'll show that the uniqueness of fixed point. Let κ and μ be two fixed point of h, that is, $h\kappa = \kappa$ and $h\mu = \mu$ with $\kappa \neq \mu$. By the virtue of ϕ there exists $s > 0$ such that $\Gamma_{\kappa,\mu,a}(\phi(s)) > 0$ for all $a \in S$. Then, by (1), we have

$$\begin{aligned} \frac{1}{\Gamma_{h\kappa,h\mu,a}(\phi(s))} - 1 &\leq \alpha(\kappa, \mu, \eta)(\frac{1}{\Gamma_{h\kappa,h\mu,a(\phi(s))}} - 1) \\ &\leq \min(\frac{1}{\Gamma_{\kappa,\mu,a}(\phi(\frac{s}{c}))} - 1, \frac{1}{\Gamma_{\kappa,h\kappa,a}(\phi(\frac{s}{c}))} - 1, \\ &\quad \frac{1}{\Gamma_{\mu,h\mu,a}(\phi(\frac{s}{c}))} - 1) \\ &= \min(\frac{1}{\Gamma_{\kappa,\mu,a}(\phi(\frac{s}{c}))} - 1, \frac{1}{\Gamma_{\kappa,\kappa,a}(\phi(\frac{s}{c}))} - 1, \\ &\quad \frac{1}{\Gamma_{\mu,\mu,a}(\phi(\frac{s}{c}))} - 1) \\ &= \min(\frac{1}{\Gamma_{\kappa,\mu,a}(\phi(\frac{s}{c}))} - 1, 0, 0) \\ &= 0. \end{aligned}$$

Hence $\Gamma_{h\kappa,h\mu,a}(\phi(s)) \geq 1$, for all $a \in S$, and it implies $\kappa = \mu$.

If we replace $\phi(\eta)$ by t and $\alpha(\kappa, \mu, \eta) = 1$, in the above theorem, we get the following result.

Corollary 19. *Let (S, Γ, Δ) be a complete 2-Menger space and $h : S \to S$ be a mapping satisfying the following inequality for all $\kappa, \mu, a \in S$,*

$$\frac{1}{\Gamma_{h\kappa,h\mu,a}(\eta)} - 1 \leq \min(\frac{1}{\Gamma_{\kappa,\mu,a}(\frac{\eta}{c})} - 1, \frac{1}{\Gamma_{\kappa,h\kappa,a}(\frac{\eta}{c})} - 1, \frac{1}{\Gamma_{\mu,h\mu,a}(\frac{\eta}{c})} - 1) \quad (20)$$

where $\eta > 0$, $0 < c < 1$. Then h has a unique fixed point in S.

Next we give an example to support our results.

Example 20. Let $S = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$, the t-norm Δ is a minimum t-norm and Γ be defined as

$$\Gamma_{\kappa_1,\kappa_2,\kappa_3}(\eta) = \Gamma_{\kappa_1,\kappa_2,\kappa_4}(\eta) = \begin{cases} 0, & \text{if } \eta \leq 0, \\ 0.50, & \text{if } 0 < \eta \leq 5, \\ 1, & \text{if } \eta > 5. \end{cases}$$

and

$$\Gamma_{\kappa_1,\kappa_3,\kappa_4}(\eta) = \Gamma_{\kappa_2,\kappa_3,\kappa_4}(\eta) = \begin{cases} 0, & \text{if } \eta \leq 0, \\ 1, & \text{if } \eta > 0, \end{cases}$$

Then (S, Γ, Δ) is a complete 2-Menger space. If we define $h : S \to S$ as follows: $h\kappa_1 = \kappa_4, h\kappa_2 = \kappa_3, h\kappa_3 = \kappa_4, h\kappa_4 = \kappa_4$, then the mapping h satisfies all the conditions of the theorem where $\phi(\eta) = \eta$, $c \in (0, 1)$ and $\alpha(\kappa, \mu, \eta) = 1$. Then κ_4 is the unique fixed point of h in S.

CONCLUSION

In recent research work, it is clear that contraction mappings play vital roles. The contraction is supposed to appear in probabilistic analysis also. Many researchers have concentrated their works on these spaces. Some authors also showed that PM spaces are applicable also in nuclear fusion. G. Verdoolage et. al Verdoolage et al. (2012) may be noted in this respect. The authors have showed that PM spaces have an important role to identify confinement regimes and plasma disruption. The distribution function plays the role of metric in these spaces. Also it may be noted that t-norm has an important role. Here we use the minimum t-norm. But we think that different types t-norm may be used here. These problems may be taken up as future open problems.

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