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APPROXIMATE PROCEDURE FOR CALCULATION OF SHEAR STRESSES σ_{xz} AND σ_{yz}

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For thick and moderately thick plates which have middle plane parallel to plane (x,y) state of the stress in cross section, which is described with component of the stress in plane (x,y), doesn't give a realistic presentation of state of the stress. In order to define state of the stress more „real" it is necessary to define shear stresses in planes (x,z) and (y,z).

This paper shows one approximate procedure that can be used to calculate components of shear stresses σxz and σyz. Procedure is applied to example for calculation of shear stresses for laminated composite plates. Theoretical bases are based on Layerwise theory, and solutions of theory equations are determined in closed form. At the end of paper there are the results of procedure application for plates with geometrical and material characteristics which are given graphically, using charts.

Keywords: shear stresses, composite plate, layerwise theory

GENERAL

Layered composite plates consist of orthotropic layers with arbitrary angle of orientation of fibbers. Fibber and basic mass of each layer can have different physical and mechanical characteristics. By combining layers of different characteristics we can get plates of significantly higher bending stiffness, as well as higher axial and shear stiffness.

In order to solve the problem of layered composite plates single-layer ESCT theories are available as well as Layer-wise theories (GLPT). It is proved that the accuracy of the calculation impact of these plates, using single-theory, decreases with increasing of the thickness. Due to this, classical 2D slab models cannot provide "real" state of stress and deformations of thick and moderately thick plates, especially of plates with highly anisotropic characteristics, such as layered composite plates and plate with delamination.

In recent decades, the Layer-wise theory is developed based on the assumption that the components of displacement through the thickness of the plate can be shown by one-dimensional interpolation functions. Combining 2D plate model, in the plane (x, y), and 1D model, perpendicular to the plane of the plate, we get a model that can be used in resolving thick and moderately thick plates as well as plates with various forms of geometric and material nonlinearities. The above theory allows the introduction of geometric imperfections by expanding the Layer-wise theory, i.e., by simply upgrading or adding new members.

Layer-wise theory is based on an analysis of the layer as a part of composite plate. For each of the composite layers, single-layer theory or ESC theory is applied, and then the layers are connected by introducing the assumed interpolation function through the thickness of the plate. Depending on the selected interpolation functions solutions with more or less accuracy are obtained. The assumption that each layer has a different change of shifts allows the obtaining of more "realistic" effects in arbitrary cross-section [1], [2].

LAYER-WISE THEORY

The mathematical model of linear Layer-wise theory is based on the foundations of the Layer-wise theory which neglects the deformation perpendicular to the plane of the plate ε_{zz}=0, while the shear components ε_{xy}, ε_{xz}, ε_{yz} differ from the zero [3]. Deformation tensor obtains the following form:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & 0 \end{bmatrix} \quad (1)$$

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which, because of the symmetry properties can be shown as a vector:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{bmatrix} \tag{2}$$

Relation of components of deformation tensor and components of displacement vector are defined with:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{bmatrix} = \begin{bmatrix} u,_x \\ v,_y \\ v,_x + u,_y \\ v,_z + w,_y \\ u,_z + w,_x \end{bmatrix} \tag{3}$$

Having in mind the above assumption it is concluded that component of displacement, perpendicular to the plane of the plate, is constant along the thickness of the plate, i.e., *w(x,y,z)=w(x,y)*. In that case, displacement of arbitrary point of the plate can be shown as the sum of displacement of the point of middle plane of the plate *U(x,y)*, *V(x, y)* and *W(x,y)* as well as additional displacement along the thickness of the plate *U(x,y,z)* and *V(x,y,z)*:

$$\begin{aligned} u(x,y,z) &= u(x,y) + U(x,y,z) \\ v(x,y,z) &= v(x,y) + V(x,y,z) \\ w(x,y,z) &= w(x,y) \end{aligned} \tag{4}$$

Additional displacements are displayed in the form of the sum as a product of appropriate componential displacement of j-point along the thickness and interpolation function $\psi^J(z)$:

$$\begin{aligned} U(x,y,z) &= \sum_{J=1}^{N} u^J(x,y)\psi^J(z) \\ V(x,y,z) &= \sum_{J=1}^{N} v^J(x,y)\psi^J(z) \end{aligned} \tag{5}$$

If, along the two adjacent layers, a linear interpolation function is adopted, then the number of nodes along the thickness of the plate - N is for one higher than the number of layers – n.

Relations of the stresses and deformations are defined for each viewed layer j:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix}^j = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & & \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & & \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & & \\ & & & \overline{Q}_{44} & \overline{Q}_{45} \\ & & & \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}^j \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{bmatrix}^j \tag{6}$$

where $\overline{Q}_{ij}{}^j$ is the transformed material stiffness of j orthotropic layer. It is necessary to make the transformation of material characteristics because each of the layers may have fibbers that carry in different directions, i.e., the fibbers can cover arbitrary angle in the plane (x,y). After defining the variable , the stiffness of the composite is determined, as a system of layers, taking into account the impact of one layer to another, [4].

ANALYTICAL SOLUTION

For simply supported rectangular laminated composite plate, which contains n orthotropic layers, displacements are shown in the following form:

$$\begin{aligned} u &= \sum_{m,n}^{\infty} X_{mn} \cos\alpha x \sin\beta y \\ v &= \sum_{m,n}^{\infty} Y_{mn} \sin\alpha x \cos\beta y \\ w &= \sum_{m,n}^{\infty} W_{mn} \sin\alpha x \sin\beta y \\ U^J &= \sum_{m,n}^{\infty} R_{mn}^J \cos\alpha x \sin\beta y \\ V^J &= \sum_{m,n}^{\infty} S_{mn}^J \sin\alpha x \cos\beta y \end{aligned} \tag{7}$$

where: $\alpha = \dfrac{m\pi}{a}$; $\beta = \dfrac{n\pi}{b}$ $J = 1,\ldots,N$

The equations of equilibrium are obtained in the following form:

$$\begin{bmatrix} [K] & [K^j] \\ [K^j]^T & [K^{ji}] \end{bmatrix} \begin{Bmatrix} X_{mn} \\ Y_{mn} \\ W_{mn} \\ R_{mn}^J \\ S_{mn}^J \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ 0 \\ 0 \end{Bmatrix} \tag{8}$$

Q_{mn} - coefficients of the load.

Coefficient matrix contains sub-matrixes K whose parts are the function of material stiffness of layered composite plate and the coefficients of α and β. The solutions of the system of equations (8) are unknown coefficients X_{mn}, Y_{mn}, W_{mn}, R_{mn}^{J} i S_{mn}^{J} in the same number as the number of equations (3+2N).

Once determined the solutions, using relations (7) the values of componential displacement are defined, and then the components of deformation tensor (3). Stress components are calculated using relation of (6) [3], [5].

APPROXIMATE PROCEDURE FOR CALCULATION OF SHEAR STRESSES σ_{XZ} AND σ_{YZ}

Shearing stresses in the planes (x, z) and (y, z) are inter-layers stresses that are important for the analysis of thick and moderately thick plates as well as plates that contain some form of delamination. The shown procedure is approximate procedure that can be applied in the process of solving numerical equations of the problems [3], [5], [6].

From the constitutive equations (6), using solutions of equations (8) and relations (7), (4) and (3), along each layer j, constant values of shear stresses for an arbitrary point in the plane (x, y) are obtained:

$$\sigma_{xz}{}^{j} = \overline{Q}_{45}{}^{j} \sum_{m,n} \left\{ \left[\frac{1}{h^{j}} \left(S_{nm}{}^{j+1} - S_{nm}{}^{j} \right) + \beta\, Z_{nm} \right] \sin(\alpha x)\cos(\beta y) \right\} + \overline{Q}_{55}{}^{j} \sum_{m,n} \left\{ \left[\frac{1}{h^{j}} \left(R_{nm}{}^{j+1} - R_{nm}{}^{j} \right) + \alpha\, Z_{nm} \right] \cos(\alpha x)\sin(\beta y) \right\}$$

$$\sigma_{xz}{}^{j} = \overline{Q}_{44}{}^{j} \sum_{m,n} \left\{ \left[\frac{1}{h^{j}} \left(S_{nm}{}^{j+1} - S_{nm}{}^{j} \right) + \beta\, Z_{nm} \right] \sin(\alpha x)\cos(\beta y) \right\} + \overline{Q}_{45}{}^{j} \sum_{m,n} \left\{ \left[\frac{1}{h^{j}} \left(R_{nm}{}^{j+1} - R_{nm}{}^{j} \right) + \alpha\, Z_{nm} \right] \cos(\alpha x)\sin(\beta y) \right\} \tag{9}$$

Assuming parabolic shear stress change along each layer j:

$$\bar{\sigma}^{j}_{xz}(z) = N_1(z)\, \bar{f}^{j}_{1x} + N_2(z)\, \bar{f}^{j}_{2x} + N_3(z)\, \bar{f}^{j}_{3x}$$
$$\bar{\sigma}^{j}_{yz}(z) = N_1(z)\, \bar{f}^{j}_{1y} + N_2(z)\, \bar{f}^{j}_{2y} + N_3(z)\, \bar{f}^{j}_{3y} \tag{10}$$

where z is coordinate of the local coordinate system of j layer of observed plate. The coefficients $\bar{f}^{j}_{1x}, \bar{f}^{j}_{2x}, \bar{f}^{j}_{3x}, \bar{f}^{j}_{1y}, \bar{f}^{j}_{2y}$ i $\bar{f}^{j}_{3y}$ represent the shear stresses $\boldsymbol{\sigma_{xz}}$ and $\boldsymbol{\sigma_{yz}}$ at the ends and in the middle of observed layer, while $N_{1(z)}$, $N_{2(z)}$ and $N_{3(z)}$ are single-dimensional rectangular interpolation functions defined by the following expression:

$$N_1(z) = 1 - \frac{3z}{h_j} + \frac{2z^2}{h_j^2} \quad N_2(z) = \frac{4z}{h_j} - \frac{4z^2}{h_j^2} \quad N_3(z) = -\frac{z}{h_j} + \frac{2z^2}{h_j^2} \tag{11}$$

where *hj* is thickness of the layer *j*.

It is obvious that for n layers there will be per 3n of unknown coefficients $\bar{f}^{j}_{1x}, \bar{f}^{j}_{2x}, \bar{f}^{j}_{3x}$ for the stress σ_{XZ} and 3n of unknown coefficients $\bar{f}^{j}_{1y}, \bar{f}^{j}_{2y}$ i $\bar{f}^{j}_{3y}$ for stresses σ_{XZ}. To determine the unknown it is necessary to write 3n equations for the required stresses σ_{XZ} and 3n for stresses σ_{YZ} . Display of approximate procedure is given for the shear stresses in the plane (x,z) with unknown coefficients $\bar{f}^{j}_{1x}, \bar{f}^{j}_{2x}, \bar{f}^{j}_{3x}$

To determine the 3n of unknown coefficients we set the following conditions:

1) The shear stresses at the top and bottom fibers of the plate are zero. Applying this condition only two equations can be written:
   $\sigma_{XZ}{}^{n} = 0$ za $z=h_n$ (12)
   $\sigma_{XZ1} = 0$ za $z=0$ (13)
2) In the connections of layers shear stresses have equal values. This condition gives the (n-1) equation:
   σ_{xzj} (z=0) = σ_{xz} j-1(z=h_j-1) za j=2,3,...,n (14)
3) Along each layer j shear stresses have constant values defined by the relation (9). From this condition n equation is obtained.

4) Change of derivative of shear stresses σ_{xz},z in the connections of layers defines (n-1) equation It is concluded that in total (2+(n-1)+n+(n-1)=3n equations can be written, i.e., as many as the number of unknown coefficients $\bar{f}^j_{1x}, \bar{f}^j_{2x}, \bar{f}^j_{3x}$.

For the adopted parabolic change (10) and (11) from the first condition (12) it is obtained that:

$$\bar{f}_1^1 = 0$$

$$\bar{f}_3^N = 0 \tag{15}$$

From the second group of equations (14) (n-1) equation forms are obtained:

$$\bar{f}_1^j - \bar{f}_3^{j-1} = 0 \tag{16}$$

The third group of equations are the constitutive relations (9) which are j=1,...,n.
From the fourth group of equations it is obtained:

$$-\frac{1}{h_j}\bar{f}_1^j - \frac{3}{h_{j+1}}\bar{f}_1^{j+1} + \frac{4}{h_j}\bar{f}_2^j + \frac{4}{h_{j+1}}\bar{f}_2^{j+1} - \frac{3}{h_j}\bar{f}_3^j - \frac{1}{h_{j+1}}\bar{f}_3^{j+1} = \sigma^{j+1}_{xz,z} - \sigma^j_{xz,z} \quad j = 1, n-1 \tag{17}$$

where and are the first derivatives of the shear stresses obtained from constitutive equations. For the analytical solution difference between first derivatives has the following form:

$$\begin{aligned} J_x^j = \sigma^j_{xz,z} - \sigma^{j-1}_{xz,z} = {} & 2\left(\bar{Q}_{13}^{j-1} - \bar{Q}_{13}^j\right)\left(u_{,xy} + \sum_{k=1}^N U^k_{,xy}\psi^k\right) + \left(\bar{Q}_{11}^{j-1} - \bar{Q}_{11}^j\right)\left(u_{,xx} + \sum_{k=1}^N U^k_{,xx}\psi^k\right) + \\ & \left(\bar{Q}_{33}^{j-1} - \bar{Q}_{33}^j\right)\left(u_{,yy} + \sum_{k=1}^N U^k_{,yy}\psi^k\right) + \left(\bar{Q}_{13}^{j-1} - \bar{Q}_{13}^j\right)\left(v_{,xx} + \sum_{k=1}^N V^k_{,xx}\psi^k\right) + \\ & \left(\bar{Q}_{12}^{j-1} + \bar{Q}_{33}^{j-1} - \bar{Q}_{12}^j - \bar{Q}_{33}^j\right)\left(v_{,xy} + \sum_{k=1}^N V^k_{,xy}\psi^k\right) + \left(\bar{Q}_{13}^{j-1} - \bar{Q}_{13}^j\right) \\ & \left(v_{,xx} + \sum_{k=1}^N V^k_{,xx}\psi^k\right) + \left(\bar{Q}_{23}^{j-1} - \bar{Q}_{23}^j\right)\left(v_{,yy} + \sum_{k=1}^N V^k_{,yy}\psi^k\right) \end{aligned} \tag{18}$$

for j=2,n.

After determination of solutions from the above system of equations $\bar{f}^j_{1x}, \bar{f}^j_{2x}, \bar{f}^j_{3x}$, j=1,n, using the expression (10) and (11) values of the stresses can be calculated σ_{xzj} per the thickness of the plate. $\bar{f}^j_{1y}, \bar{f}^j_{2y}$ i $\bar{f}^j_{3y}$

Shear stresses in the plane (y, z) are obtained applying the same procedure with unknown coefficients $\bar{f}^j_{1y}, \bar{f}^j_{2y}$ i $\bar{f}^j_{3y}$

NUMERICAL EXAMPLES

Applying the approximate procedure in Figures 1 to 4, for the adopted number of layers and adopted characteristics of layers, the change in shear stress is given σ_{yz} and σ_{xz} in dimensionless form:

$$\bar{\sigma}_{xz} = \frac{1}{sq}\sigma_{xz} \quad \bar{\sigma}_{yz} = \frac{1}{sq}\sigma_{yz}$$

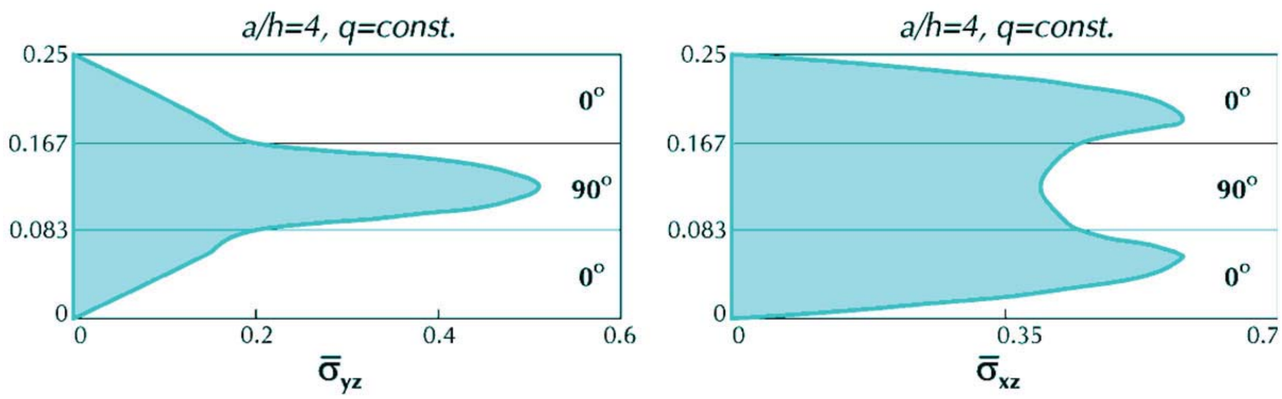

Figure 1: Distribution of stress and along the thickness of the plate with layers 0°/90°/0°

A square plate of symmetrical and anti-symmetrical arrangement of layers is considered. Layers of the plate contain fibbers that with the x axis form the angles 00 and 900. The load is evenly distributed, and the material characteristics of layers are: E_1/E_2=25, E_2=1, G_{12}=G_{23}=0.5, G_{23}=0.2, V_{12}=V_{13}=0.25.

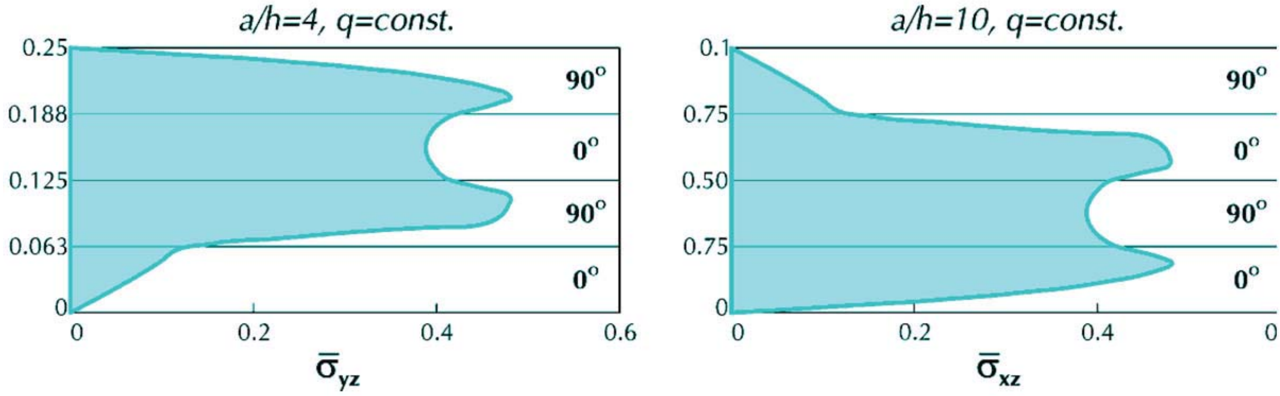

Figure 2: Distribution of stress $\bar{\sigma}_{yz}$ and $\bar{\sigma}_{xz}$ along the thickness 0°/90°/0°/90°

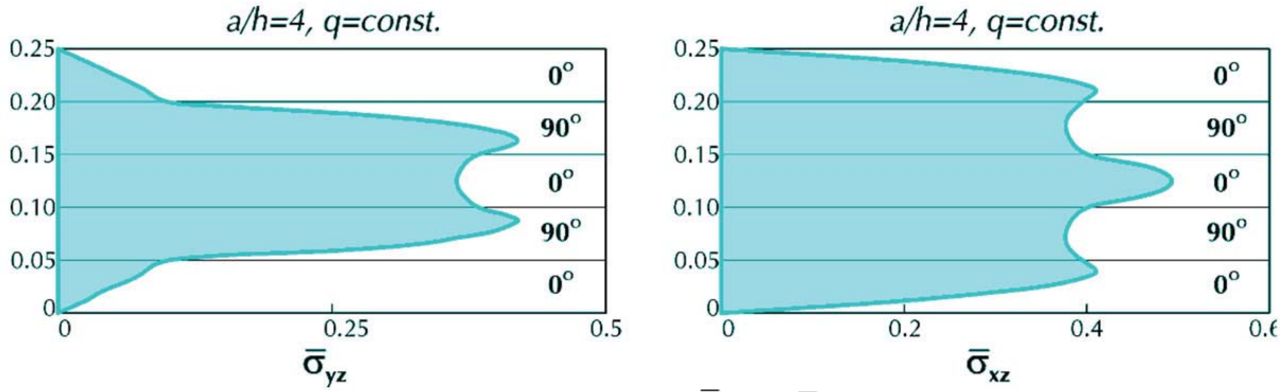

Figure 3: Distribution of stress $\bar{\sigma}_{yz}$ and $\bar{\sigma}_{xz}$ along the thickness 0°/90o/0°/90°/0°

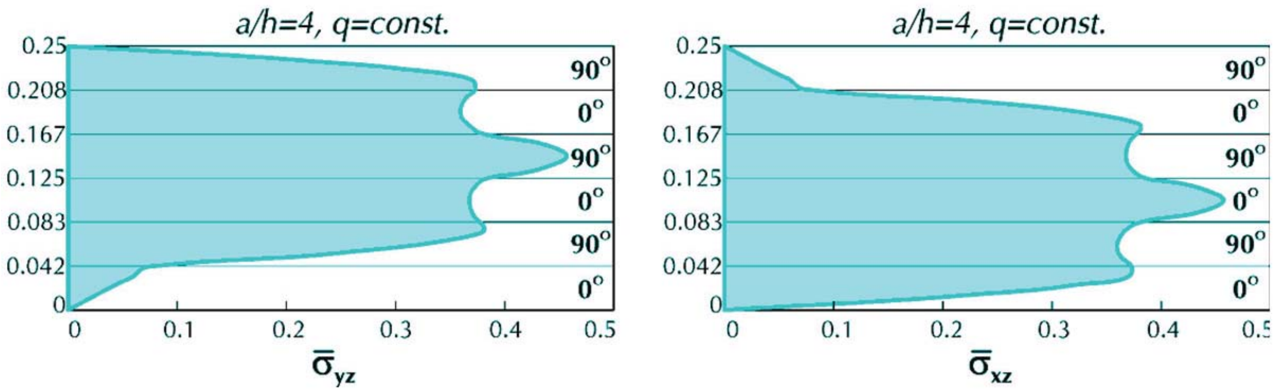

Figure 4: Distribution of stress $\bar{\sigma}_{yz}$ and $\bar{\sigma}_{xz}$ along the thickness 0o/90o/0°/90°/0°/90°

CONCLUSION

Using the constitutive relations, defined for the j layer of composite plate, constant values of shear stresses σ_{xzj} and σ_{yzj} are obtained along the thickness. In this case, in the connection points of two adjacent layers there are different stress values, while in the top and bottom fibbers of the plate the shear stresses are different from zero. It is concluded that using constitutive relations of the stresses and deformations a real change of the stress σ_{xz} and σ_{yz} are not obtained along the thickness of laminated composite plates.

It is possible, by shown approximate procedure to obtain parabolic change of shear stresses along the thickness of each layer of laminated composite plate. Numerical examples given in this paper were obtained using tailor made program prepared by the author. The stresses along each layer are defined in three points, two at the ends and one in the middle of each layer. Defining realistic distribution of shear stresses σ_{xz} and σ_{yz} is especially important for thick and moderately thick plates, as well as for plates containing some of the forms of delamination.

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