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THE APPROXIMATE AND NUMERICAL SOLUTION OF ROMANOVSKIJ LINEAR PARTIAL INTEGRAL EQUATIONS

Anatolij Semenovich Kalitvin[1*], Vladimir Anatoljevich Kalitvin[1]
[1]Lipetsk State Pedagogical P. Semenov-Tyan-Shansky University, Russian Federation

The study of Markov chains with two-way coupling leads to the solution of linear partially integral equations of the second kind in the space of functions continuous on the square. A characteristic feature of the equations is the permutation of variables for the unknown function under the integral sign and integration over part of the variables. Equations of such types are not Fredholm integral equations and for their study a well-developed theory of Fredholm integral equations of the second kind can't be directly applied. The equations considered in the article we call partially integral equations of Romanovskij, who first obtained them in the study of Markov chains with two-way coupling and studied these equations in the case of continuous kernels. An explicit solution of partially integral Romanovskij equations can be found in rare cases, and therefore the problem of studying approximate and numerical methods for solving such equations is vital. When using approximate and numerical methods, it should be taken into account that the linear partially integral operator in the Romanovskij equation is not completely continuous, and the direct application of methods associated with the complete continuity of operators for its solution requires justification. The justification of approximate and numerical methods for solving linear partially integral equations of Romanowskij is given in the annotated paper. The paper contains theorems on the solvability of equations, results on various approximate and numerical methods for their solution, the theorem on the solution of linear partially integral equations by Romanovskij, using the method of mechanical quadratures, together with an estimate of the rate of convergence of a numerical solution to an exact solution of this equation.

Key words: Linear integral equations, Linear equations of Romanovskij, Partial integrals, Approximate and numerical methods

INTRODUCTION

Markov chains play an extremely important role in the study of various problems of technology, genetics, physics and other problems.

The problems of doubly connected and multiply connected Markov chains are reduced to integral equations first studied in the case of continuous given functions by V.I. Romanovskij [01]. Solutions of such equations can be found explicitly in rare cases, so it is important to develop approximate and numerical methods for their solution.

Approximate and numerical methods for solving the inhomogeneous integral Romanovskij equation:

$$x(t,s) = \int_a^b m(t,s,\sigma)x(\sigma,t)d\sigma + f(t,s) \equiv (Mx)(t,s) + f(t,s) \quad (1)$$

are being considered in this article.

The singularity of equation (1) is connected with the rearrangement of variables for the unknown function under the integral sign and integrating it in one of two variables. Because of this, the operator *M* in equation (1) is not integral (since the criterion of A.V. Bukhvalov [02]) on the integral representation of a bounded linear operator is not fulfilled) and is not completely continuous.

Fundamentals of the theory of equation (1) with a continuous kernel are constructed by V.I. Romanovskii [01], using methods analogous to the method of Fredholm determinants, in [03] studied more general classes of linear integral equations of Romanovskii type with partial integrals and kernels of more general types. We note that the Fredholm property of equation (1), whose kernel is a continuous function by (t,s) with values in the space of summable functions, is established in [04] for the case of a space of continuous functions.

The conditions for the equivalence of equation (1) to the Fredholm integral equation of the second kind and the conditions for the invertibility of these equivalent equations in the space of functions continuous on the square are contained in Theorem 1; in Theorem 2 conditions for the unique solvability in the space of continuous functions of equation (1) and the conditions for which equation (1) either has no continuous solutions, or has more than one continuous solution are given.

Next, we study the approximate solution of the invertible equation (1). This equation is replaced by a linear partially integral Romanovskij equation with a degenerate continuous kernel close to the kernel m. The solution of the approximate equation is constructed explicitly, and the estimate of the error of the approximate solution of equation (1) is given.

For other methods of approximate solution, equation (1) is reduced to a system of Fredholm integral equations of the second kind with an additional condition for solutions or is transformed into a two-dimensional integral Fredholm equation of the second kind with a degenerate kernel.

**Lipetsk State Pedagogical P. Semenov-Tyan-Shansky University, Lenina str. 42, Lipetsk 398020, Russia; akalitvin@inbox.ru*

Numerical schemes for solving equation (1) with a continuous kernel are constructed. The integral equation (1) is replaced by a system of linear algebraic equations, whose solution tends to the solution of equation (1) for an unrestricted refinement of a segment [a, b] .

Another method of numerical solution of equation (1) is associated with the replacement of this equation by a system of linear integral equations of the second kind with an additional condition and the replacement of this system by a system of linear algebraic equations with an additional condition on the solution. Theorem 3 shows the conditions under which the solution of equation (1) can be found by the method of mechanical quadratures, and the rate of convergence of the numerical solution to the exact solution is given.

RESULTS AND DISCUSSION

CONDITIONS ON THE SOLVABILITY OF EQUATION (1)

Let D= [a, b]x[a, b], C=C(D), be the space of functions continuous on D, L^1, be the space of summable functions on [a, b], $C(L^1)$ be the space of continuous vector-valued functions (t, s) $\rightarrow$ z(t, s, ., .) $\in$ L^1 (D) and let m $\in$ $C(L^1)$.

By [03], the operator M acts and is bounded in C. As noted above, the operator M is not a completely continuous integral operator in C. However, the operator M^2 is a completely continuous integral operator in the space C [05].

Indeed, applying the Fubini theorem, we establish the equality:

$$(M^2x)(t,s)=\int_a^b\int_a^b m(t,s,\sigma)m(\sigma,t,\sigma_1)x(\sigma_1,\sigma)d\sigma_1 d\sigma.$$

It can be verified directly that $m(t,s,\cdot)m(\cdot,t,\cdot)\in C(L^1(D)),$ where $C(L^1(D))$ denotes the space of continuous vector-valued functions $(t,s)\rightarrow z(t,s,\cdot,\cdot)\in L^1(D)$. Then M^2 is a completely continuous integral operator in C.

Consequently, Fredholm's theorems hold for equation (1) with kernel from $C(L^1)$.

Suppose that 1 is not an eigenvalue of the operator M^2. By the theorem on the spectrum map $\pm 1$is not a point of the spectrum of the operator $M:\pm 1\notin\sigma(M)$ Therefore, (1) has a unique solution x(t, s). Therefore, x(t, s) is the unique solution of the integral equation:

$$x(t,s)=(M^2x)(t,s)+(Mf)(t,s)+f(t,s). \qquad (2)$$

Conversely, let equation (4) have a unique solution x(t, s). Equation (2) can be written in the form:

$$(I-M^2)x=Mf+f\Leftrightarrow(I+M)(I-M)x=(I+M)f, \qquad (3)$$

where I is the identity operator on C. Since equation (3) with a completely continuous operator M^2 has a unique solution, then $1\notin\sigma(M)$. Then the operator I+M is invertible in C .

Applying the operator $(I+M)^{-1}$ to both sides of equation (3) to the left and taking into account that $\pm 1\notin\sigma(M)$ we get that x(t,s) is the unique solution of equation (1). Thus, in the case under consideration, equations (1) and (2) are equivalent and have a unique solution x(t,s).

If, however, 1 is an eigenvalue of the operator M^2 and $\pm 1\notin\sigma(M)$ then by (3) the operator I+M is invertible in C. Applying the operator $(I+M)^{-1}$ to both sides of equation (3), we obtain the equivalent equation (1).

Thus, it is valid

Theorem 1. Let m ϵ $C(L^1)$ and f be an arbitrary function in C. Then the following assertions hold:

a. if $1\notin\sigma(M^2)$ then in C the equation (1) and the Fredholm integral equation of the second kind (2) are equivalent and invertible;
b. if $1\in\sigma(M^2)$ and $1\notin\sigma(M^2)$ then in C the equations (1) and (2) are equivalent.

We note that the equation x=Ax+f with linear bounded operator A in C is considered (here and below) invertible in C if the operator I-A is invertible on C.

Suppose that the condition of Theorem 1 is satisfied. Then in C equations (1) and (2) are equivalent.

Let $k(t,s,\sigma,\sigma_1)=m(t,s,\sigma)m(\sigma,t,\sigma_1)$ where mϵ$C(L^1)$. Since the kernel $k(t,s,\sigma,\sigma_1)\in C(L^1(D)$ where mϵ$C(L^1)$ is the space of continuous vector-valued functions $(t,s)\rightarrow z(t,s,\cdot,\cdot)\in L^1(D)$ and $C(L^1(D))$ is realized as a tensor product of the spaces C and $L^1(D)$ with a cross-norm, which coincides with the norm in $C(L^1(D))$ [06], and the set of continuous functions from the space C(DxD) is everywhere dense in $C(L^1(D))$, then:

$$k(t,s,\sigma,\sigma_1)=k_0(t,s,\sigma,\sigma_1)+\sum_{j=1}^{n}k_j(t,s)\tilde{k}_j(\sigma,\sigma_1), \qquad (4)$$

where:

$$\sup_{(t,s)\in D}\int_a^b\int_a^b|k_0(t,s,\sigma,\sigma_1)|d\sigma d\sigma_1<1,$$

and k_j and $\tilde{k}_j$ are continuous functions on D and the functions k_j are linearly independent, and the functions $\tilde{k}_j$ are orthonormal in the usual sense.

Substituting (4) into (2) we obtain,

$$x(t,s)=(K_0x)(t,s)+(Kx)(t,s)+g(t,s), \qquad (5)$$

where

$$(K_0x)(t,s)=\int_a^b\int_a^b k_0(t,s,\sigma,\sigma_1)x(\sigma_1,\sigma)d\sigma_1 d\sigma,$$

$$(Kx)(t,s)=\sum_{j=1}^{n}k_j(t,s)\int_a^b\int_a^b\tilde{k}_j(\sigma,\sigma_1)x(\sigma_1,\sigma)d\sigma_1 d\sigma,$$

$$g(t,s)=\int_a^b m(t,s,\sigma)f(\sigma,t)d\sigma+f(t,s). \qquad (6)$$

By virtue of [07]

$$\|K_0\|=\sup_{(t,s)\in D}\int_a^b\int_a^b|k_0(t,s,\sigma,\sigma_1)|d\sigma d\sigma_1<1. \qquad (7)$$

Therefore, the operator $I\text{-}K_0$ has a bounded inverse operator in C and

$$(I-K_0)^{-1}x(t,s)=x(t,s)+\int_a^b\int_a^b r(t,s,\sigma,\sigma_1)x(\sigma_1,\sigma)d\sigma_1 d\sigma, \qquad (7)$$

where

$$r(t,s,\sigma,\sigma_1)=\sum_{i=1}^{\infty}k_0^{(i)}(t,s,\sigma,\sigma_1),\ k_0^{(1)}(t,s,\sigma,\sigma_1)=k_0(t,s,\sigma,\sigma_1)$$

and

$$k_0^{(i)}(t,s,\sigma,\sigma_1)=\int_a^b\int_a^b k_0^{(i-1)}(t,s,\tilde\sigma,\tilde\sigma_1)k_0(\tilde\sigma,\tilde\sigma_1,\sigma,\sigma_1)d\tilde\sigma_1 d\tilde\sigma (i=2,3,..).$$

Then equation (5) can be written in the form

$$x(t,s)=(I-K_0)^{-1}(Kx)(t,s)+(I-K_0)^{-1}g(t,s). \qquad (8)$$

Taking (7) into account, we obtain an integral equation with a degenerate kernel:

$$x(t,s)=\int_a^b\int_a^b\sum_{j=1}^{n}q_j(t,s)\tilde k_j(\sigma,\sigma_1)x(\sigma,\sigma_1)d\sigma_1 d\sigma+h(t,s), \qquad (9)$$

where

$$q_j(t,s)=k_j(t,s)+\int_a^b\int_a^b r(t,s,\sigma,\sigma_1)k_j(\sigma,\sigma_1)x(\sigma,\sigma_1)d\sigma_1 d\sigma,$$

$$h(t,s)=g(t,s)+\int_a^b\int_a^b r(t,s,\sigma,\sigma_1)g(\sigma,\sigma_1)d\sigma_1 d\sigma.$$

Assuming

$$x_j=\int_a^b\int_a^b \tilde k_j(\sigma,\sigma_1)x(\sigma,\sigma_1)d\sigma_1 d\sigma,\ e_{jk}=\int_a^b\int_a^b \tilde k_j(\sigma,\sigma_1)q_k(\sigma,\sigma_1)d\sigma_1 d\sigma,$$

$$h_j=\int_a^b\int_a^b \tilde k_j(\sigma,\sigma_1)h(\sigma,\sigma_1)d\sigma_1 d\sigma\ (j,k=1,\dots,n),$$

we obtain a system of linear algebraic equations

$$x_j=\sum_{k=1}^{n}e_{jk}x_k+h_j\ (j=1,\dots,n). \qquad (10)$$

Thus, it is valid.

Theorem 2. Let m ϵ C(L^1) and f ϵ C. Then the following assertions hold:

a. if the principal determinant of system (10) is not equal to zero, then equation (1) has a unique solution continuous on D;
b. if the principal determinant of system (10) is zero, then in C equation (1) either has no solutions, or has a finite number of linearly independent solutions.

AN APPROXIMATE SOLUTION OF EQUATION (1)

An approximate solution in C of equation (1) with a kernel from C(L^1) and a continuous function f(t,s) is a rather effective replacement of the kernel by a degenerate one.

We assume that equation (1) with kernel m ϵ C(L^1) and function f ϵ C is invertible in C. By virtue of the stability of the invertibility of equations with respect to sufficiently small perturbations [08], there is a ϵ>0, such that equation:

$$\tilde x(t,s)=\int_a^b\tilde m(t,s,\sigma)x(\sigma,t)d\sigma+f(t,s)\equiv(\tilde Mx)(t,s)+f(t,s) \qquad (11)$$

is invertible for $\|M-\tilde M\|<\varepsilon.$

Solutions of equations (1) and (11) can be written in the form:

$$x=(I-M)^{-1}f,\ \tilde x=(I-\tilde M)^{-1}f.$$

By virtue of

$$(I-M)^{-1}-(I-\tilde M)^{-1}=(I-M)^{-1}(I-(I-M)(I-\tilde M)^{-1})=$$
$$(I-M)^{-1}(I-\tilde M)^{-1}((I-\tilde M)-(I-M))=(I-M)^{-1}(I-\tilde M)^{-1}(M-\tilde M)$$

we get

$$\|(I-M)^{-1}-(I-\tilde M)^{-1}\|\le\|(I-M)^{-1}\|\|(I-\tilde M)^{-1}\|\|M-\tilde M\|\le c\|M-\tilde M\|, \qquad (12)$$

where c is a constant. If now

$$|\,m(t,s,\sigma)-\tilde m(t,s,\sigma)\,|<\frac{\varepsilon}{c(b-a)}, \qquad (13)$$

then from (12), (13) and the formulas for the norm of the Romanowskij operator [03], we have

$$\|(I-M)^{-1}-(I-\tilde M)^{-1}\|\le c\|M-\tilde M\|<\varepsilon.$$

Thus, the operators $(I-M)^{-1}$ and $(I-\tilde M)^{-1}$ differ little in norm, if the kernels of equations (3) and (13) are sufficiently close.

Taking (12) into account, we obtain the following estimate:

$$\|x-\tilde x\|\le c\|M-\tilde M\|\|f\|, \qquad (14)$$

It shows that the number $\|x-\tilde x\|$ is sufficiently small if (13) holds. The application of the estimate (14) is related to the estimate of the constant c. In the general case, effective estimates of the constant c are unknown. However, any known upper bounds for the numbers $(I-M)^{-1}$ and $(I-\tilde M)^{-1}$ lead to an upper estimate for the constant c.

Let us cite one such estimate. Similarly to [03], the equalities

$$(I-M)^{-1}f(t,s)=f(t,s)+\int_a^b r_1(t,s,\sigma)f(\sigma,t)d\sigma+\int_a^b\int_a^b r_2(t,s,\sigma,\sigma_1)f(\sigma,\sigma_1)d\sigma d\sigma_1,$$
$$(I-\tilde M)^{-1}f(t,s)=f(t,s)+\int_a^b \tilde r_1(t,s,\sigma)f(\sigma,t)d\sigma+\int_a^b\int_a^b \tilde r_2(t,s,\sigma,\sigma_1)f(\sigma,\sigma_1)d\sigma d\sigma_1,$$

where $r_1,\tilde r_1\in C(L^1),\ \ r_2,\tilde r_2\in C(L^1(D))$ are some functions.

If now

$$|r_1(t,s,\sigma)|\le a_1(t,s,\sigma),$$
$$|r_2(t,s,\sigma,\sigma_1)|\le a_2(t,s,\sigma,\sigma_1),$$
$$|\tilde r_1(t,s,\sigma)|\le \tilde a_1(t,s,\sigma),$$
$$|\tilde r_2(t,s,\sigma,\sigma_1)|\le \tilde a_2(t,s,\sigma,\sigma_1),$$

where the known functions

$a_1, \widetilde{a}_1 \in C(L^1), \quad a_2, \widetilde{a}_2 \in C(L^1(D)),$
then by virtue of the estimate of the norm of an operator of Romanovskij type with partial integrals in C [03], from (14) implies the estimate

$$c \le \sup_{(t,s)\in D}\left(1+\int_a^b |a_1(t,s,\sigma)|\,d\sigma + \int_a^b\int_a^b |a_2(t,s,\sigma,\sigma_1)|\,d\sigma d\sigma_1\right)\times \sup_{(t,s)\in D}\left(1+\int_a^b |\widetilde{a}_1(t,s,\sigma)|\,d\sigma + \int_a^b\int_a^b |\widetilde{a}_2(t,s,\sigma,\sigma_1)|\,d\sigma d\sigma_1.\right)$$

We show that when the kernel of equation (1) is replaced by a degenerate kernel, the Romanovskij equation is obtained, whose solution is constructed explicitly.

Suppose that in (11)

$$\widetilde{m}(t,s,\sigma) = \sum_{j=1}^{n} l_j(t) m_j(s) n_j(\sigma), \qquad (15)$$

where l_j, m_j, n_j are continuous functions on [a, b].

Substituting (15) into (11), we obtain

$$\widetilde{x}(t,s) = \sum_{j=1}^{n} l_j(t) m_j(s) \int_a^b n_j(\sigma)\widetilde{x}(\sigma,t)d\sigma + f(t,s). \qquad (16)$$

We set

$$\widetilde{x}_j(t) = \int_a^b n_j(\sigma)\widetilde{x}(\sigma,t)d\sigma \ (j=1,\dots,n). \qquad (17)$$

Then

$$\widetilde{x}(t,s) = \sum_{j=1}^{n} l_j(t) m_j(s)\widetilde{x}_j(t) + f(t,s). \qquad (18)$$

Substituting (18) into (17), we obtain the system

$$\widetilde{x}_j(t) = \sum_{k=1}^{n} m_k(t)\int_a^b l_{jk}(\sigma)\widetilde{x}_k(\sigma)d\sigma + f_j(t) \ (j=1,\dots,n), \qquad (19)$$

where

$$l_{jk}(\sigma) = l_k(\sigma)n_j(\sigma), f_j(t) = \int_a^b n_j(\sigma) f(\sigma,t)d\sigma.$$

Assuming

$$\widetilde{x}_{jk} = \int_a^b l_{jk}(\sigma)\widetilde{x}_k(\sigma)d\sigma, \qquad (20)$$

in view of (19) we obtain

$$\widetilde{x}_j(t) = \sum_{p=1}^{n} m_p(t)\widetilde{x}_{jp} + f_j(t) \ (j=1,\dots,n). \qquad (21)$$

Substituting (21) into (20), we obtain the system

$$\widetilde{x}_{jk} = \sum_{p=1}^{n} \widetilde{x}_{jp} d_{pjk} + f_{jk} \ (j,k=1,\dots,n), \qquad (22)$$

where

$$d_{pjk} = \int_a^b l_k(\sigma)m_p(\sigma)n_j(\sigma)d\sigma, \ f_{jk} = \int_a^b l_k(\sigma)n_j(\sigma)f_j(\sigma)d\sigma.$$

Thus, the integral equation of Romanowskij (11) with the degenerate kernel (15) reduces to system (22), whose solution can be found as the union of the solutions of systems obtained from (22) for each fixed j=1,...,n. Since equation (11) has a unique solution, each of these systems has a unique solution. Consequently, system (22) has a unique solution. Substituting this solution of system (22) into (21), we obtain $\widetilde{x}_j(t) \quad (j=1,\dots,n)$. The only solution of equation (11) is now obtained by substitution of the found $\widetilde{x}_j(t) \quad (j=1,\dots,n).$ in (18).

Another method of approximate solution of equation (1) is connected with the transition to an equivalent problem for the system of linear integral equations of Fredholm of the second kind with a parameter and the subsequent approximate solution of this problem.

Indeed, let

$$y(t,s) = \frac{1}{2}(x(t,s)+x(s,t)), z(t,s) = \frac{1}{2}(x(t,s)-x(s,t)),$$
$$g(t,s) = \frac{1}{2}(f(t,s)+f(s,t)), h(t,s) = \frac{1}{2}(f(t,s)-f(s,t)),$$
$$k(t,s,\sigma) = \frac{1}{2}(m(t,s,\sigma)+m(s,t,\sigma)), l(t,s,\sigma) = \frac{1}{2}(m(t,s,\sigma)-m(s,t,\sigma)).$$

Then

$$x = y + z, f = g + h, m = k + l, y(t,s) = y(s,t), z(t,s) = -z(s,t),$$
$$g(t,s) = g(s,t), h(t,s) = -h(s,t),$$
$$k(t,s,\sigma) = k(s,t,\sigma), l(t,s,\sigma) = -l(s,t,\sigma),$$

and equation (1) can be written in the form of the system

$$\begin{cases} y(t,s) = \int_a^b k(t,s,\sigma)y(t,\sigma)d\sigma - \int_a^b l(t,s,\sigma)z(t,\sigma)d\sigma + g(t,s), \\ z(t,s) = \int_a^b l(t,s,\sigma)y(t,\sigma)d\sigma - \int_a^b k(t,s,\sigma)z(t,\sigma)d\sigma + h(t,s) \end{cases} \qquad (23)$$

Fredholm integral equations with parameter *t* in which the unknown function satisfies the additional condition

$$y(t,s) = y(s,t), \ z(t,s) = -z(s,t). \qquad (24)$$

If $1 \notin \sigma(M)$ then problem (23) / (24) is equivalent to equation (1), has a unique continuous on D solution (y(t, s), z(t, s)).

Thus, under the condition $1 \notin \sigma(M)$ the approximate solution of equation (1) reduces to an approximate solution of the system (23) and verification of equalities (24), understood as approximate equalities.

Another method of approximate solution of equation (1) for $\pm 1 \notin \sigma(M)$ is associated with the transition to the Fredholm integral equation of the second kind (2) and the replacement in (2) of the kernels by formula (4), in which the kernel $k_0(t,s,\sigma,\sigma_1)$ is chosen equal to zero, and the sum is chosen so that:

$$\sup_{(t,s)\in D)}\int_a^b\int_a^b |k(t,s,\sigma,\sigma_1) - \sum_{j=1}^{n} k_j(t,s)\widetilde{k}_j(\sigma,\sigma_1)|\,d\sigma d\sigma_1 < \varepsilon,$$

where $\varepsilon > 0$ is an arbitrarily small number. As a result, we obtain the Fredholm integral equation of the second kind with a continuous degenerate kernel:

$$x(t,s) = \int_a^b\int_a^b \sum_{j=1}^{n} k_j(t,s)\widetilde{k}_j(\sigma,\sigma_1)x(\sigma_1,\sigma)d\sigma_1 d\sigma + g(t,s), \qquad (25)$$

where the function g(t, s) is determined by the formula (6). For sufficiently small $\varepsilon > 0$, equation (25) is invertible and is solved in the standard way.

NUMERICAL SOLUTION OF EQUATION (1)

We consider equation (1) with continuous given functions *f(t, s)* and $m(t,s,\sigma)$ where $t,s,\sigma \in [a,b]$

The following approximation scheme is justified by V.I. Romanovskij in [01] and can be used for the numerical solution of equation (1).

The segment [a, b] is divided into parts of length δ by points:

$$t_i = s_i = \sigma_i \ (i = 0,\ldots,n; t_0 = s_0 = \sigma_0 = a, t_n = s_n = \sigma_n = b).$$

We set

$$x_{kl} = x(t_k, s_l), f_{kl} = f(t_k, s_l), m_{hkl} = m(t_k, s_l, \sigma_h)$$

and let Δ denote the determinant of the system of linear equations

$$x_{kl} = f_{kl} + \delta \sum_{h=1}^{n} x_{hk} m_{hkl} \ (k,l = 1,\ldots,n). \qquad (26)$$

If now $n \to \infty$, then just as in Fredholm theory, system (26) approximates equation (1), and its solution tends to the solution of equation (1) [01].

Thus, an approximate numerical solution of equation (1) can be found as a solution of system (26). We note that this solution (26) is obtained under the condition $\Delta \neq 0$. For sufficiently large n this condition means that $1 \notin \sigma(M)$

Another method for the numerical solution of equation (1) is based on the numerical solution of problem (25)/(26) with the use of quadrature formulas. For example, using the formula of left rectangles, the segment [a,b] splits into *n* equal parts by points

$$t_i = s_i = \sigma_i = a + ih,$$

where

$$h = (b-a)/n, i = 0,1,\ldots,n,$$

and the system (25) is replaced by the system

$$\begin{cases} y_{ij}(n) = h\left(\sum_{p=0}^{n-1} k_{ijp} y_{ip}(n) - \sum_{p=0}^{n-1} l_{ijp} z_{ip}(n)\right) + g_{ij}, \\ z_{ij}(n) = h\left(\sum_{p=0}^{n-1} l_{ijp} y_{ip}(n) - \sum_{p=0}^{n-1} k_{ijp} z_{ip}(n)\right) + h_{ij}, \end{cases} \qquad (27)$$

where

$$g_{ij} = g(t_i, s_j), h_{ij} = h(t_i, s_j), k_{ijp} = k(t_i, s_j, \sigma_p), l_{ijp} = l(t_i, s_j, \sigma_p)$$

$$(i, j, p = 0,1,\ldots,n-1).$$

The system (27) is solved for each fixed i=0,1,...,n-1, its solution reduces to solving n systems of linear algebraic equations [09]. Since for each fixed tϵ[a,b] the system (23) is a system of linear integral equations with completely continuous integral operators, then for $n \to \infty$ the solution $(y_{\bar{j}}^{(n)}, z_{\bar{j}}^{(n)})$ of the system (27) tends to $(y_{\bar{j}}, z_{\bar{j}})$ where $y_{\bar{j}} = y(t_i, s_j)$ $z_{\bar{j}} = z(t_i, s_j)$

The verification of equality (24) reduces to estimating the smallness of the number $\delta = \max_{ij}(|y_{ij} - y_{jj}| + |z_{ij} + z_{jj}|)$.

The approximate values of the solution of equation (1) are calculated by the formula:

$$x(t_i, s_j) = y(t_i, s_j) + z(t_i, s_j)(i, j = 0,1,\ldots,n-1)$$

by sufficiently small δ

We note that the direct application of quadrature formulas to equation (1) with continuous given functions f(t,s) and m(t,s,σ) causes difficulties due to the fact that the operator *M* in equation (1) is not completely continuous, and the well-known arguments of the mechanical quadrature method for Fredholm integral equations use the complete continuity of integral operators, which determine such equations.

However, if $1 \notin \sigma(M^2)$ then the method of mechanical quadratures is applied not to equation (1), but to the equivalent reversible equation (2) with a completely continuous integral operator M^2. This uses the cubature formula:

$$\int_a^b \int_a^b z(t,s)dtds = \sum_{i=1}^{Q} \gamma_{ijPQ} z(t_i, s_j) + r_{PQ}(z), \qquad (28)$$

where

$$a \le t_1 < t_2 < \ldots < t_P \le b, a \le s_1 < s_2 < \ldots < s_Q \le b.$$

It is assumed that the quadrature process (28) converges: for any function fϵC(D) the condition

$$r_{PQ}(z) = \int_a^b \int_a^b z(t,s)dtds - \sum_{i=1}^{P} \sum_{j=1}^{Q} \gamma_{ijPQ} z(t_i, s_j) \to 0 \ \ for \ P,Q \to \infty$$

be realized.

Equation (2) can be written in the form

$$x(t,s) = \int_a^b \int_a^b k(t,s,\sigma,\sigma_1) x(\sigma_1,\sigma) d\sigma_1 d\sigma + g(t,s), \qquad (29)$$

where

$$k(t,s,\sigma,\sigma_1) = m(t,s,\sigma) m(\sigma,t,\sigma_1)$$

and g(t,s) is a function (8). Setting $t=t_p$, $s=s_q$ in (29) and replacing the integral by the formula:

$$\int_a^b \int_a^b k(t_p, s_q, \sigma, \sigma_1) x(\sigma_1, \sigma) d\sigma_1 d\sigma = \sum_{i=1}^{P} \sum_{j=1}^{Q} \gamma_{ijPQ} k_{pqij} x(t_i, s_j) + r_{pqPQ},$$

where

$$k_{pqij} = k(t_p, s_q, t_i, s_j),$$

and r_{pqPQ} is the remainder, we get the system, after discarding the remainders in the equations of which we will have a system of equations

$$x_{pq} = \sum_{i=1}^{P} \sum_{j=1}^{Q} \gamma_{ijPQ} k_{pqij} x_{ij} + g(t_p, s_q)(p = 1,\ldots,P; q = 1,\ldots,Q), \qquad (31)$$

where $x_{ij} = x(t_i, s_j)$.

By [10], we have

Theorem 3. Let the following conditions hold:

1. for each P and Q the coefficients γ_{ijPQ} of formula (30) are positive and there exists a number G such that $\gamma_{ijPQ} \le G$
2. the process (28) converges;
3. $X_0 \epsilon C$ is a solution of equation (29).

Then for sufficiently large P and Q the system (32) has the solution

$$x_{pq}(p=1,\ldots,P;q=1,\ldots,Q),$$
$$\max_{1\le p\le P, 1\le q\le Q} |x_{pq} - x_0(t_p,s_q)| \to 0 \, for P,Q \to \infty,$$

and the rate of convergence is estimated by inequalities

$$c_1 R_{PQ} \le \max_{1\le p\le P, 1\le q\le Q} |x_{pq} - x_0(t_p,s_q)| \le c_2 R_{PQ} ,$$

where c_1 and c_2 are positive constants,

$$R_{PQ} = \max_{1\le p\le P, 1\le q\le Q} |r_{PQ}(z_{pqPQ})|, \, z_{pqPQ}(t,s) = k(t_p,s_q,t,s)x_0(t,s).$$

The analytic approximation of $x_{pq}(t,s)$ to the solution $\tilde{x}(t,s)$ of equation (29) is naturally defined by:

$$x_{pq}(t,s) = hg \sum_{i=1}^{P} \sum_{j=1}^{Q} \gamma_{ijPQ} k(t,s,t_i,s_j) x_{ij} + g(t,s).$$

We note that Theorem 3 was established in [11].

CONCLUSION

To solve the partially Romanovskij integral equation (1), it is possible to use other methods of numerical solution of integral equations. However, when applying such methods directly to the equation (1), one should take into account the absence of complete continuity for the operator M. If the applied method is connected with the complete continuity of the integral operator, then by applying this method to the Romanovskij integral equation (1) this method next substantiate for equation (1) directly, or apply it to equation (2) with a completely continuous integral operator, or apply it to the numerical solution of problem (23) / (24) for the system of Fredholm integral equations second kind with parameter t and the completely continuous integral operators.

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