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ENGINEERING METHOD OF FAULT -TOLERANT SYSTEM SIMULATIONS

Evgenii V. Larkin[1], Aleksandr N. Privalov[2]*
[1]Tula State University, Russia
[2]Tula State Lev Tolstoy Pedagogical University, Russia

Autonomous fault-tolerant systems, operated at hard environment, are considered in this paper. It is shown that common method of units failure compensation, based on an introduction to the system a structural redundancy, leads to the increase of weight/size factor and energy consumption, but sometime does not prolongs its lifetime. The new approach to fault/recovery process modeling, based on use of fundamental apparatus of parallel semi-Markov process, in which ordinary processes simulate the life-cycle of individual units, and the complex process, assembled from ordinary processes, simulates reliability system as a whole, is proposed. Dependences for calculation of time intervals and probabilities of wandering through ordinary semi-Markov processes, with use of the recursive method are obtained. It is shown, that when there is rather complex model of unit life-cycle, semi-Markov process would be replaced with more coarse Markov process. Notions of complex semi-Markov process, such as functional states and semi-Markov matrices Cartesian product are introduced. Theoretical results obtained are confirmed by the practical calculation of the reliability indicators of the system with passive redundancy.

Key words: fault-tolerance, semi-Markov process, structural and functional states.

INTRODUCTION

The assurance problem can be regarded as a main problem of complex systems, operated autonomously during long time in unfavorable environment [1], [2], [3], [4], [5], [6]. The inimical environment increases failures rate of equipment, so it is necessary to utilize equipment units with higher reliability and/or endurance to external coercions, or to compensate failures by means of fault-tolerant system creation [7], [8], [9], [10]. Common principle of fault-tolerance supposes introducing redundancy into the system structure to substitute failed unit during operation, increases system weight, dimensions, energy consumption, etc. [27], [28], [29], [30]. Per se redundant units themselves are the source of faults. This is why preliminary redundant system simulation is of interest.

There is the common approach to the system reliability simulation, based on the Markov [11], [12], [13], [14] or semi-Markov [14], [15], [16] processes theory, which rather of widely used when it is necessary to describe sequence of faults/recoveries in one separate unit, subassembly or system, which are considered as a unit [17]. Due to the approach, states of Markov chain are abstract analogues of unit states, and Markov switching are just alike real faults/recoveries of the element. Therefore, wandering through Markov chain characterizes the life-cycle of the unit.

When interacting units assemble is considered, simple Markov or semi-Markov process is insufficient to describe their operation in the failure tolerance regime. Destructive/restorative processes in elements, gathered in assembles, develop independently. Moreover, processes compete [16], [18], [19] between them for a current fault/recovery events. Therefore, it is necessary to have mathematical apparatus, which, from one-side permits to describe fault/recovery process inside separate element, and from other side permits to simulate the competition effect in assembles. Apparatus would allow evaluate probabilities and time intervals of wandering through parallel Markov/semi-Markov process [16]. On determination, mentioned characteristics mean reliability factors of fault-tolerant system as a whole [5], [6], [7], [8], such as failure rate, mean time between failures, mean time to recovering, etc.

METHODS

The approach to simulation of fault-tolerant systems

Let us consider fault-tolerant system, which includes M units, m-th of which in terms of fault tolerance modeling is described with semi-Markov process μ^m. Processes in in units develop in parallel, so together processes μ^m, $1 \leq m \leq M$ form complex M-parallel semi-Markov process [16, 20] as follows

$$\boldsymbol{\mu} = \left[\mu_1, \dots, \mu_m, \dots, \mu_M\right] \quad (1)$$

Semi-Markov process μ^m includes set of structural states (below “states”) A_m and semi-Markov matrix: $h_m(t)$

$$\mu_m = \left\{A_m, \boldsymbol{h}_m(t)\right\}, \quad (2)$$

where t is the time.
Set of states may be represented as conjunction

$$A = \bigcup_{m=1}^{M} A_m \quad (3)$$

**elarkin@mail.ru*

in which $A_m \cap A_n = \varnothing$, when $m \neq n$; $A_m=\{a_{0(m)},..., a_{j(m)},..., a_{J(m)}\}$; $a_{0(m)}$ is the starting state of the m-th semi-Markov process, meaning the start of *m*-th unit exploitation (element is surely able to work); $a_{j(m)}$ is mathematic analogue m-th unit state (able to work, unable to work, short-time failed, under repair, etc.); $a_{J(m)}$ is absorbing state of semi-Markov process, which is mathematic analogue of fully destroyed element.

Semi-Markov matrix $\boldsymbol{h}_m(t)$ is as follows:

$$\boldsymbol{h}_m(t) = \left[h_{j(m),k(m)}(t)\right] = \boldsymbol{p}_m \otimes \boldsymbol{f}_m(t) \quad (4)$$

where $\boldsymbol{p}_m$ is the $[J(m)+1] \times [J(m)+1]$ stochastic matrix; *fm(t)* is the $[J(m)+1] \times [J(m)+1]$ matrix of time densities of residence process μ_m in states of set A_m;

$h_{j(m),k(m)}(t)=p_{j(m),k(m)} \cdot f_{j(m),k(m)}(t)$; $p_{j(m),k(m)}$ is the priori probability of switching from the state $a_{j(m)}$ to the state $a_{k(m)}$ when wandering through states semi-Markov process μ_m; $f_{j(m),k(m)}(t)$ is time density of residence the process in the state $a_{j(m)}$, when there was the decision about switching to the state $a_{k(m)}$.

Due to assumptions, that $a_{0(m)}$ is the starting state, and $a_{J(m)}$ is absorbing state, semi-Markov matrix has the next features (fig. 1 a):

elements of $\boldsymbol{h}_m(t)$ main diagonal are equal to zeros, that describes graph without loops, physically it means, that during exploiting unit physical condition currently switches from one state to another, and after switching unit does not remain in the same condition, as before switching;

probabilities of stochastic matrix $\boldsymbol{p}_m$ reflect those or that cause of element fail, or possibility of repair element with those or that result;

densities of matrix $\boldsymbol{f}_m(t)$ describe how many times lasts period till fail with concrete cause or how many times will be spent till element restoration;

for elements of rows from $0(m)$-th till $[J(m)-1]$-th the next expression is true:

$$\sum_{k(m)=1(m)}^{J(m)} \int_0^{\infty} h_{j(m),k(m)}(t)dt = 1,\ 0(m) \leq j(m) \leq J(m); \quad (5)$$

for elements of rows from $0(m)$-th till $[J(m)-1]$-th the next expression is true:

both probabilities of $\boldsymbol{p}_m$-matrix and parameters of densities of $\boldsymbol{f}_m(t)$-matrix (expectation, dispersion, initial and central moments of higher orders) depend on the substance, of which the element is made, a quality of element manufacturing and assembling, an exploiting conditions, side effects, etc, and define parameters of wandering through the semi-Markov process μ_m.

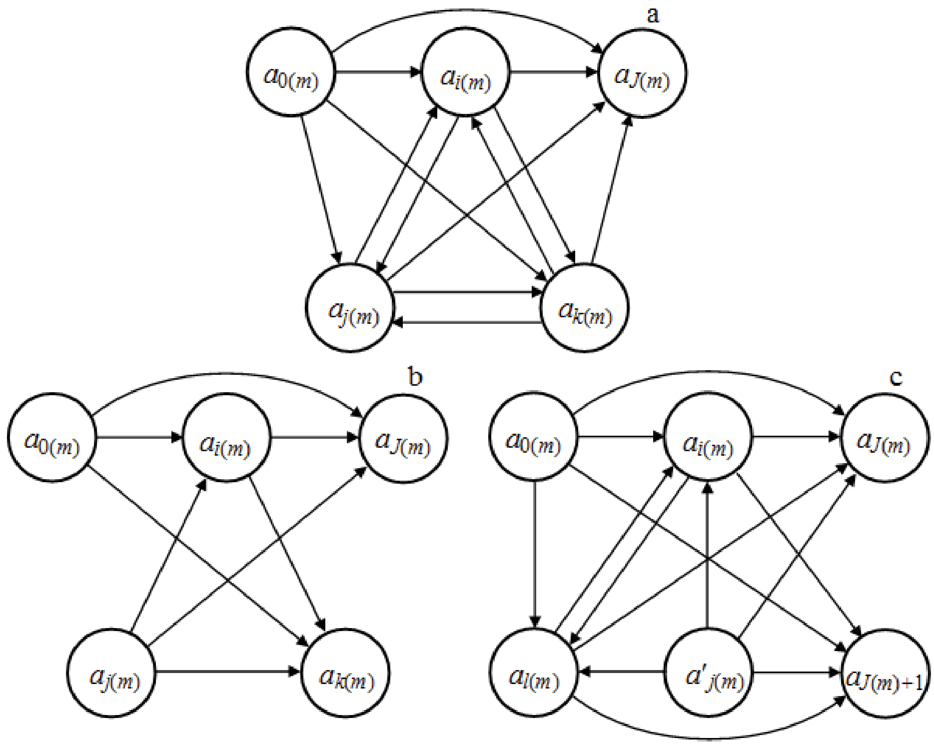

Figure 1: The structure of elementary failure/recovery process

elements of the matrix $\boldsymbol{h}_m(t)$ zero column are all equal to zeros, so during wandering it is impossible to return to the state $a_{0(m)}$, which describes the case of surely workable *m*-th unit;

elements of *J(m)*-th row are all equal to zeros, so fully destroyed unit can not be returned to operation;

Single element reliability parameters calculation

As it is known [16], [21], time density of wandering through the semi-Markov process (2) from the state $a_{0(m)}$ to the state $a_{J(m)}$ is as follows:

$$\tilde{f}_{0(m),J(m)}(t) = L^{-1}\left[I^R_{0(m)} \cdot \sum_{w=1}^{\infty} \{L[\boldsymbol{h}_m(t)]\}^w \cdot I^C_{J(m)} \right] \quad (6)$$

where $I^R_{0(m)}$ is the [$J(m)+1$]-size row-vector, in which 0(m)-th element is equal to one, and other elements are equal to zeros; $I^Ñ_{0(m)}$ is the [$J(m)+1$]-size column-vector, in which $J(m)$-th element is equal to one, and other elements are equal to zeros; L[...] and L^{-1}[...] are direct and inverse Laplace transforms, correspondingly.

The interim Laplace transform in (6) is necessary to replace semi-Markov matrix convolution operation, which is not defined, with characteristic matrix multiplication operation, which is well known. The dependence (6) defines pure (not weighed) density of time of reaching the state $a_{J(m)}$ from the state $a_{0(m)}$, by definition of matrix product operation [22], and due to the fact, that there is the only starting state $a_{0(m)}$ and only absorbing state $a_{J(m)}$ in the semi-Markov process.

Mean "lifetime" of m-th element in the system and probability of the fact, that m-th element remains operable during time θ, are as follows:

$$\tilde{T}_{0(m),J(m)} = \int_0^{\infty} t \cdot \tilde{f}_{0(m),J(m)}(t)dt \quad (7)$$

$$\tilde{p}_{0(m),J(m)}(\theta) = \int_0^{\theta} \tilde{f}_{0(m),J(m)}(t)dt \quad (8)$$

Besides the task of time density $\tilde{f}_{0(m),J(m)}(t)$ definition fault-tolerant system designer would to solve, the task of definition of time density of reaching arbitrary state $a_{k(m)}$ from the state $a_{j(m)}$,$1(m) \le j(m) \le J(m)$, so both states are no startin, not absorbing. When $j(m) \ne k(m)$, the task may be interpreted as definition the time interval till failure, or definition the repair time. When $j(m) = k(m)$, the task may be interpreted as definition the time interval between failures, or time interval between repairing. The only restriction, imposed onto wandering trajectories, is that neither state $a_{j(m)}$ nor state $a_{k(m)}$ process should fall twice. In other to satisfy the restriction state $a_{j(m)}$ should get status starting one, and state $a_{j(m)}$ should get status absorbing one. When $j(m) = k(m)$, the state $a_{j(m)}$ should be split onto starting and absorbing substates.

First case is shown on the fig. 1 b, where is shown the graph, fig. 1 a, from which all arks, leading to the state $a_{j(m)}$, and all arks, leading from the state $a_{k(m)}$, are deleted. To form such structure in semi-Markov matrix $\boldsymbol{h}_m(t)$ all elements of $j(m)$-th column and $k(m)$-th row should be replaced to zeros. Elements $h_{i(m),l(m)}(t)$ should be recalculated as follows:

$$h'_{i(m),l(m)}(t) = \frac{h_{i(m),l(m)}(t)}{\sum\limits_{\substack{k(m)=0(m),\\ k(m)\ne j(m)}}^{J(m)} p_{i(m),k(m)}}; \quad (9)$$

$$0(m) \le i(m) \le J(m);\ i(m) \ne k(m);$$

So $\boldsymbol{h}_m(t)$ is transformed to matrix $\boldsymbol{h}'_m(t)$ as follows:

$$\boldsymbol{h}_m(t) \rightarrow \boldsymbol{h}'_m(t). \quad (10)$$

Stochastic summation of densities, formed on all possible wandering trajectories gives next expression:

$$\tilde{h}'_{j(m),k(m)}(t) = \boldsymbol{I}^R_{j(m)} \cdot L^{-1}\left[\sum_{w=1}^{\infty} \{L[\boldsymbol{h}'_m(t)]\}^w \right] \cdot \boldsymbol{I}^C_{k(m)}, \quad (11)$$

where $I^R_{j(m)}$ is the row-vector, in which $j(m)$-th element is equal to one, and other elements are equal to zeros; $I^Ñ_{k(m)}$ is the column-vector, in which $k(m)$-th element is equal to one, and other elements are equal to zeros.

In the semi-Markov process $h'_m(t)$ there are as minimum two absorbing states: namely $a_{k(m)}$ and $a_{j(m)}$, so group of events of reaching from is not full and in common case $\tilde{h}'_{j(m),k(m)}(t)$ is weighted, but not pure density. The state $a_{k(m)}$ from the state $a_{j(m)}$ may be reached with probability

$$\tilde{p}'_{j(m),k(m)} = \int_0^{\infty} \tilde{h}'_{j(m),k(m)}(t)dt. \quad (12)$$

Pure time density of wandering from the state $a_{j(m)}$ to the state $a_{k(m)}$ may be defined as follows:

$$\tilde{f}'_{j(m),k(m)}(t) = \frac{\tilde{h}'_{j(m),k(m)}(t)}{\tilde{p}'_{j(m),k(m)}}. \quad (13)$$

Second case is shown on the fig. 1 c, where state $a_{j(m)}$ is split onto starting state $a_{j(m)}$ and absorbing state $a_{J(m)+1}$. To form such structure in semi-Markov $\boldsymbol{h}_m(t)$ next changes should be done:

one row and one column should be added to the matrix; added, [$J(m)$+1]-th, row should be fulfilled with zeros; $j(m)$-th column at first should be carried to the [$J(m)$+1]-th column, and then it should be fulfilled by zeros.

So $\boldsymbol{h}_m(t)$ is transformed to [$J(m)+1$] × [$J(m)+1$] matrix $\boldsymbol{h}''(t)$ as follows:

$$\boldsymbol{h}_m(t) \rightarrow \boldsymbol{h}''_m(t). \quad (14)$$

Stochastic summation of densities, formed on all possible wandering trajectories gives next expression:

$$\tilde{h}''_{j(m),k(m)}(t) = \boldsymbol{I}^R_{j(m)} \cdot L^{-1}\left[\sum_{w=1}^{\infty} \{L[\boldsymbol{h}''_m(t)]\}^w \right] \cdot \boldsymbol{I}^C_{J(m)+1}, \quad (15)$$

where $I^R_{j(m}$ is the [J(m)+2]-size row-vector, in which $j(m)$-th element is equal to one, and other elements are equal to zeros; $I^Ñ_{J(m)+1}$ is the [J(m)+2]-size column-vector, in which [$J(m)$+1]-th element is equal to one, and other elements are equal to zeros.

In the semi-Markov process $\boldsymbol{h}''(t)$ there are two absorbing states: namely $a_{J(m)}$ and $a_{J(m)+1}$, so group of events of reaching $a_{J(m)+1}$ from $a_{j(m)}$ is not full and in common case is weighted, but not pure density. The state $a_{J(m)+1}$ from the state $a_{j(m)}$ may be reached with probability

$$\tilde{p}''_{j(m),J(m)+1} = \int_0^\infty \tilde{h}''_{j(m),J(m)+1}(t)dt, \tag{16}$$

and pure time density of wandering from the state $a_{j(m)}$ to the state $a_{k(m)}$ may be defined as follows:

$$\tilde{f}''_{j(m),J(m)+1}(t) = \frac{\tilde{h}''_{j(m),J(m)+1}(t)}{\tilde{p}''_{j(m),J(m)+1}}. \tag{17}$$

From (13) and (17) mean time of reaching $a_{k(m)}$ from $a_{j(m)}$ probability of reaching $a_{k(m)}$ from $a_{j(m)}$ during time θ, mean time of returning to the state $a_{j(m)}$ and probability of returning to $a_{j(m)}$ during time θ may be obtained as follows [23]:

$$\tilde{T}'_{j(m),k(m)} = \int_0^\infty t\cdot\tilde{f}'_{j(m),k(m)}(t)dt. \tag{18}$$

$$\tilde{p}'_{j(m),k(m)}(\theta) = \int_0^\theta \tilde{f}'_{j(m),k(m)}(t)dt. \tag{19}$$

$$\tilde{T}''_{j(m),j(m)} = \int_0^\infty t\cdot\tilde{f}''_{j(m),J(m)+1}(t)dt. \tag{20}$$

$$\tilde{p}''_{j(m),j(m)}(\theta) = \int_0^\theta \tilde{f}''_{j(m),J(m)+1}(t)dt. \tag{21}$$

As it follows from (6), (13), (17) computational complexity of estimation of $\tilde{f}_{0(m),J(m)}(t)$, $\tilde{f}'_{j(m),k(m)}(t)$ and $\tilde{f}''_{j(m),J(m)+1}(t)$ is extremely high, which is linked with necessity of execution of direct and inverse Laplace transform, exponentiation of semi-Markov matrix to infinite degree, extraction from the result the only weighed density element and estimation of its probability and pure density. To reduce computational complexity it is necessary to take advantage of B.Grigelionis theorem [24] and recursive procedure [21].

Due to the theorem by B.Grigelionis combination of non-Poisson fluxes approximately converge to the fluxes with Poisson properties [25]. So, both $\tilde{f}_{0(m),J(m)}(t)$, and $\tilde{f}'_{j(m),k(m)}(t)$, or $\tilde{f}''_{j(m),J(m)+1}(t)$ may be correspondingly approximated with exponential distribution densities:

$$\tilde{f}_{0(m),J(m)}(t) \cong \hat{f}_{0(m),J(m)}(t) = \frac{1}{\tilde{T}_{0(m),J(m)}}\exp\left(-\frac{t}{\tilde{T}_{0(m),J(m)}}\right); \tag{22}$$

$$\hat{f}'_{j(m),k(m)}(t) \cong \tilde{f}'_{j(m),k(m)}(t) = \frac{1}{\tilde{T}'_{j(m),k(m)}}\exp\left(-\frac{t}{\tilde{T}'_{j(m),k(m)}}\right); \tag{23}$$

$$\tilde{f}''_{j(m),J(m)+1}(t) \cong \hat{f}''_{j(m),J(m)+1}(t) \cong \frac{1}{\tilde{T}''_{j(m),j(m)}}\exp\left(-\frac{t}{\tilde{T}''_{j(m),j(m)}}\right). \tag{24}$$

In such a way, description of time intervals between events, based on the semi-Markov processes, are substituted by description, based on pure Markov processes. Of course, such substitutions coarsen the model [26], but permit to substantially simplify math calculations, when one would investigate and/or solve practical problems of system fault-tolerant design. As seen from (22), (23), (24), there is only parameter in exponential density, namely expectation, which one should to have for comprehensive description of the events flux. So the method should be oriented on accelerated numerical calculation the expectations namely. After all, method should not include matrix raising into degree, which tends to infinity, as it is in (6), (11), (15).

The recursive method of expectation calculation

For description the method let us consider abstract Markov process, which is described with matrix

$$\boldsymbol{x}(t) \in \{\boldsymbol{h}_m(t), \boldsymbol{h}'_m(t), \boldsymbol{h}''_m(t)\}. \tag{25}$$

Abstract Markov process may be represented with the stochastic matrix $\boldsymbol{q} = (q_{r,s})$ and expectation matrix $\boldsymbol{Y} = (Y_{r,s})$, obtained from $\boldsymbol{x}(t)$ as follows:

Abstract Markov process may be represented with the stochastic matrix $\boldsymbol{q} = (q_{r,s})$ and expectation matrix $\boldsymbol{Y} = (Y_{r,s})$, obtained from $\boldsymbol{x}(t)$ as follows:

$$\boldsymbol{q} = \int_0^\infty \boldsymbol{x}(t)dt = (q_{r,s}); \tag{26}$$

$$\boldsymbol{Y}' = \int_0^\infty t\cdot\boldsymbol{x}(t)dt = (Y'_{r,s}); \tag{27}$$

$$\boldsymbol{Y} = \boldsymbol{Y}'/\boldsymbol{q} = \left(\frac{Y'_{r,s}}{q_{r,s}}\right) = (Y_{r,s}), \tag{28}$$

where $\boldsymbol{Y}'$ is the matrix of weighted expectations; $\boldsymbol{Y}'/\boldsymbol{q}$ is the operation of direct (element-by-element) division of matrices.

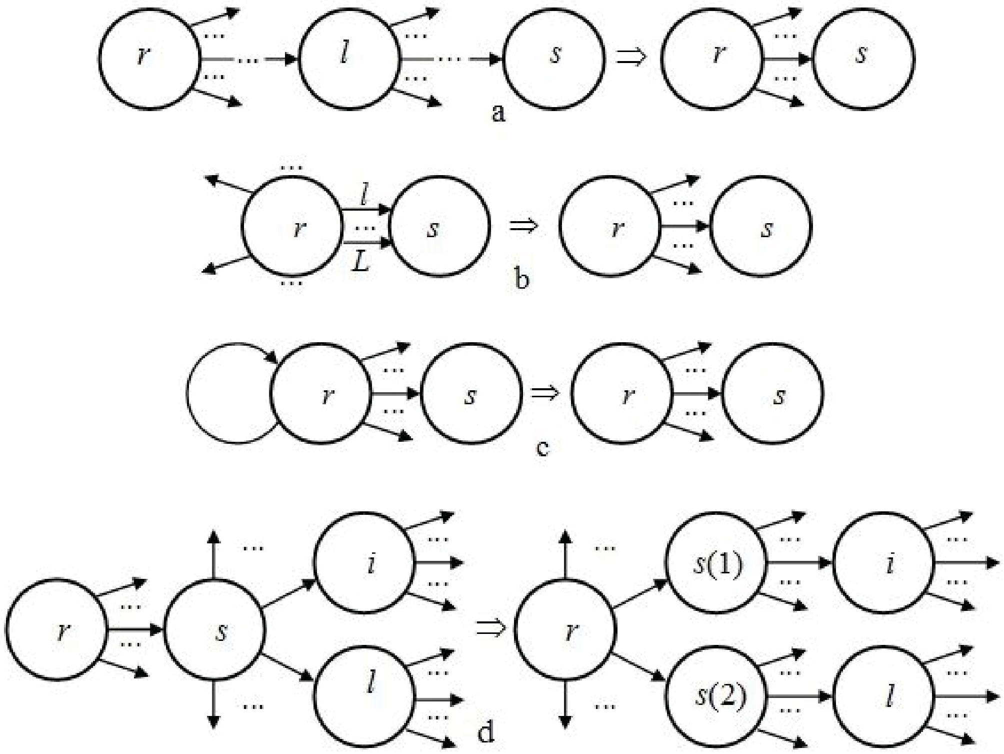

Figure 2: Operations of Markov sub-process reducing

For realization of recursive method [21] let us to introduce four elementary transformations of Markov process structure, which are shown on fig. 2.

On the fig. 2 a the integration of sequential situated states is shown. Probability and expectation of time of wandering from a_r to a_s are determined with the next dependencies:

$$\tilde{q}_{r,s} = \prod_{l=r}^{s-1} q_l \, ; \tilde{Y}_{r,s} = \sum_{l=r}^{s-1} Y_l \, . \tag{29}$$

where $q_l \in (q_{r,s})$, when $l \in \{r, r+1, ..., l, ..., s-1, s\}$, are probabilities of switching from previous state, belonging to wandering trajectory, shown on the fig. 2 a, to the next state, belonging the same trajectory; $Y_l \in (Y_{r,s})$ are time expectations of residence in states, belonging to wandering trajectory, shown on the fig. 2.

On the fig. 2 b the integration of parallel arcs is shown. Probability and time expectation of switching from *r* to *s* are determined by the next dependencies:

$$\tilde{q}_{r,s} = \sum_{l=1}^{L} q_{r,l,s} \, ;$$

$$\tilde{Y}_{r,s} = \frac{\sum_{k=1} q_{r,l,s} \cdot T_{r,l,s}}{\tilde{q}_{r,s}} \, , \tag{30}$$

where *l* is the number of arc leading from *r* to *s*; $q_{r,\, l,\, s}$, $T_{r,\, l,\, s}$ are probabilities, and expectations of wandering from *r* to *s* with *l*-th arc.

Fig. 2 c shows elimination of loop. Probability and time expectation of switching from *r* to *s* are as follows:

$$\tilde{q}_{r,s} = \frac{q_{r,s}}{1 + q_{r,r}} \, ;$$

$$\tilde{Y}_{r,s} = Y_{r,s} + \frac{Y_{r,r} \cdot q_{r,r}}{1 - q_{r,r}} \, . \tag{31}$$

It is necessary to admit, that at first in Markov process ***x(t)*** there is neither parallel arks, nor loops. Named structural elements appear during recursive transformations of the process (25).

Fig. 2 c shows splitting the state s onto states s_1 and s_2. Probabilities and time expectations, when splitting, are as follows:

$$\tilde{q}_{r,s(1)} = q_{r,s} \cdot q_{s,i} \, ; \tilde{q}_{r,s(2)} = q_{r,s} \cdot q_{s,l} \, ;$$
$$\tilde{Y}_{r,s(1)} = Y_{r,s} \, ; \tilde{Y}_{r,s(2)} = Y_{r,s} \, ; \tilde{Y}_{s(1),i} = Y_{s,i} \, ; \tilde{Y}_{s(2),l} = Y_{s,l} \, . \tag{32}$$

In (29), (30), (31), (32) symbols with tilde means parameter's value after transformation; symbols without tilde means parameter's value before transformation.

The recursive transformation presupposes sequential elimination of states from s-th, till necessary number. Let on the first step of recursion graph be the full S-states one with no loops. On the discussed step of recursion only s states remain in the graph. States are re-numerated in comparison with numeration of (25) in such a way, that initial and destination states have numbers one and two, correspondingly, that is necessary for simplification of recursion procedure indexation.

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Procedure starts from double splitting the state with highest current number s is it is shown on the fig. 3. The first splitting gives the set $A_s = \{A_{s,1}, ..., A_{s,r}, ..., A_{s,s-1}\}$, r-th element of which, namely $A_{s,r}$ performs, in turn, the set of states, which are obtained after second splitting as follows: $A_{s,r} = \{a_{s,r,1}, ..., A_{s,r,r}, ..., A_{s,r,s-1}\}$. Common number of states, which perform the state s after double splitting, is equal to $(s-1)^2$. Indexation after splitting is as follows: first index performs number of state under splitting, second index mean state number, from which arc, leading to state after splitting issues; third index means state number, to which arc, leading from the state after splitting, falls.

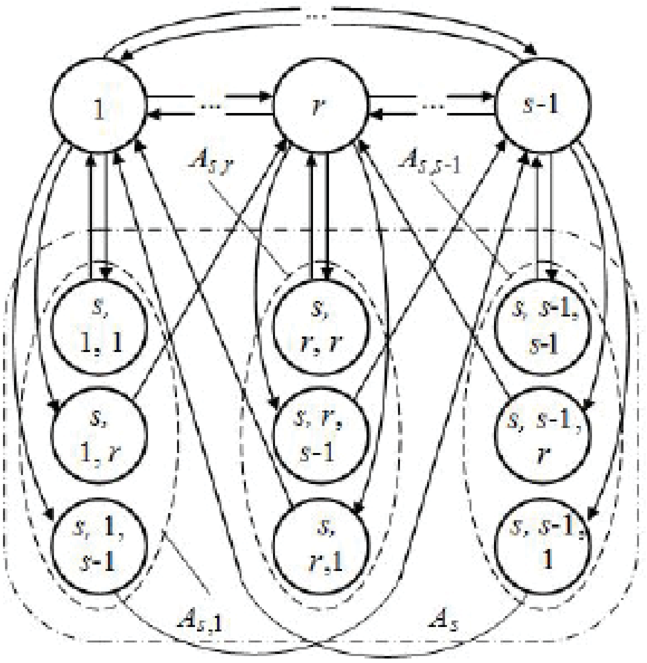

Figure 3: Double splitting the state s

In other to ensure equivalency of transforms, probabilities and expectations must be recalculated in accordance with dependencies (32) and (29):

$$\tilde{q}_{r,l} = q_{r,s} \cdot q_{s,l}; \quad \tilde{Y}_{r,l} = Y_{r,s} + Y_{s,l}, \; 1 \le r, l \le s - 1. \tag{33}$$

On second step, in accordance with (30), parallel arcs should be integrated, and both probabilities, and expectations should be integrated as follows:

$$\hat{q}_{r,l} = \tilde{q}_{r,l} + q_{r,l}; \quad \hat{Y}_{r,l} = \frac{\tilde{Y}_{r,l}\tilde{q}_{r,l} + Y_{r,l}q_{r,l}}{\hat{q}_{r,l}}, \quad 1 \le r, l \le s - 1, \; l \ne r. \tag{34}$$

On third step, loops, which emerges after double splitting, should be eliminated and probabilities and expectations should be recalculated in accordance with (31):

$$q_{r,l} = \frac{\hat{q}_{r,l}}{1 + \hat{q}_{r,r}}; \quad Y_{r,l} = \hat{Y}_{r,l} + \frac{\tilde{Y}_{r,r} \cdot \tilde{q}_{r,r}}{1 - \tilde{q}_{r,r}}, \quad 1 \le r, l \le s - 1, \; l \ne r \tag{35}$$

After deleting of loops in the graph stay *s* - 1 states only and it is ready for next recursion stage. Recursion one need to continue, until in te reduced Markov process stay only two (in the case of $\boldsymbol{h}_m(t)$) or four (in the cases of $\boldsymbol{h}'_m(t)$, $\boldsymbol{h}''_m(t)$) states, including states under investigation. The result of calculation is time expectations of Markov processes (22), (23) or (24), which can be used when investigate competition in fault-tolerant system.

Results and Discussions

Interaction in fault-tolerant system

In real fault-tolerant system (1) semi-Markov every processes μ^m do not operates separately, by itself, but processes really interact between them [16, 18, 19]. So, there may be created such abstraction, as complex *M*-parallel semi-Markov process [16]

$$^M\mu = \left\{^M A, {}^M\boldsymbol{h}(t)\right\}, \tag{36}$$

where $^M A$ - is the set of functional states; $^M \boldsymbol{h}(t)$- is the semi-Markov matrix.

Functional states are formed from structural states, mentioned in (1), by mean of Cartesian multiplication of sets A_m, $1 \le m \le M$, namely

$$^M A = \coprod_{m=1}^{M} A_m = \left\{\alpha_{1(\alpha)}, ..., \alpha_{j(\alpha)}, ..., \alpha_{J(\alpha)}\right\} = \left\{[a_{0(1)}, ..., a_{0(m)}, ..., a_{0(M)}], ..., [a_{j(1)}, ..., a_{k(m)}, ..., a_{i(M)}], ..., [a_{J(1)}, ..., a_{J(m)}, ..., a_{J(M)}]\right\}, \tag{37}$$

where $\alpha_{j(\alpha)} = [\alpha_{j(1)}, ..., \alpha_{k(m)}, ..., \alpha_{i(M)}]$ is the functional state; $\coprod$ is the symbol of group Cartesian product; $J(\alpha) = \prod |A_m| = \prod J(m)$.

To define semi-Markov matrix $^M \boldsymbol{h}(t)$, semi-Markov matrices Cartesian multiplication operation should be introduced, in which matrices $\boldsymbol{h}_m(t)$ are considered as specifically ordered sets:

$$^M\boldsymbol{h}(t) = \coprod_{m=1}^{M} \boldsymbol{h}_m(t). \tag{38}$$

As it follows from (18), indices of $^M \boldsymbol{h}(t)$ rows and columns are vectors,

$$0(\alpha) = [j(1), ..., j(m), ..., j(M)] \le j(\alpha) = [j(1), ..., j(m), ..., j(M)] \le J(\alpha) = [j(1), ..., j(m), ..., j(M)] \tag{39}$$

Let semi-Markov process $^M\mu$ switches into the state $\alpha_{j(\alpha)}$. During sojourn of processes μ_m $1 \le m \le M$ in the states $\alpha_{j(1)}, ..., \alpha_{j(m)}, ..., \alpha_{j(M)}$, vectors of weighted densities $\boldsymbol{h}_{j(m)}(t) = [h_{j(m),0(m)}(t), ..., h_{j(m),k(m)}(t), ..., h_{j(m),J(m)}(t)]$ for $1 \le m \le M$ compete between them. There are $J(\alpha)$ possible directions of competition. Direction $\alpha_{k(\alpha)} = [\alpha_{k(1)}, ..., \alpha_{k(m)}, ..., \alpha_{k(M)}]$ would be selected with probability

$$p_{j(\alpha),k(\alpha)} = \prod_{m=1}^{M} p_{j(m),k(m)} \cdot \quad (40)$$

Within the selected direction weighted time of winning the competition by the process , with taking into account probability of selected direction, is as follows:

$${}^{M}h_{j(\alpha),[j(1),\dots,j(m-1),k(m),j(m+1),\dots,j(M)]}(t) = = f_{j(m),k(m)}(t)\cdot \sum_{\substack{l=1,\\ l\neq m}}^{M}\left[1-F_{j(l),k(l)}(t)\right]\cdot \prod_{l=1}^{M} p_{j(l),k(l)}, \quad (41)$$

where *l* is auxiliary index.

Elements ${}^{M}h_{j(\alpha),[j(1),\dots,j(m-1),k(m),j(m+1),\dots,j(M)]}(t)$, 1 $1 \le m \le M$, ful fill proper cells of the semi-Markov matrix ${}^{M}\boldsymbol{h}(t)$ $j(\alpha)$ -th row. The remaining cells of this row are fulfilled with zeros. In such a way Cartesian multiplication (38) is executed.

It is necessary to admit the next.

1) Permutation of factors in Cartesian product (38) leads only to permutation in rows and in columns in semi-Markov matrix ${}^{M}\boldsymbol{h}(t)$, and not change matrix as a whole. Complex semi-Markov process obtained is just alike ordinary semi-Markov process with set of states (37) and semi-Markov matrix (38). To solve the problem of evaluation of fault-tolerant parameters, such as: time of obtaining the state, in which *l* elements of *M* are in workable state; probability the situation, in which at current time *m* elements of *M* are in repaired condition, etc. one should to use method, proposed in section 3, with .using as basic abstraction the model of complex semi-Markov process (36).

2) Number of columns and rows in semi-Markov matrix ${}^{M}\boldsymbol{h}(t)$ increases with geometric progression in accordance with dimensions of matrices $\boldsymbol{h}_m(t)$, $1 \le m \le M$, so, computational complexity of fault-tolerant system analysis problem depends on degree of matrices $\boldsymbol{h}_m(t)$ simplification with use methods, discussed in section 3. In limiting case semi-Markov process ${}^{M}\boldsymbol{h}(t)$ would include only two states, and semi-Markov matrix $\boldsymbol{h}_m(t)$ is 2×2 dimension matrix, so rows and columns of ${}^{M}\boldsymbol{h}(t)$ may be numerated with binary code, that facilitates the problem solution.

Example

The system, which includes units, fault/recovery structure of which is shown on the fig. 2 with right without side arcs. Semi-Markov model of the unit fault/recoveries process is as follows:

$$\mu = \left\{ \{r, s\}, \begin{bmatrix} p\cdot g(t) & 0 \\ 0 & (1-p)\cdot g(t) \end{bmatrix} \right\}. \quad (42)$$

where *r,s* are states, simulated operation between failures and overall destruction of unit, correspondingly; *g(t)* is the time density with expectation *T; p* is the probability of short-time failure; (1 - *p*) is the probability of unit destruction.

In accordance with method, discussed in section 3, semi-Markov process (42) is transformed into the process $\tilde{\mu}$, structure of which is shown on the fig. 2 c, left without side arcs.:

$$\tilde{\mu} = \left\{ \{r, s\}, \begin{bmatrix} 0 & \frac{1-p}{T}\exp\left(-\frac{1-p}{T}t\right) \\ 0 & 0 \end{bmatrix} \right\}, \quad (43)$$

where $\frac{T}{1-p}$ is a unit lifetime.

Conformity of lifetime interval to exponential law was verified with use direct Monte-Carlo method for the case, when *g(t)* is uniform distribution, *g(t) = 1,* when *t* (*t* - 1) ≤ 0,5. Result of verification is shown on the fig. 5, where experimental histogram, which is just alike exponential law, is shown. Experimental expectation is equal to 1,96 grades of time, (error less then 1,5 % of theoretical 2 grades of time).

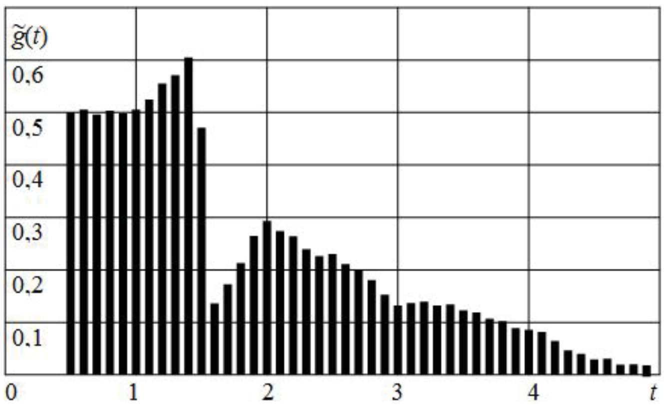

Figure 4: Density of unit "lifetime"

Into fault-tolerant system *M* units enable in parallel, to provide a passive redundancy [7], [8], [9], as it is shown on the fig. 5. Structural states of semi-Markov model (43) fulfill functional states, shown on the fig. 5. As an absorbing state in the structure the boundary of functional state is used. State the system as a whole is divided onto hierarchical levels. On the upper, *M*-th level all *M* units are in workable state. Functional state $\alpha_{M,1}$ is marked as vector [11, ..., 11], including *M* "ones". When *m*-th semi-Markov process switches, proper "one" transforms to "zero". In the vector stays *M* - 1 "ones" and system as a whole decrements on the (*M* - 1)-th level, where there are (*M* - 1) workable units. Functional state $\alpha_{M-1,m}$ is marked as vector [11, ..., 11], including (*M* - 1) "ones". Bottom level includes only absorbing states, nominated as 0, and it means, that no workable units there are in the system.

Let on the *L*-th level functional state there are *L* of *M* workable units, which just stay "alive" after severe failures of other units. Due to the fact, that on *L*-th hierarchical level may exist its specific unit loading conditions, which change probabilities *p* and time intervals between events *T*, index *L* is used, i.e. on this level $p = p_L$, $T = T_L$. So the time density $\tilde{f}_L(t)$ and the time expectation $\tilde{T}_L$ of decline system from *L*-th to (*L* - 1)-th level are as follows:

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$$\tilde{f}_L(t) = L\frac{1-p_L}{T_L}\exp\left(-\frac{1-p_L}{T_L}Lt\right);\ \tilde{T}_L = \frac{T_L}{(1-p_L)L}. \quad (44)$$

Common time to failure of the fault-tolerant system, as a whole, is as follows:

$$T_{tf} = \sum_{L=M}^{1}\tilde{T}_L. \quad (45)$$

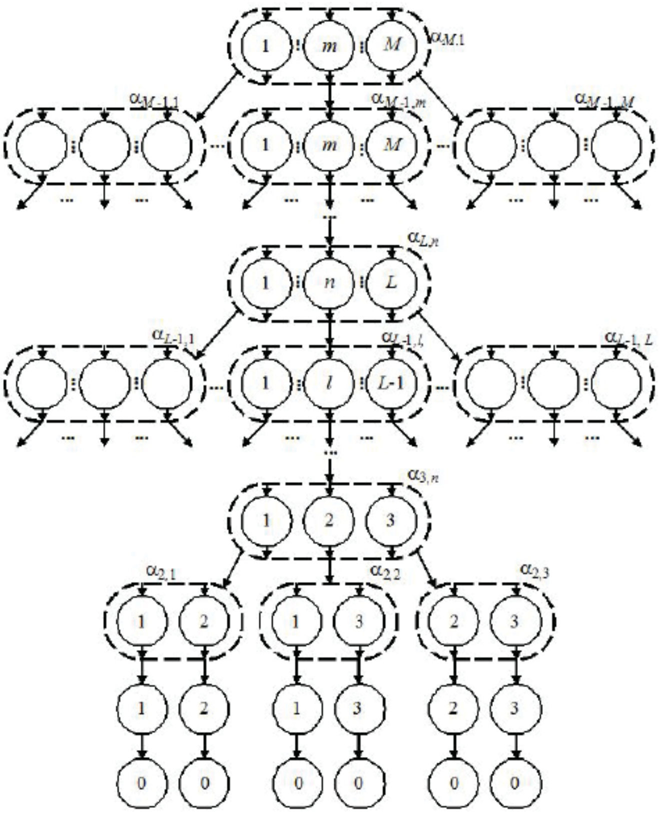

Figure 5: Model of fault-tolerant system with passive redundancy

Backup efficiency may be evaluated as $\frac{T}{1-p} - \tilde{T}_L$,

where T_0 is time to failure of only unit in non-overloading regime of exploitation.

Models, based on the classic semi-Markov process theory [20, 31, 32], are rather cumbersome, but permit quite exactly to describe number of effects emerging in systems under investigation. Theory of parallel semi-Markov processes is not in common use when solving urgent fault-tolerance problems that explains necessity of proposed mathematical apparatus development.

CONCLUSION

To sum up, dependences for calculation of time intervals and probabilities of wandering through ordinary semi-Markov processes, with use of the recursive method are obtained. Notions of complex semi-Markov process such as functional states and semi-Markov matrices Cartesian product are introduced.

The simulation of fault-tolerant systems has been proposed to be divided into three stages. On the first stage ordinary semi-Markov models of separate units lifecycles should be developed, and this models should be simplified till semi-Markov processes with minimal number of states, in the limit, till two-state Markov processes. At the second stage one should gather ordinary semi-Markov processes, operating in parallel, to complex semi-Markov process with functional states. parameters of residence in which are calculated with use operation of Cartesian multiplication of semi-Markov matrices. At the third stage the abstraction - complex semi-Markov process - is used for estimation of reliability parameters of the fault-tolerant system as a whole. If the system has a more complex hierarchical structure, in which blocks of the next level are assembled from units of previous level, then describing blocks complex semi-Markov processes may be considered as ordinary processes from which complex process of next hierarchical level may be formed, etc. The approach proposed permits to create model of redundant system with any degree of complexity.

Further research in this area may be directed to simulation the great number of practical redundant systems with complex interactions between components and complex algorithms of lifecycle. Also method of fault-tolerant system optimization, based on Petri-Markov nets approach may be worked out too.

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