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# OPTIMAL CONTROL OF PROPELLANT CONSUMPTION DURING INSERTION OF ROCKET INTO A CIRCLE ORBIT OF THE EARTH

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**Key words:** *ballistics, inverse problems, regularization method, variation principle, Euler equation*
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# *OPTIMAL CONTROL OF PROPELLANT CONSUMPTION DURING INSERTION OF ROCKET INTO A CIRCLE ORBIT OF THE EARTH*

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*The problem of launching a rocket into the Earth's orbit has already been solved using the regularization method in previous studies. But the regularization method remains relevant for application to solving integral equations of the first kind, which determine the components of speed and acceleration. The problem of optimal control of propellant consumption during the insertion of a rocket into a circle orbit of the Earth is solved using regularized solutions of integral equations of the first kind which are solutions of corresponding Euler equations on discrete-time net. The influence of the regularization parameter and some additional parameters on precision of discredited problem is investigated. Calculations are carried out for existing chemical rocket engine and promising plasmic one. Considered algorithm is summed up easily to problem of suborbital flights by setting desired coordinate system and modifying motion equations. Conclusions were drawn about the required speed for the lowest fuel consumption, as well as about the problem for a single-stage rocket. Thus, the development of a plasma rocket engine with an exhaust velocity is more than ten times higher than that of a chemical one.*

*Key words: ballistics, inverse problems, regularization method, variation principle, Euler equation*

## *INTRODUCTION*

A problem of the trajectory optimization of a rocket or a spacecraft with a rocket engine belongs to a class of the dynamic systems optimization problems. Its solution leads to searching for the local or global extremum of a beforehand defined functional determined on the set of the solutions of the controlled dynamic system satisfying some conditions [1-3]. As a rule, the conditions can be both internal and boundary to the control process. Thus, we consider a rocket or a spacecraft to be the controlled dynamic system. Applying some restrictions to it we have some formulation of the optimization problem [4-6]. It is well known that its solution is found with the maximum principle by Pontryagin transferring the optimization problem to the boundary problem [7-9]. Besides, we have to determine explicitly the performance criteria and restrictions [1, 4, 10].

There are two models of rocket engine performance [11-13]. The first of them matches the non-controlled engine when the reactive force and the relative velocity of exhaust gazes are considered to be constant [14-16]. The engine just can be turned on or off. That is the most realistic model. The second of them matches the ideal limited power engine when the power of the engine is constant [17]. Under this restriction, we can vary the reactive force and the exhaust velocity [14]. In this work, we vary both the reactive force and the power of the rocket engine by varying the consumption of propellant and keeping the exhaust gases velocity. The optimal control problem is to find the trajectory corresponding to the minimal consumption of propellant.

A problem of insertion of a rocket into an orbit of the Earth at the height *h1* with the first orbital velocity *Y1* during the time *T1* supplying minimal propellant consumption is considered. A similar problem has been solved using the regularization method in [18-20]. In this work, the regularization method is applied to solve integral equations of the first kind determining components of the velocity and the acceleration. If there are the horizontal component of the velocity $\upsilon_x(\tau)$ and the vertical one $\upsilon_y(\tau)$ then the set of the equation of motion of a body with the varying mass $m(\tau)$ in atmosphere is [18] (Eq. 1) with the initial conditions (Eqs. 2-3). Where $\mu \le m(\tau) \le m$ is the variable mass of a rocket with propellant, *kg*; $\mu$ is the mass of construction of a rocket, *kg*; $\upsilon(\tau)$ is the velocity of a rocket; $w(\tau)$ is the control function equal to the consumption of propellant trough one second, *kg/s*; *a=const=2500 m/s* is the relative velocity of exhaust gases; $0 \le c[h(\tau)] \le 0.2\cdot10^{-7}$ *kg/m* is the generalized ballistic coefficient of air; $g$=9.81 m/s$^2$ is the free-fall acceleration.

$$\begin{Bmatrix} \dfrac{d\upsilon_x(\tau)}{d\tau} = \dfrac{1}{m(\tau)}\left[a_x w(\tau) - c[h(\tau)]\upsilon_x^2(\tau)\right], \\ \dfrac{d\upsilon_y(\tau)}{d\tau} = \dfrac{1}{m(\tau)}\left[a_y w(\tau) - c[h(\tau)]\upsilon_y^2(\tau)\right] - g, \\ \dfrac{dm}{d\tau} = - w(\tau) \end{Bmatrix} \tag{1}$$

$$\upsilon_x(0) = \Upsilon_{0x} \tag{2}$$

$$\upsilon_y(0) = \Upsilon_{0y} \tag{3}$$

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The optimal control function $w\tilde{}(\tau)$ must be positive at a time interval $0 \le \tau \le T_1$. Gradual decrease in the consumption of the mass of propellant begins at the time instant $\tau=0$ when the velocity is equal to $\Upsilon_0$. The optimal control function $w\tilde{}(\tau)$ and the time instant $T_1$ when burning of propellant is stopped are desired while (Eq. 4) is the velocity of a rocket equal to the first orbit velocity $\Upsilon_1$ at the height $h_1$ reached at the instant $T_1$ (Eq. 5). Where $G= 6.6743\cdot 10^{-11} m^3 s^{-2} kg^{-1}$ is the gravitational constant; $M=5.97\cdot 10^{24} kg$ is the mass of the Earth; $R_0= 6.371\cdot 10^6 m$ is the radius of the Earth [21-23].

$$\upsilon(T_1) = \Upsilon_1 \tag{4}$$

$$\Upsilon_1 = \sqrt{\frac{GM}{R_0 + h_1}} \tag{5}$$

If $T_1$ is the time instant in which the velocity becomes equal to the first orbit one then the velocity of the lifting of a rocket depends on the height of circle orbit $h_1$ according to the integral equation of the first kind (Eq. 6) with the boundary conditions (Eqs. 7-8), while the horizontal component of the acceleration depends on the velocity of a rocket at the time instant when propellant burning is stopped according to the (Eq. 9) with the boundary conditions (Eqs. 10-11), where (Eq. 12) is the horizontal component of the acceleration of a rocket, *m/s.*

$$\int_0^{T_1} \upsilon_y(\tau)d\tau = h_1 \tag{6}$$

$$\upsilon_y(0) = \Upsilon_{0y} = 0 \tag{7}$$

$$\upsilon_y(T_1) = \Upsilon_{1y} = 0 \tag{8}$$

$$\int_0^{T_1} \upsilon_x^{'}(\tau)d\tau = \Upsilon_{1x} = \Upsilon_1 \tag{9}$$

$$\upsilon_x^{'}(0) = \Upsilon_{0x'} = 0 \tag{10}$$

$$\upsilon_x^{'}(T_1) = \Upsilon_{1x'} = 0 \tag{11}$$

$$\upsilon_x^{'}(\tau) = dv_x(\tau)/d\tau \tag{12}$$

From the set of (Eq. 13) the differential equation for the varying mass $m(\tau)$ is gotten which is connected with the velocity $\upsilon(\tau)$ with the initial condition (Eq. 14). As far as (Eqs. 15-17) there is (Eq. 18).

$$\frac{dm(\tau)}{d\tau} + \frac{1}{a_x(\tau) + a_y(\tau)} \times \left\{\left[\frac{d\upsilon_x(\tau)}{d\tau} + \frac{d\upsilon_y(\tau)}{d\tau} + g\right] m(\tau) + c[h(\tau)](\upsilon_x^2(\tau) + \upsilon_y^2(\tau))\right\} = 0 \tag{13}$$

$$m(0) = m_0 \tag{14}$$

$$[\upsilon(\tau)]^2 = [\upsilon_x(\tau)]^2 + [\upsilon_y^2(\tau)]^2 \tag{15}$$

$$a_x = a\upsilon_x/\upsilon \tag{16}$$

$$a_y = a\upsilon_y/\upsilon \tag{17}$$

$$\frac{dm(\tau)}{d\tau} + \frac{\upsilon(\tau)}{a(\upsilon_x(\tau) + \upsilon_y(\tau))}\left\{\left(\frac{d\upsilon_x(\tau)}{d\tau} + \frac{d\upsilon_y(\tau)}{d\tau} + g\right) m(\tau) + c[h(\tau)]\upsilon^2(\tau)\right\} = 0 \tag{18}$$

The consumption of propellant is found from the same set of equations as (Eq. 19).

$$w(\tau) =- \frac{dm(\tau)}{d\tau} \tag{19}$$

The procedure of searching for the optimal consumption of propellant using solutions of the integral equations of the first kind is the next. The consequence of couples of the numbers $\{h^{(n)}_1, T^{(n)}_1\}$ is set, and to each the height $h^{(n)}_1$ there is the first space velocity $\Upsilon^{(n)}_1$. For each couple of the numbers using the regularization method, the integral equations of the first kind (Eq. 6) in the velocity $\upsilon_y(\tau)$ and (Eq. 9) in the acceleration $\upsilon'_x(\tau)$ are solved. The acceleration (Eq. 20) can be calculated and the velocity $\upsilon'_x(\tau)$ from the ordinary differential equation (21) with the initial condition (Eq. 22) can be found [24-26].

$$\upsilon_y^{'}(\tau) = d\upsilon_y(\tau)/d\tau \tag{20}$$

$$\frac{d\upsilon_x(\tau)}{d\tau} = \upsilon_x^{'}(\tau) \tag{21}$$

$$\upsilon_x(0) = \Upsilon_{0x} = 0 \tag{22}$$

Substituting the functions $\upsilon_x(\tau)$, $\upsilon'_x(\tau)$, $\upsilon_y(\tau)$, $\upsilon'_y(\tau)$ into the (Eq. 18) the mass $m(\tau)$ and the consumption $w(\tau)$ from the (Eq. 19) are found. Then from the sequence of couples of the numbers $\{h^{(n)}_1, T^{(n)}_1\}$ such a couple $\{h^{(m)}_1, T^{(m)}_1\}$ is found on which the propellant consumption (Eq. 19) reaches its minimum (Eq. 23):

$$w^{(m)}\left[h_1^{(m)}, T_1^{(m)}\right] = \inf_{\Upsilon_1^{(n)}, T_1^{(n)}} w^{(n)}\left[h_1^{(n)}, T_1^{(n)}\right] \tag{23}$$

As a result, the functions $w^{(m)}, h^{(m)}_1, T^{(m)}_1$ are gotten which are considered to be approximate regularized solution of the problem of optimal control.

## *DETERMINATION OF VERTICAL COMPONENT OF THE VELOCITY AND THE ACCELERATION*

For each couple of the numbers $\{h^{(n)}_1, T^{(n)}_1\}$ the right-hand side of the (Eq. 6) is put approximately, and $h^{(n)}_1 \in [h^{(0)}_1, h^{(N)}_1]$ where (Eq. 24). The integral equations (25-26) has the kernel $K(h_1,\tau)=1$ and the function (27):

$$N = m + r, r \ge 0 \tag{24}$$

$$A\upsilon_y \equiv \int_0^{T_1} K(h_1,\tau)\upsilon_y(\tau)d\tau = u_\delta(h_1) \tag{25}$$

$$h_1 \in [h_1^{(0)}, h_1^{(N)}] \tag{26}$$

$$u_\delta(h_1) = h_1 \tag{27}$$

The required approximate (regularized) solution of the (Eq. 25), $A\upsilon_y=u_\delta$, is the function $\upsilon_y(\tau)$ which is the solution of the integrodifferential equation (28) of Euler [18], where $K(h_1,\tau)=1$, (Eqs. 29-30):

$$\int_0^{T_1} \bar{K}(\tau,t)\upsilon_y(t)dt + \gamma\left\{q(\tau)\upsilon_y(\tau) - \frac{d}{d\tau}\left(p(\tau)\frac{d\upsilon_y}{d\tau}\right)\right\} = \int_{h_1^{(0)}}^{h_1^{(N)}} K(h_1,\tau)u_\delta(h_1)\, dh_1 \tag{28}$$

$$\bar{K}(\tau,t) = \int_{h_1^{(0)}}^{h_1^{(N)}} K(h_1,\tau)K(h_1,t)\, d\Upsilon_1 = \int_{h_1^{(0)}}^{h_1^{(N)}} dh_1 = = h_1^{(N)} - h_1^{(0)} \quad (29)$$

$$\int_{h_1^{(0)}}^{h_1^{(N)}} K(h_1,\tau)u_\delta(h_1)\, dh_1 = \int_{h_1^{(0)}}^{h_1^{(N)}} h_1\, dh_1 = \frac{1}{2}\left[\left(h_1^{(N)}\right)^2 - \left(h_1^{(0)}\right)^2\right] \quad (30)$$

And if $F_1$ is a set of the functions $\upsilon(\tau)$ continuous on the interval $[0, T_1]$ and having the first order derivatives $d\upsilon(\tau)/d\tau$ square integrable on $[0, T_1]$, then for the functions $\upsilon(\tau)\epsilon F_1$ the stabilizing functional is determined as [18] (Eq. 31) where $q(\tau)$, $p(\tau)$ are defined nonnegative functions such that for every $\tau\epsilon[0, T_1]$. There are (Eq. 32) and $p(\tau)\geq p_0 > 0$ where $p_0$ is a number. One of these functionals is chosen (Eq. 32).

$$\Omega[\upsilon] = \int_0^{T_1}\left\{q(\tau)\upsilon^2(\tau) + p(\tau)\left(\frac{d\upsilon}{d\tau}\right)^2\right\}d\tau \quad (31)$$

$$q^2(\tau)+p^2(\tau) \neq 0 \quad (32)$$

Minimizing the functional (Eq. 31) is a conditional extremum problem. It is solved by the method of undetermined Lagrange multipliers; the function $\upsilon_y(\tau)$ is found minimizing the smoothing functional (Eq. 33) where [18] (Eq. 34).

$$M^\gamma[\upsilon,u_\delta] = \rho_{L_2}^2(A\upsilon,u_\delta) + \gamma\Omega[z] \quad (33)$$

$$\rho_{L_2}(u_1,u_2) = \left\{\int_c^d [u_1(x) - u_2(x)]^2 dx\right\}^{\frac{1}{2}} \quad (34)$$

This is an unconditional extremum problem, in which the regularization parameter is determined from the (Eq. 35) with the solution (Eq. 36) depending on the discrepancy $\delta$.

$$\rho_{L_2}(A\upsilon,u_\delta) = \delta \quad (35)$$

$$\gamma = \gamma(\delta) \quad (36)$$

The parameter $\gamma$ may be determined both by the discrepandy (Eq. 35) and other ways [18, 27]. Consequently:

$$\left(h_1^{(N)} - h_1^{(0)}\right)\int_0^{T_1}\upsilon_y(t)dt + \gamma\left\{q(\tau)\upsilon_y(\tau) - \frac{d}{d\tau}\left(p(\tau)\frac{d\upsilon_y(\tau)}{d\tau}\right)\right\} = = \frac{1}{2}\left[\left(h_1^{(N)}\right)^2 - \left(h_1^{(0)}\right)^2\right] \quad (37)$$

This equation is solved with one of the boundary conditions following from the equality to zero of the solution or its first derivative on the bounds of the interval $[0, T_1]$ [18] (Eqs. 38-41):

$$\upsilon(0) = 0, \upsilon(T_1) = 0 \quad (38)$$

$$\upsilon(0) = 0, \upsilon'(T_1) = 0 \quad (39)$$

$$\upsilon'(0) = 0, \upsilon(T_1) = 0 \quad (40)$$

$$\upsilon'(0) = 0, \upsilon'(T_1) = 0 \quad (41)$$

As (Eqs. 7-8) there is a need not to pass on to a new function $\tilde{\upsilon}(\tau)$ satisfying to the boundary conditions $\tilde{\upsilon}(0)=0$, $\tilde{\upsilon}(T_1)=0$ [18]. Now (Eqs. 42-43) are put. Then (Eq. 44):

$$q(\tau) = q = const > 0 \quad (42)$$

$$p(\tau) = p = const > 0 \quad (43)$$

$$\left(h_1^{(N)} - h_1^{(0)}\right)\int_0^{T_1}\upsilon_y(t)dt + \gamma q\upsilon_y(\tau) - \gamma p\frac{d^2\upsilon_y(\tau)}{d\tau^2} = = \frac{1}{2}\left[\left(h_1^{(N)}\right)^2 - \left(h_1^{(0)}\right)^2\right] \quad (44)$$

A difference analogue of the (Eq. 44) is written down on a uniform net with the time increment $\Delta\tau$. The interval $[0, T_1]$ is divided into $M$ equal parts and set the ends of got intervals as nodes of the net (Eqs. 45-46):

$$\tau_i = i\Delta\tau, i = 1,2\ldots,M \quad (45)$$

$$\Delta\tau = \frac{T_1}{M} \quad (46)$$

Replacing the integral in the left-hand side of the (Eq. 44) by the integral sum corresponding to it according to the formula of rectangles, for example, and $\upsilon''_y(\tau)$ by corresponding difference expression, there is [18] (Eqs. 47-48):

$$\left(h_1^{(N)} - h_1^{(0)}\right)\Delta\tau\sum_{j=1}^{M}\left(\upsilon_y\right)_j + \gamma q\cdot\left(\upsilon_y\right)_i + + \gamma p\frac{2\left(\upsilon_y\right)_i - \left(\upsilon_y\right)_{i-1} - \left(\upsilon_y\right)_{i+1}}{\Delta\tau^2} = f_i \quad (47)$$

$$f_i = \frac{1}{2}\left[\left(h_1^{(N)}\right)^2 - \left(h_1^{(0)}\right)^2\right], i = 1,2,\ldots,M \quad (48)$$

The values of the right-hand side $f_i$ are calculated analytically. At the same time, the numbers $N$, $M$ of the net points on the coordinates $h_1$, $\tau$ are independent. If $i=1$, $i=M$ then there undefined values $(\upsilon_y)_0$ and $(\upsilon_y)_{M+1}$ are in the set of linear algebraic equations (48) for the vector (49). To satisfy the boundary conditions (50), (51) are put.

$$\upsilon_y = \left((\upsilon_y)_1, (\upsilon_y)_2,\ldots,(\upsilon_y)_M\right) \quad (49)$$

$$(\upsilon_y)_0 = \upsilon_y(0) = \Upsilon_{0y} = 0 \quad (50)$$

$$(\upsilon_y)_{M+1} = (\upsilon_y)_M = \upsilon_y(T_1) = \Upsilon_{1y} = 0 \quad (51)$$

Thus, the problem of searching for approximate (regularized) solution of the equation (26), (52), leads to solving the set of linear algebraic equations for the vector (53).

$$A\upsilon_y = u_\delta \quad (52)$$

$$\upsilon_y = \left((\upsilon_y)_1, (\upsilon_y)_2,\ldots,(\upsilon_y)_M\right) \quad (53)$$

### DETERMINATION OF HORIZONTAL COMPONENT OF THE VELOCITY AND THE ACCELERATION

For each couple $\{Y^{(n)}_1, T^{(n)}_1\}$ the right-hand side of the (Eq. 9) is known approximately, and $Y^{(n)}_1 \in [Y^{(0)}_1, Y^{(N)}_1]$ where (Eq. 24) [28, 29]. The integral equation (54) has the kernel $K(Y_1,\tau)=1$ and the function (55).

$$A\upsilon_x' \equiv \int_0^{T_1} K(\Upsilon_1,\tau)\upsilon_x'(\tau)d\tau = u_\delta(\Upsilon_1), \Upsilon_1 \in \left[\Upsilon_1^{(0)},\Upsilon_1^{(N)}\right] \quad (54)$$

$$u_\delta(\Upsilon_1) = \Upsilon_1 \quad (55)$$

The required approximate (regularized) solution of the (Eq. 54), $A\upsilon'_x=u_\delta$, is the function $\upsilon'_x(\tau)$ which is the solution of the integro-differential equation (56) of Euler [18], where $K(Y_1,\tau)=1$, (Eqs. 57-58). Consequently, (Eq. 59).

$$\int_0^{T_1} \bar{K}(\tau,t)\upsilon_x'(t)dt + \gamma\left\{q(\tau)\upsilon_x'(\tau) - \frac{d}{d\tau}\left(p(\tau)\frac{d\upsilon_x'}{d\tau}\right)\right\} = \int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} K(\Upsilon_1,\tau)u_\delta(\Upsilon_1)\,d\Upsilon_1, \quad (56)$$

$$\bar{K}(\tau,t) = \int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} K(\Upsilon_1,\tau)K(\Upsilon_1,t)\,d\Upsilon_1 = \int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} d\Upsilon_1 = \Upsilon_1^{(N)} - \Upsilon_1^{(0)} \quad (57)$$

$$\int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} K(\Upsilon_1,\tau)u_\delta(\Upsilon_1)\,d\Upsilon_1 = \int_{\Upsilon_1^{(0)}}^{\Upsilon_1^{(N)}} \Upsilon_1\,d\Upsilon_1 = \frac{1}{2}\left[\left(\Upsilon_1^{(N)}\right)^2 - \left(\Upsilon_1^{(0)}\right)^2\right] \quad (58)$$

$$\left(\Upsilon_1^{(N)} - \Upsilon_1^{(0)}\right)\int_0^{T_1}\upsilon_x'(t)dt + \gamma\left\{q(\tau)\upsilon_x'(\tau) - \frac{d}{d\tau}\left(p(\tau)\frac{d\upsilon_x'(\tau)}{d\tau}\right)\right\} = \frac{1}{2}\left(\Upsilon_1^{(N)} - \Upsilon_1^{(0)}\right), \quad (59)$$

As (Eqs. 10-11) there is a need not to pass on to a new function $\tilde{\upsilon}(\tau)$ satistying to the boundary conditions $\tilde{\upsilon}(0)=0$, $\tilde{\upsilon}(T_1)=0$ [18]. (Eq. 38-39) are put. Then:

$$\left(\Upsilon_1^{(N)} - \Upsilon_1^{(0)}\right)\int_0^{T_1}\upsilon_x'(t)dt + \gamma q\upsilon_x'(\tau) - \gamma p\frac{d^2\upsilon_x'(\tau)}{d\tau^2} = \frac{1}{2}\left[\left(\Upsilon_1^{(N)}\right)^2 - \left(\Upsilon_1^{(0)}\right)^2\right], \quad (60)$$

A difference analog of the (Eq. 60) is written down on a uniform grid with step $\Delta\tau$. The interval [0, $T_1$] is divided into $M$ equal parts and sets the ends of got intervals as nodes of the net (Eqs. 45-46). Replacing the integral in the left-hand side of the (Eq. 60) by the integral sum corresponding to it according to the formula of rectangles, for example, and $[\upsilon'_x(\tau)]''$ by corresponding difference expression, there is [18]:

$$\left(\Upsilon_1^{(N)} - \Upsilon_1^{(0)}\right)\Delta\tau\sum_{j=0}^{M}\left(\upsilon_x'\right)_j + \gamma q\cdot\left(\upsilon_x'\right)_i + \gamma p\frac{2\left(\upsilon_x'\right)_i - \left(\upsilon_x'\right)_{i-1} - \left(\upsilon_x'\right)_{i+1}}{\Delta\tau^2} = f_i \quad (61)$$

$$f_i = \frac{1}{2}\left[\left(\Upsilon_1^{(N)}\right)^2 - \left(\Upsilon_1^{(0)}\right)^2\right], i = 1,2,\ldots,M \quad (62)$$

The values of the right-hand side $f_i$ are calculated analytically. At the same time, the numbers $N$, $M$ of the net points on the coordinates $Y_1$, $\tau$ are independent. If $i=1$, $i=M$ then there undefined values $(\upsilon'_x)_0$ and $(\upsilon'_x)_{M+1}$ are in the set of linear algebraic equation (62) for the vector $\upsilon'_x=((\upsilon'_x)_1, (\upsilon'_x)_2,\ldots,(\upsilon'_x)_M)$. To satisfy the boundary conditions (Eqs. 63-64) are put.

$$\left(\upsilon_x'\right)_0 = \upsilon_x'(0) = \Upsilon_{0x}' = 0 \quad (63)$$

$$\left(\upsilon_x'\right)_{M+1} = \left(\upsilon_x'\right)_M = \upsilon_x'(T_1) = \Upsilon_{1x}' = 0 \quad (64)$$

Thus, the problem of searching for approximate (regularized) solution of the (Eq. 26), $A\upsilon'_x=u_\delta$, leads to solving the set of linear algebraic equations for the vector (65). Then the vector (66) is found solving the ordinary differential equation (21).

$$\upsilon_x' = \left(\left(\upsilon_x'\right)_1,\left(\upsilon_x'\right)_2,\ldots,\left(\upsilon_x'\right)_M\right) \quad (65)$$

$$\upsilon_x = \left((\upsilon_x)_1,(\upsilon_x)_2,\ldots,(\upsilon_x)_M\right) \quad (66)$$

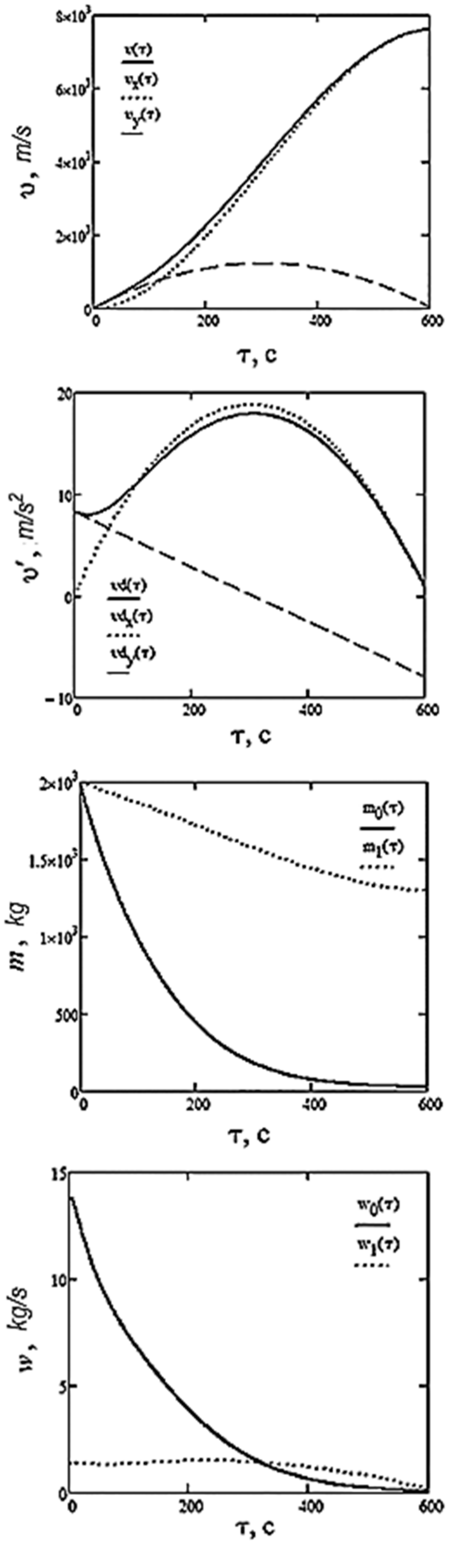
### RESULTS OF CALCULATION

The problem of injection into a circle orbit at the height $h_1$=500 *km* during the time $T_1$=600 *s* of a one-stage rocket with the total mass of its construction and payload $\mu$=1000 *kg*, and the mass of propellant $\Delta m$=1000 *kg* is considered. Consequently, the start mass of a rocket is equal to $m_0$=2000 *kg*. The velocity $\upsilon_y(\tau)$ is an approximate solution of the integral equation (6) which is found from the Euler equation (37) transformed into the set of linear algebraic equations (48). If the regularization parameter $\gamma=const$ then the distribution of the velocity corresponds to the right-hand side $u_\delta(h_1)=h_1$ of the integral equation (26) within an accuracy of the solution of the set of algebraic equations (48). A solution $(\upsilon_y)_i$ $(i=0,1,\ldots,M)$ has the homogeneous boundary conditions $(\upsilon_y)_0=\upsilon_y(0)=0$; $(\upsilon_y)_M=\upsilon_y(T_1)=0$ with the values $q(\tau)=q=10^{-6} m\cdot s$; $p(\tau)=p=1 m\cdot s^3$ (Fig. 1a). The acceleration $\upsilon'_y(\tau)$ is found by differentiating $\upsilon_y(\tau)$ numerically (Fig. 1b):

$$\left(\upsilon_y'\right)_i = \frac{\left(\upsilon_y\right)_i - \left(\upsilon_y\right)_{i-1}}{\tau_i - \tau_{i-1}}, i = 1,2,\ldots,M \quad (67)$$

$$\left(\upsilon_y'\right)_0 = \left(\upsilon_y'\right)_1 \quad (68)$$

$$\left(\upsilon_y'\right)_{M+1} = \left(\upsilon_y'\right)_M \quad (69)$$




(a)

(b)

(c)

(d)

*Figure 1: Distribution in the time of the properties of a chemical rocket ($a=2.5 \cdot 10^3 m/s$) and a plasmic rocket ($a=2.5 \cdot 10^4 m/s$) with the start mass $m_0$=200 kg injected into a circle orbit of the Earth at the height $h_1$=500 km and time $T_1$=600 s: the velocity υ, m/s, (a); the acceleration υ', $m/s^2$, (b); the mass of a rocket $m(m_0(\tau))$ is the mass of a chemical rocket, $m_1(\tau)$ is the mass of a plasmic rocket), kg, (c); the propellant consumption $w(w_0(\tau))$ is the consumption of a chemical rocket, $w(\tau)$ is the consumption of a plasmic rocket), kg/s, (d)*

Then the velocity $(\upsilon_y)_{M+1}$ necessary to solve numerically the ordinary differential equation (18) in the mass $m_i(i= 0,1,\ldots,M)$ is:

$$\left(\upsilon_y\right)_{M+1} = \left(\upsilon_y\right)_M + \left(\upsilon_y^{'}\right)_M (\tau_{M+1} - \tau_M) \qquad (70)$$

The acceleration $\upsilon'_x(\tau)$ is an approximate solution of the integral equation (9) which is found from the Euler equation (59) transformed into the set of linear algebraic equations (62). If the regularization parameter *γ=const* then the distribution of the velocity corresponds to the right-hand side $u_\delta(Y_1)=(Y_1)$ of the integral equation (53) within an accuracy of the solution of the set of algebraic equations (62). A solution $(\upsilon'_x)_i$ $(i=0,1,\ldots,M)$ has the homogeneous boundary conditions $(\upsilon_y)_0=\upsilon_y(0)=0$, $(\upsilon_y)_M=\upsilon_y(T_1)=0$ with the values $q(\tau)=q=10^{-6}m$; $p(\tau)=p=1\ m{\cdot}s^2$ (Fig. 1b). The velocity $\upsilon_x(\tau)$ is found from the acceleration $\upsilon'_x(\tau)$ by solving numerically the ordinary differential equation (21) on the time net $(\upsilon_x)_i$ $(i=0,1,\ldots,M)$ (Fig. 1a). Then the velocity $(\upsilon_x)_{M+1}$ necessary to solve numerically the ordinary differential equation (18) in the mass $m_i(i=0,1,\ldots,M)$ is:

$$(\upsilon_x)_{M+1} = (\upsilon_x)_M + \left(\upsilon_x^{'}\right)_M (\tau_{M+1} - \tau_M) \qquad (71)$$

A one-stage chemical rocket with the velocity of exhaust gazes $a=2.5 \cdot 10^3 m/s$ is able to inject into a circle orbit just its own propellant with minimal mass of construction (Fig. 1c). Therefore, one uses multi-stage chemical rockets. To analyze a one-stage rocket engine demonstrative enough another kind of a rocket engine promising at the present is considered. There are projects of plasmic rocket engines with the velocity of exhaust gazes $a=2.5 \cdot 10^4 m/s$ reducing by 10 times the consumption of propellant *w* and keeping the reactive force *aw*. The consumption of propellant of one-stage plasmic engine injecting into a circle orbit at the height $h_1$ during the time $T_1$ a rocket with the start mass is analyzed (72) (Fig. 1d).

$$m_0 = \mu + \Delta m \qquad (72)$$

The Euler equation for the integral equation (6) in the velocity $\upsilon_y(\tau)$ corresponds to the right-hand part $h^{(n)}_1 \epsilon [h^{(0)}_1, h^{(N)}_1]$ where $h^{(0)}_1=450{\cdot}10^3 m$ $(Y_1=7.643{\cdot}10^3 m/s)$, $h^{(N)}_1=550{\cdot}10^3 m$ $(Y_1 =7.588{\cdot}10^3 m/s)$. The first orbital velocity $Y_1$ is the right-hand part of the equation (9) in the acceleration $\upsilon'_x(\tau)$. A consequence of the heights $h^{(n)}_1 \epsilon [h^{(0)}_1, h^{(N)}_1]$ and a consequence of the times of injection are set to find a couple $\{h^{(m)}_1, T^{(m)}_1\}$ supplying a minimal consumption of propellant (Eq. 19). The least propellant consumption 669.4 *kg* is for the couple $\{h_1$= 550 km, $T_1$= 540 s$\}$ and the start mass of a rocket 2000 *kg* (Table 1) as the first orbital velocity decreases when the height increases according to (5).

To solve a problem of keeping predetermined distribution of the velocity $\upsilon_x(\tau)$, $\upsilon_y(\tau)$, and consequently the acceleration $\upsilon'_x(\tau)$, $\upsilon'_y(\tau)$, corresponding to the trajectory $x(\tau)$, $y(\tau)$, the regularization parameter $\gamma_x(\tau)$, from the (Eq. 62) and $\gamma_y(\tau)$, from the (Eq. 48) has to be found. The regularization parameter is able to be found analytically by the method of simple iteration or the iteration-variation

*Table 1: The consumption of propellant of a plasmic rocket ($a=2.5 \cdot 10^4 m/s$) equal to the difference of the masses $m(0)-m(T_1)$ for couples of the numbers $\{h_1, T_1\}$ (the mass of an empty rocket $\mu=10^3 kg$, the mass of propellant $\Delta m=10^3 kg$)*

| $h_1$, km, $T_1$, s | 540 | 570 | 600 | 630 | 660 |
|---|---|---|---|---|---|
| 450 | 680.3 | 695.5 | 710.6 | 725.6 | 740.3 |
| 475 | 677.2 | 692.4 | 707.4 | 722.2 | 737.0 |
| 500 | 674.4 | 689.4 | 704.3 | 719.1 | 733.8 |
| 525 | 671.8 | 686.7 | 701.5 | 716.2 | 730.8 |
| 550 | 669.4 | 684.2 | 698.8 | 713.4 | 727.9 |

method [30, 31]. In that case the right-hand side of the integral equation (6) in the form of the height of the orbit and the integral equation (9) in the form of the first orbital velocity $Y_1$ will deviate from the predetermined one. Similar problems may be formulated and solved to transport a rocket with a payload into desired point. Polar of spherical coordinate system can be also used.

## CONCLUSIONS

The problem of insertion of a rocket into the desired orbit in the view of minimal consumption of propellant leads to solving the set of two ordinary differential equations in the components of the velocity (when a movement is in the plane $x_y$ and two integral equations. Summarizing the differential equations, the ordinary differential equation in the mass of a rocket from the time connecting it with the free-fall acceleration, the ballistic coefficient of atmosphere depending on the height, the components of the velocity of exhaust gases, and a rocket are gotten. The integral equations follow from the laws of mechanics: $\upsilon_y(\tau)=dh(\tau)/dh,=>\upsilon_y(\tau)d\tau=dh(\tau)$, and $\upsilon'_x(\tau)=d\upsilon_x(\tau)/d\tau,=>\upsilon'_x(\tau)d\tau=dv_x(\tau)$. The integral equations are solved using the regularization method and an Euler equation on a time net as the set of linear algebraic equations in the velocity $\upsilon_y(\tau)$ or the acceleration $\upsilon'_x(\tau)$.

In the right-hand side of the integral equation in the vertical component of the velocity there is the height of an orbit. In the right-hand side of the integral equation in the horizontal component of the acceleration is the first orbital velocity depending on the height of an orbit. Searching for the least consumption of propellant a sequence of couples of the numbers are set: the first one is the time of propellant burning, the second one is the height of an orbit with the first orbital velocity. The numbers must belong to admissible intervals of the flight time and the orbit height. For each of the couples corresponding integral equations are solved. From these equations the vertical component of the velocity determining the vertical acceleration, and the horizontal component of the acceleration determining the horizontal velocity dependent on the time are found.

The Euler equation for the integral equation in the vertical component of the velocity includes the regularization parameter $\gamma=\gamma(\tau)>0$ and the functions $q=q(\tau)>0$ $m{\cdot}s$; $p=p(\tau)>0$ $m{\cdot}s^3$. Keeping $\gamma=const$ and changing the functions $q,p$ the velocity distribution in the time to the homogeneous boundary conditions is brought to supply the desired height of an orbit within an accuracy of solution of the Euler equation. The Euler equation for the integral equation in the horizontal component of the acceleration includes the regularization parameter $\gamma=\gamma(\tau)>0$ and the functions $q=q(\tau)>0$ $m$; $p=p(\tau)>0$ $m{\cdot}s^2$. Keeping $\gamma=const$ and changing the functions $q,p$ the acceleration distribution in the time to the homogeneous boundary conditions is brought to supply the first orbital velocity corresponding to the desired height within an accuracy of solution of the Euler equation.

The problem of insertion of a multy-stage rocket into desired orbit in the view of minimal consumption of propellant is analogous to the problem for a one-stage rocket. But a one-stage rocket injects just itself without any payload. Therefore, working out a plasmic rocket engine with the velocity of exhaust gases more tenfold than chemical one has is promising. Problems of suborbital and interplanetary flights can be solved using the procedure in the spherical or polar coordinate system. Today there are used low power ion-plasma rocket engines for suborbital flights. Manned flights are reasonable on the basis of high power plasmic rocket engines with the reactive force comparable to chemical ones. To search for a solution of the integral equations closed to known distributions of the velocity and acceleration in the time there is a need to find the regularization parameter in the time according to those functions. In that case the right-hand sides of the integral equations deviate from desired values.

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