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AN APPROXIMATION TO THE INVERSE OF LEFT-SIDED TRUNCATED GAUSSIAN CUMULATIVE NORMAL DENSITY FUNCTION USING POLYA'S MODEL TO GENERATE RANDOM VARIATES FOR SIMULATION APPLICATIONS

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Key words: *gaussian distribution, normal distribution, random variate generation, cumulative density function, mathematical approximation, truncated normal distribution*
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AN APPROXIMATION TO THE INVERSE OF LEFT-SIDED TRUNCATED GAUSSIAN CUMULATIVE NORMAL DENSITY FUNCTION USING POLYA'S MODEL TO GENERATE RANDOM VARIATES FOR SIMULATION APPLICATIONS

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The Gaussian or normal distribution is vital in most areas of industrial engineering, including simulation. For example, the inverse of the Gaussian cumulative density function is used in all simulation software (e.g., ARENA, ProModel) to generate a group of random numbers that fit Gaussian distribution. It is also used to estimate the life expectancy of new devices. However, the Gaussian distribution that is truncated from the left side is not defined in any simulation software. Estimation of the expected life of used devices needs left-sided truncated Gaussian distribution. Additionally, very few works examine generating random numbers from left-sided truncated Gaussian distribution. A high accuracy mathematical-based approximation to the left-sided truncated Gaussian cumulative density function is proposed in the current work. Our approximation is built based on Polya's approximation of the Gaussian cumulative density function. The current model is beneficial to approximate the inverse of the left-sided truncated Gaussian cumulative density function to generate random variates, which is necessary for simulation applications.

Key words: gaussian distribution, normal distribution, random variate generation, cumulative density function, mathematical approximation, truncated normal distribution

INTRODUCTION

Random variate generation is an important method in simulation and statistical computing. Statisticians developed different methods, tools, models, approximations, and algorithms to generate random variates over many decades. The difficulty of variate generation depends on the complexity of the cumulative probability mass function (CDF). The simplest method is inverse transform sampling (ITS), a method for pseudo-random number sampling which randomly generates variates using the CDF inverse. In the literature, ITS appears in different names, such as inversion sampling (e.g., [1-2]), inverse probability integral transform (e.g., [3-4]), inverse transformation method (e.g., [5-6]), Smirnov transform (e.g.,[7-8]) and golden rule (e.g., [9-10]). ITS method can be performed in two steps: 1. By generating random numbers based on the uniform distribution over the domain [0:1] (i.e., N~U[0:1]). 2. And by substituting each generated random number into the inverse of the CDF. The generated variates should fit the probability mass function (PDF) of a particular distribution. Theoretically, this method can be used with any distribution, as long as the curve of CDF can be sketched. In addition, with using ITS, random variates can be generated easily from discrete distributions. However, we need to compute CDF by integrating the PDF for continuous distributions, which is mathematically impossible for many distributions (e.g., Gaussian distribution). Therefore, engineers and statisticians developed other methods. As a possible solution to the irreversible CDF, the inverse may be approximated mathematically. In this paper, ITS will be used to generate variates from a left-sided truncated Gaussian distribution by approximate the inverse mathematically. Besides the ITS method, other methods are introduced in the literature, such as the imperial method [11-13] and acceptance-rejection methods [14-19]. Romano [20] developed an algorithm to generate random variates from the Madland–Nix fission energy spectrum, assuming a constant compound nucleus cross-section is given based on physics considerations. Then, he wrote a program to generate variates using the developed algorithm. Favaro et al. [21] presented how to generate random variates for a more general class of tilted α-stable distributions, referred to as the class of Laguerre-type exponentially tilted α-stable distributions. Recalling the ITS method, if a mathematical approximation function of a CDF inverse is available, random variates can be generated from that approximation. The fitness of the generated random variates to the original distribution depends on the approximation accuracy [24]. For example, Polya developed a model to approximate the CDF of Gaussian distribution. As a result, the ITS method can generate Gaussian distribution-based random variates by applying on the inverse of Polya's model. This will be explained in detail in the following sections. In this paper,

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an approximation to left-sided truncated Gaussian distribution is derived from Polya's normal distribution approximation. The derived approximation will be inversed and then used to generate random variates. In general, CDF of truncated Gaussian distribution is difficult to estimate, so engineers usually used either an approximation program or special software to do this job.

POLYA'S APPROXIMATION

George Polya (1887-1985) developed Polya's approximation, the Hungarian-American mathematician and father of problem-solving in mathematics education. Polya's approximation to the Gaussian distribution CDF is represented by Equation 1 [22]. After examining the approximation equation, some findings and notes are provided in Equation 2-8 and Figure 1.

$$\Phi(z) \cong \frac{1}{2}\left[1+\left\{1-exp\left(\frac{-2(z)^2}{\pi}\right)\right\}^{\frac{1}{2}}\right], z\epsilon[0:\infty] \quad (1)$$

Although Polya's approximation is one of the first introduced models, it is one the best. Its simplicity and the possibility to be inversed are big advantages, besides the high accuracy. In Equation 2, Polya's approximation is re-written on the domain of [-∞:∞] using the fact, Φ(-z) = 1 - Φ(z).

$$\Phi(z) \cong \begin{cases} 1-\frac{1}{2}\left[1+\left\{1-exp\left(\frac{-2(z)^2}{\pi}\right)\right\}^{\frac{1}{2}}\right], z<0 \\ \frac{1}{2}\left[1+\left\{1-exp\left(\frac{-2(z)^2}{\pi}\right)\right\}^{\frac{1}{2}}\right], z\geq 0 \end{cases} \quad (2)$$

A deviation function, D(z) is defined to assess the Polya's approximation accuracy, as addressed in Equation 3:

$$D(z) = \frac{1}{2}\left[1+\left\{1-exp\left(\frac{-2(z)^2}{\pi}\right)\right\}^{\frac{1}{2}}\right] - \Phi(z), z\epsilon[0:\infty] \quad (3)$$

The property mentioned above, Φ(-z) = 1 - Φ(z), leads to estimate D(z) in the negative domain of Z score, as in Equation 4.

$$\boldsymbol{D}(-\boldsymbol{z}) = -\boldsymbol{D}(\boldsymbol{z}) \quad (4)$$

Figure 1 shows the deviation of Polya's approximation from real values. The maximum deviation is 0.00314583 over the region of [0:∞], and this maximum value is located at Z=1.654. According to Equation 4, the deviation curve must reach the same but negative value (i.e., -0.00314583) at Z=-1.654. Thus, Polya's approximation maximum deviation is 0.00314583 for the entire domain, [-∞:∞]. Figure 1 and Equation 1 also show a comparison between Polya's approximation and Cadwell's approximation. Polya's approximation is more accurate than Cadwell's approximation of maximum deviation from the real over the domain [0:∞]. The same matter is concluded over the domain [-∞:0], by applying Equation 4. Further, by comparing the equations, Polya's approximation is much simpler than Cawell's approximation. The selection of Cadwell's approximation to be compared with Polya's is due to three reasons: 1. Both models share most equation terms and the only difference between them is the exponent, 2. Popularity of Cadwell's approximation and 3. Cadwell's approximation introduced after Polya's of few years by the well-known worldwide mathematician, J. H. Cadwell.

$$\Phi(z) \cong \begin{cases} 1-\frac{1}{2}\left[1+\left\{1-\exp\left(\frac{-2Z^2}{\pi}-\frac{2(\pi-3)Z^4}{3\pi^2}\right)\right\}^{\frac{1}{2}}\right] & Z<0 \\ \frac{1}{2}\left[1+\left\{1-\exp\left(\frac{-2Z^2}{\pi}-\frac{2(\pi-3)Z^4}{3\pi^2}\right)\right\}^{\frac{1}{2}}\right] & Z\geq 0 \end{cases} \quad (5)$$

Cadwell's Approximation

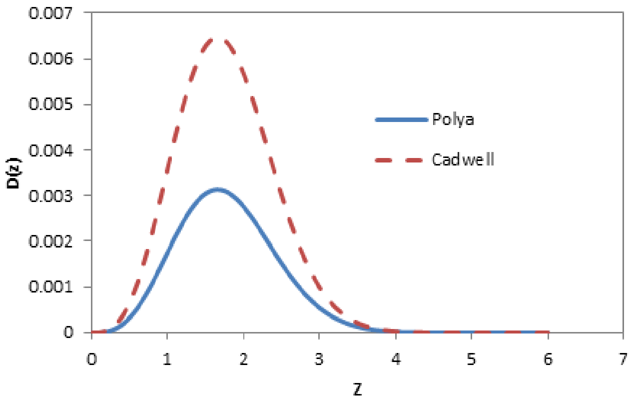

Figure 1: Change of deviation, D(z) vs Z score for both Polya's and Cadwell's approximations

Polya's approximation can be differentiated to estimate PDF as shown in Equation 6. Equation 7 is a simplification of Equation 6. This equation is used to build the left-sided truncation Gaussian distribution.

$$\emptyset(z) \cong \begin{cases} \frac{-z\times\exp\left(\frac{-2z^2}{\pi}\right)}{\pi\left[1-\exp\left(\frac{-2z^2}{\pi}\right)\right]}, z<0 \\ \frac{z\times\exp\left(\frac{-2z^2}{\pi}\right)}{\pi\left[1-\exp\left(\frac{-2z^2}{\pi}\right)\right]}, z\geq 0 \end{cases} \quad (6)$$

$$\emptyset(z) \cong \frac{-z^2\times\exp\left(\frac{-2z^2}{\pi}\right)}{|z|\pi\left[1-\exp\left(\frac{-2z^2}{\pi}\right)\right]} \quad (7)$$

One of the most useful properties of Polya's approximation is the ability to inverse the CDF equation. If U(R) = F^{-1} (Φ(z)) and R = Φ(z), then U(R) can be presented as in the following equation (Equation 8).

$$U(R)=\begin{cases}-\left[\frac{-\pi}{2}\ln(1-[1-2R]^2)\right]^{\frac{1}{2}}, R\epsilon[0:0.5)\\ \left[\frac{-\pi}{2}\ln(1-[2R-1]^2)\right]^{\frac{1}{2}}, R\epsilon[0.5:1]\end{cases} \quad (8)$$

APPROXIMATION OF LEFT-SIDED TRUNCATED GAUSSIAN DISTRIBUTION CDF.

In this section, an approximation to CDF of left-sided truncated Gaussian distribution is proposed. Truncated Gaussian distribution has been found in almost all industries. Once a practitioner truncates a Gaussian distributed population from left, right or both sides for any reason, the truncated population fits truncated Gaussian distribution. There are many situations where the population is truncated, such as truncation of less than acceptable level of a test score, weight, height, and specimen strength. Truncation is primarily limited to estimating used devices' reliability and truncation of unfit products (i.e., exceed the upper specification level or beneath the lower specification level). The truncated Gaussian distribution due to scrapping/reworking unfit products is well-detailed in the literature, such as [23-28]. Despite the plenty of industrial problems on truncated Gaussian distribution, the theoretical research, such as estimating properties/criteria or approximating PDF and CDF is still immature. The reason behind that could be the mathematical complexity of the distribution. However, applications on truncated Gaussian distribution have been found in the literature intensively. Figure 2 shows left-sided truncated Gaussian distributions at truncation points of ZL=-1 and ZL=0 concerning the untruncated Gaussian distribution. Since the area under the PDF curve refers to the summation of all probabilities and must equal 1, the truncated area should be added to the untruncated area to keep its area equal to 1. This makes the PDF curve scaled up.

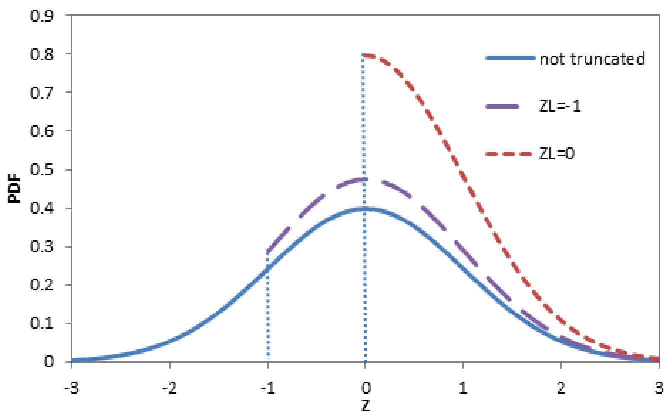

Figure 2: A comparison between the left-sided truncated standard Gaussian distribution at two different truncation points and original distribution

The PDF of the general left-sided truncated distribution ((i.e., $f_T(x,x_L)$)) is shown in Equation 9. According to the equation, the value of $f_T(x,x_L)$ is normalized original PDF value (i.e., f(x) in Equation 9). To normalize the truncated curve, we divide the original curve values by the area under the original PDF curve over the newly defined domain of [xL:∞]. Equation 10 represents the CDF of the general left-sided truncated distribution.

$$f_T(x,x_L)=\frac{f(x)}{\int_{x_L}^{\infty}f(x)\,dx}dx \quad (9)$$

$$F_T(x,x_L)=\int_{x_L}^{x}\frac{f(x)}{\int_{x_L}^{\infty}f(x)dx}dx \quad (10)$$

The CDF of the left-sided truncated standard Gaussian distribution is addressed in Equation 11.

$$\Phi_T(z,z_L)=\int_{z_L}^{z}\frac{\emptyset(z)}{\int_{z_L}^{\infty}\emptyset(z)dz}dz \quad (11)$$

We can approximate the left-sided truncated standard Gaussian distribution, $\Phi_T(z,z_L)$ by substituting ø(z) approximation (i.e., Equation 7) in Equation 11, as addressed in Equation 12

$$\Phi_T(z,z_L)\cong\int_{z_L}^{z}\frac{\dfrac{-z^2\times\exp\left(\frac{-2z^2}{\pi}\right)}{|z|\pi\left[1-\exp\left(\frac{-2z^2}{\pi}\right)\right]}}{\int_{z_L}^{\infty}\dfrac{-z^2\times\exp\left(\frac{-2z^2}{\pi}\right)}{|z|\pi\left[1-\exp\left(\frac{-2z^2}{\pi}\right)\right]}dz}dz \quad (12)$$

In this paper, the truncation area is assumed to be equal or less than half (i.e., zL<0). The solution of Equation 12 is led to Equation 13. If Z=0, $\Phi_T(z,z_L)$ can be solved using the upper or lower part of the equation, leading to the exact estimate.

$$\Phi_T(z,z_L)\cong\begin{cases}\dfrac{\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{1/2}-\left\{1-exp\left(\frac{-2z^2}{\pi}\right)\right\}^{1/2}}{1+\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{1/2}}, z_L\le z\le 0\\ \dfrac{\left\{1-exp\left(\frac{-2z^2}{\pi}\right)\right\}^{1/2}+\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{1/2}}{1+\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{1/2}}, z_L\le 0\le z\end{cases} \quad (13)$$

APPROXIMATION OF THE INVERSE OF THE LEFT-SIDED TRUNCATED GAUSSIAN DISTRIBUTION.

In this section, an approximation to the inverse of the left-sided truncated standard Gaussian distribution is proposed. Since Equations 13 has only one term of z, the equation can be inversed. If $U(R) = F^{-1}(\Phi(z,z_L))$and $R=\Phi(z,z_L)$, then U(R) can be written as addressed in Equation 14.

$$U(R) \cong \begin{cases} -\sqrt{\frac{-\pi}{2} Ln\left\{1-\left(\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{\frac{1}{2}} - R\left[1+\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{\frac{1}{2}}\right]\right)^2\right\}}, & z_L \le z \le 0 \\ \sqrt{\frac{-\pi}{2} Ln\left\{1-\left(R\left[1+\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{\frac{1}{2}}\right] - \left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{\frac{1}{2}}\right)^2\right\}}, & z_L \le 0 \le z \end{cases} \quad (14)$$

By generating random numbers of R~U[0,1] and substituting them in Equation 14, we generate random numbers based on left-sided truncated standard Gaussian distribution. Moreover, using the formula, z=(U(R)-*μ)/*σ, we can generate random numbers based on the non-standard distribution (i.e., *μ ≠0, *σ ≠0) where *μ and *σ are the mean and the standard deviation of the original Gaussian distribution before truncation, respectively.

ACCURACY OF THE INTRODUCED APPROXIMATIONS

The fitness to the original of the introduced approximation of the CDF and its inverse is presented in this section. The accuracy is estimated in terms of the maximum deviation of the approximation results from the true results. A new deviation function is defined, as addressed in Equation 15. The function consists of two parts according to the domain of Z value (i.e., Z≤0 or Z≥0). We can use any part of the equation for Z=0, as the function is continuous at this point. The deviation function is related to the change in deviation with a Z score.

$$D_T(z) = \begin{cases} \frac{\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{1/2} - \left\{1-exp\left(\frac{-2z^2}{\pi}\right)\right\}^{1/2}}{1+\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{1/2}} - \Phi(z), z_L \le z \le 0 \\ \frac{\left\{1-exp\left(\frac{-2z^2}{\pi}\right)\right\}^{1/2} + \left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{1/2}}{1+\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{1/2}} - \Phi(z), z_L \le 0 \le z \end{cases} \quad (15)$$

Figure 3 shows the deviation at five different truncation points (ZL) (i.e., -4, -3, - 2, -1 and 0). Usually, the truncation point is rarely located at a point more than ZL=-2 on the Z-score. In this study, we reported the deviation value until ZL=0. The maximum deviation at ZL=-4, ZL=-3, ZL=-2, ZL=-1, ZL=0 is 0.00314, 0.00317, 0.003357, 0.00386 and 0.006276, respectively. It is clearly noticed that the maximum of all curves maximum deviation is for ZL=0 with a value of 0.006276. In the literature, most approximations of Gaussian distribution CDF have a maximum deviation between 0.02 and 0.002. Further, we never found an approximation in the literature that is less than 0.02. This means that our introduced approximation is very accurate and definitely can be used without any worries about the results.

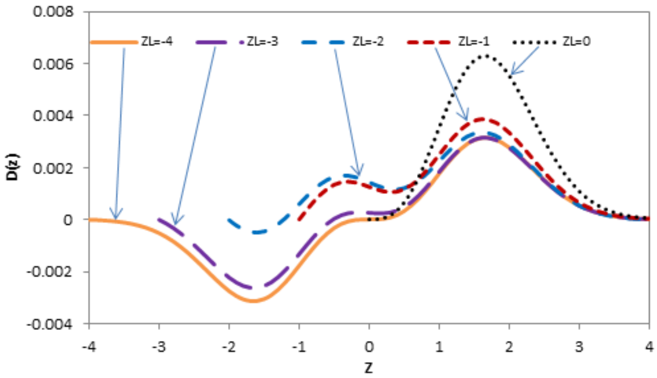

Figure 3: Deviation of the introduced approximation of the left-sided truncated Gaussian distribution CDF from the real vs. Z-score

The deviation between the introduced approximation CDF inverse and the true results is illustrated, as addressed in Equation 16. The new deviation function has two parts according to the domain of z (i.e., z≤0 or 0≤z). The R can be a random number generated from a uniform distribution over the domain [0:1].

$$D_{INV}(R, z_L) = \begin{cases} -\sqrt{\frac{-\pi}{2} Ln\left\{1-\left(\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{\frac{1}{2}} - R\left[1+\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{\frac{1}{2}}\right]\right)^2\right\}} - \Phi_T^{-1}(R, z_L), z_L \le z \le 0, R\sim U[0:1] \\ \sqrt{\frac{-\pi}{2} Ln\left\{1-\left(R\left[1+\left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{\frac{1}{2}}\right] - \left\{1-exp\left(\frac{-2z_L{}^2}{\pi}\right)\right\}^{\frac{1}{2}}\right)^2\right\}} - \Phi_T^{-1}(R, z_L), z_L \le 0 \le z, R\sim U[0:1] \end{cases} \quad (16)$$

Figure 4 shows the deviation function value versus Z score at different truncation points. In all curves, the deviation decreases with R approaching 0.5 and exponentially increases with R approaching 1. For example, if the resolution of R is 0.01, then the maximum deviation is less than 0.08 at 0.99. Since Φ^(-1) (R=1) equals infinity, we need to ignore the deviation between the approximation and the true value at R=1. It is customary in mathematics to see two models approaching infinity at the same x value and at the same time the difference between them approaching infinity as well.Further, the difference between the two approaches to infinity models is constant, if and only if the subtraction between them after removing the diminishing terms is constant. Since the inverse of the Gaussian distribution CDF approximation is a complex integral equation, it is impossible to find a solvable mathematical approximation with a constant distance from the original inverse equation when R approaches 1 (i.e., the function values approach infinity). Therefore, the deviation between the inverse of the left-sided truncated Gaussian distribution CDF and any past or future model must approach infinity when R approaches 1. Further, it is noticeable that there is a peak at near R=0. However, the deviation at the exact value of R=0 equals 0.

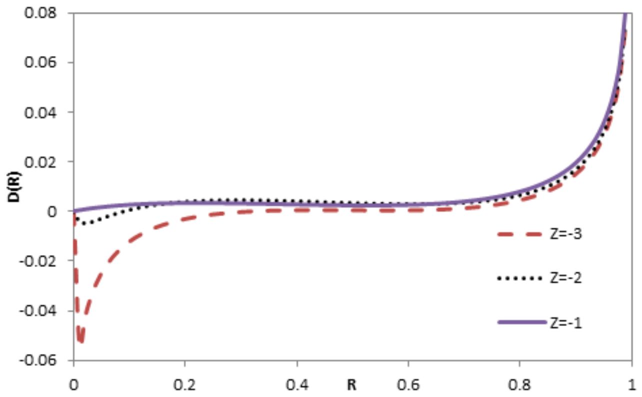

Figure 4: Deviation of introduced approximation of inverse of left-sided truncated Gaussian distribution CDF from true values vs. Z score

EXAMPLE ON GENERATING RANDOM VARIATES

In this section, two examples will be given. The first is to apply the current model as an approximation to the truncated left-sided guasian distribution. The second is the application of the current model as a generator of random variables from left-sided guasian distribution. Both are related to a real case of references. The first example is an application on production of pure alumimun foil, as addressed in [29]. A sample is collected and the foil that purity is less than 99% is discarded. The sample after discarding the unfit foils is left sided truncated guasian distribution. The sample mean and standard deviation around the mean are 0.992 and 0.0005, repectively. Consider our model, we can estimate the percent of foils that are purer than 0.993 (i.e., ZL=-4, Z=2), as requested in their case. The answer is 0.0228. In the second examples, generating random variates from left-sided truncated Gaussian distribution are provided. The example was brought from [30] to give more practicality. First, we generated 50 random numbers (i.e., numbers generated from a uniform distribution over the domain [0:1]) to generate random variables. Then, we generated random variates of Polya's Gaussian distribution approximation. Lastly, we generated random variate from five different examples of left-sided truncated Gaussian distribution using our approximation.

Table 1: Example on random variates generation from left-sided truncated Gaussian Distribution

No	RN	Variate	Variate	Variate	Variate	Variate	Variate
		Untruncated (Polya's), μ=0, σ=1	ZL=-2, *μ=0, *σ=1	ZL=-1, *μ=0, *σ=1	ZL=-2, XL=5, *μ=11, *σ=3	ZL=-1, XL=49, *μ=51, *σ=2	ZL=-1, XL=158, *μ=200, *σ=21
1	0.10	-1.27	-1.17	-0.70	7.48	49.60	185.30
2	0.14	-1.07	-1.00	-0.59	8.01	49.81	187.51
3	0.24	-0.70	-0.65	-0.36	9.04	50.28	192.46
4	0.44	-0.15	-0.12	-0.07	10.64	50.86	198.49
5	0.49	-0.01	-0.01	-0.19	10.97	50.63	196.10
6	0.70	0.52	0.53	0.66	12.60	52.32	213.81
7	0.81	0.86	0.87	0.97	13.62	52.95	220.47
8	0.67	0.44	0.46	0.58	12.37	52.17	212.27
9	0.76	0.72	0.73	0.84	13.19	52.68	217.65
10	0.05	-1.63	-1.48	-0.85	6.57	49.31	182.21
11	0.45	-0.12	-0.10	-0.09	10.71	50.82	198.07
12	0.16	-0.99	-0.92	-0.55	8.23	49.91	188.51
13	0.86	1.08	1.09	1.18	14.27	53.36	224.74
14	0.17	-0.97	-0.90	-0.53	8.29	49.93	188.78
15	0.63	0.32	0.34	0.48	12.03	51.96	210.12
16	0.44	-0.14	-0.11	-0.08	10.67	50.84	198.33
17	0.55	0.13	0.15	0.31	11.45	51.62	206.47
18	0.31	-0.50	-0.46	-0.21	9.62	50.58	195.59
19	0.02	-2.00	-1.72	-0.93	5.85	49.13	180.41
20	0.07	-1.44	-1.32	-0.78	7.04	49.45	183.68
21	0.80	0.84	0.86	0.96	13.57	52.92	220.14
22	0.79	0.80	0.81	0.92	13.44	52.83	219.26

23	0.45	-0.13	-0.10	-0.09	10.70	50.82	198.15
24	0.78	0.76	0.78	0.88	13.33	52.77	218.55
25	0.89	1.21	1.23	1.31	14.68	53.62	227.47
26	0.09	-1.34	-1.24	-0.73	7.29	49.53	184.57
27	0.82	0.90	0.92	1.02	13.75	53.03	221.32
28	0.95	1.62	1.63	1.70	15.88	54.39	235.60
29	0.04	-1.76	-1.57	-0.88	6.30	49.24	181.47
30	0.45	-0.14	-0.11	-0.08	10.67	50.84	198.30
31	0.22	-0.78	-0.73	-0.41	8.82	50.18	191.34
32	0.69	0.51	0.53	0.65	12.58	52.30	213.64
33	0.34	-0.40	-0.37	-0.13	9.90	50.73	197.20
34	0.12	-1.18	-1.09	-0.65	7.72	49.69	186.25
35	0.31	-0.49	-0.45	-0.20	9.64	50.59	195.73
36	0.74	0.66	0.67	0.78	13.01	52.57	216.47
37	0.50	0.00	-0.02	-0.20	10.93	50.61	195.86
38	0.45	-0.13	-0.10	-0.09	10.70	50.82	198.10
39	0.83	0.94	0.95	1.05	13.86	53.10	222.03
40	0.44	-0.14	-0.11	-0.08	10.66	50.85	198.38
41	0.47	-0.07	-0.05	-0.13	10.86	50.73	197.18
42	0.07	-1.45	-1.33	-0.78	7.02	49.44	183.62
43	0.10	-1.25	-1.16	-0.69	7.53	49.62	185.51
44	0.57	0.17	0.19	0.35	11.58	51.70	207.31
45	0.44	-0.15	-0.12	-0.07	10.64	50.86	198.50
46	0.53	0.08	0.10	0.26	11.30	51.53	205.52
47	0.36	-0.35	-0.31	-0.09	10.06	50.82	198.09
48	0.06	-1.50	-1.37	-0.80	6.89	49.40	183.18
49	0.20	-0.84	-0.78	-0.45	8.65	50.09	190.48
50	0.84	1.00	1.02	1.11	14.05	53.22	223.30

*μ: Original mean (before truncation), *σ: Original standard deviation (before truncation).

As appears in the previous table (Table 1), we can generate random variates from left-sided truncated Gaussian distribution with any *μ and *σ by using the transformation formula, z=(U(R)-*μ)/*σ. Further, there are 1-2 values in some columns lower than ZL (or XL) corresponding to R values near R=0. This is due to the slight deviation curve that goes to the negative side at near R=0. Since we already evaluated the accuracy of the approximation and the inverse mathematically, it is not valuable to fit the variates and see the fitness, especially that the number of the variates generated is not too much. With the simulation process, thousands of variables are usually generated, and the number of generated variates is not an issue.

CONCLUSION

Ploya's approximation of standard Gaussian distribution CDF was used to derive a left-sided truncated Gaussian distribution CDF approximation. The absolute maximum deviation of Ploya's approximation is less than 0.003. Polya's approximation accuracy was discussed and compared with the well-known Cadwell's approximation. Then his approximation is used to derive left-sided truncated gausian distribution. The inverse of Polya's approximation was easily derived, as explained previously and our introduced approximation was also derived. Our approximation has been proven to be accurate with lower than 0.007 maximum deviations from the real for the whole region [ZL:∞] for any ZL E [-∞:0]. Two real case are provided to explain the model applicability. Finally, random variates of left-sided truncated Gaussian distribution are generated used out model. We recommend practitioners/engineers use this approximation due to the high simplicity and high level of accuracy besides the feasibility of using spreadsheets.

FUTURE WORK

It was possible to study more detailed features of the present approximation. Furthermore, a more accurate approximation of left-sided truncation guasian distribution based on another guasian distribution approximation can be used to derive and generate random variables.

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