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Put ka simbolima u algebri: od realističnih situacija do simboličkog jezika

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Apstrakt *U radu autori predstavljaju okvir učenja za uvođenje slova kao oznake za nepoznatu i promenljivu, koji se zasniva na situacijama iz realnog konteksta, njihovom modelovanju i prevođenju na algebarski jezik. U empirijskom delu rada organizovano je istraživanje s ciljem da ispitaju efekti ovakvog pristupa na razvijanje algebarske simbolike kod učenika mlađih razreda osnovne škole. Istraživanje je sprovedeno na uzorku od 257 učenika četvrtog razreda osnovnih škola u Republici Srbiji, primenom eksperimentalne metode sa paralelnim grupama. Rezultati istraživanja pokazuju pozitivne efekte primenjenog pristupa na bolje razumevanje simboličke notacije i veći stepen algebarske generalizacije. Ovi nalazi potvrđuju hipotezu da uvođenje algebarskih simbola kroz realistične probleme i kontekstualno povezane aktivnosti može značajno unaprediti sposobnost učenika da pravilno interpretiraju i koriste algebarsku notaciju.*

Ključne reči: *algebra, kontekstualni pristup, matematika, nepoznata, promenljiva.*

Uvod

Prvi susret učenika sa simbolikom i matematičkim jezikom dešava se već sa početkom matematičkog obrazovanja i učenjem aritmetike. Dete sa formiranjem pojma prirodnog broja i njegovim simboličkim označavanjem počinje da upoznaje matematičke simbole i da prevodi realne konkretne situacije u matematičke zapise i rešava probleme. Poseban izazov na ovom uzrastu predstavlja uvođenje algebarske simboličke notacije i ulazak u oblast algebre. Upravo se prvi susret sa algebrom posmatra kao „prelomni trenutak za većinu ljudi u odluci da matematika nije za njih i da oni takve matematičke ideje

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ne razumeju” (Milinković i Maričić, 2022: 245). U aritmetici dete operiše brojevima, a u algebri ispoljava sposobnost da manipuliše brojevnim i simboličkim reprezentacijama, odnosno, da koristi i razume ulogu slova u matematičkom zapisu. Uvođenje simboličke algebarske notacije predstavlja veliki izazov kako za praktičare, tako i za teoretičare i istraživače matematičkog obrazovanja, koji postavljaju pitanje kako i kada uvesti matematičku simboliku (Brizuela et al., 2015; Carraher et al., 2008; Dabić Boričić, 2019; Milinković i Maričić, 2022; Radford, 2011).

Od aritmetike ka algebri u razvoju simboličkog jezika

Istraživanje Kristou i saradnika (Christou et al., 2005) pokazuje da na učenički doživljaj slovnih oznaka u algebri intenzivno utiče iskustvo koje učenici imaju sa brojevima u kontekstu aritmetike. Zapravo, uspešan prelaz na algebru, zasnovan na čvrstim aritmetičkim osnovama, može biti koristan i u procesu rešavanja problema (Akkan et al., 2011). Do sličnih rezultata dolaze Sun i saradnici (Sun et al., 2023), čije istraživanje pokazuje da se u nastavnim programima moraju pronaći sadržaji elementarne aritmetike na kojim će se uvesti algebarska simbolika. Razvojni putevi u ranom algebarskom mišljenju učenika imaju smer od „aritmetičkog mišljenja”, preko „generalizovanog algebarskog” do konačno „simboličkog algebarskog” mišljenja. Moglo bi se reći da je „matematička algebarska simbolizacija generalizacija aritmetike, dok je kognitivno tačnije reći da je simbolizacija algebre artikulacija aritmetike” (Heffernan & Koedinger, 2022: 484).

Međutim, treba imati u vidu da se jedan od razloga pogrešnog tumačenja ideje nepoznate i promenljive, kao i simbola za njihovo označavanje, krije upravo u činjenici da se slova pojavljuju i u aritmetici, ali na sasvim drugačiji način. Slovne oznake u aritmetici predstavljaju najčešće skraćenice naziva objekata i predmeta i direktno upućuju na sam pojam, dok je uloga slova u algebri potpuno drugačija. Ovakvo shvatanje i upotreba slovnih oznaka od strane učenika kasnije vodi do grešaka i dubljeg nerazumevanja simbola kao oznake za promenljivu. U rešavanju algebarskih zadatka i razvijanju algebarskog načina razmišljanja posebno se mogu izdvojiti dva značajna trenutka: 1) prelaz sa verbalnog jezika na simbolički zapis, 2) prelaz sa aritmetike na algebru. Svaki od njih uslovljen je određenim odnosima koji postoje između mišljenja učenika, nastave i karakteristika algebre (Milinković i Maričić, 2022).

Ako se u obzir uzme značenje simbola, Kijeran (Kieran, 1981) smatra da je taj proces u direktnoj vezi sa formiranjem pojma jednačine. Prema tome, polazeći od aritmetike i imajući u vidu Brunerove ravni predstavljanja (akcioni, ikonički i simbolički), učenik bi mogao da stekne intuitivno razumevanje značenja jednakosti i tek onda da ga postepeno transformiše do razumevanja pravog oblika jednačine. Na ovaj način bi njegova algebra bila usidrena u aritmetici. Ovaj proces bi polazio od aritmetičke jednakosti u kojoj bi se u prvom koraku prstom sakrio jedan od brojeva, dok bi se kasnije prst zamenio nekim od držača mesta, tako da bi se na kraju čitav proces završio uvođenjem simbola kao oznake za nepoznatu. Na ovaj način bi slovo koje skriva broj bilo nepoznata – termin koji blisko odgovara ideji o skrivenom broju koji se može odrediti. Ako se, uz sve navedeno, u obzir uzme nastava algebre, može se postaviti pitanje: *Na koji način treba početi sa uvođenjem slova i algebarske notacije u početnoj nastavi matematike i kako taj proces da teče prirodnim putem?* Da bi se dao odgovor na ovo pitanje, mora se imati u vidu složenost i apstraktnost algebarskih pojmova, ali i činjenica da učenici mlađeg školskog uzrasta imaju poteškoće u razumevanju ideje nepoznate i promenljive (Akgün & Özdemir, 2006; Carraher et al., 2008; Stephens et al., 2015), pri čemu se pojavljuju i problemi u njihovom simboličkom označavanju (Booth, 1988; MacGregor & Stacey, 1997; McNeil et al., 2010; Specht, 2005).

Zadatak koji se postavlja pred učenike na ovom uzrastu treba da bude vezan za razvijanje ideje simbola u algebarskim zapisima u obliku:

- nepoznate (skriveni broj čiju vrednost treba otkriti, kao nepoznata, ali fiksna veličina – jednačine);
- promenljive (skup brojeva koji ispunjavaju određene uslove funkcionalne zavisnosti – nejednačine, funkcije, niz);
- slobodne promenljive (simbol koji predstavlja bilo koji broj u određenom skupu brojeva - generalizacija) (Milinković i Maričić, 2022).

Realni kontekst kao polazište za uvođenje algebarske simbolike

Proces formiranja pojmova promenljive i nepoznate, zatim i simbola za njihovo označavanje, krije u jedinstvu značenja i oznake, odnosno, oznaku pojma i oznaku simbola za njegovo predstavljanje. Presudnu ulogu u tom procesu ima kontekst kao nosilac značenja pojma. Ovaj kontekst, ukoliko je izražen kroz jezik svakodnevnog života, omogućava da se generalizaciji pruži smisao, te da se tim putem tako apstraktan pojam približi razumevanju učenika. Upravo iz tog razloga, osnova za razvijanje pojma simbola, kao i njegovog značenja kroz ulogu nepoznate i promenljive, mora da potiče od realnog konteksta i problema koji su vezani za ono sa čim se učenik susreće svakodnevno. Dakle, značaj razumevanja slova u algebarskom izrazu u direktnoj je vezi sa kontekstom koji postoji u problemu. Tako Risted i saradnici (Rystedt et al., 2016) smatraju da korišćenje različitih konteksta i razgovora o njima može pomoći u prelasku primitivnih na naprednije interpretacije simbola, pri čemu je važan kontekst u izražavanju odnosa koji postoje između veličina. Sa druge strane, prema mišljenju Radforda (Radford, 2002), verbalni jezik omogućava ljudima da osmisle tekstove koji odgovaraju opisanim radnjama i koje su opremljene nizom mogućnosti da se značenja razjasne i adekvatno obeleže, što sa druge strane nije slučaj sa algebarskom notacijom. Konstruisanje simboličkog zapisa za tekstualni problem zahteva drugačiji pristup iz razloga što se problemska situacija odvija prema načinu na koji se i čita, s leva na desno, dok početna tačka u simboličkom zapisu nema stalno mesto. U simboličkom zapisu je redosled zapisivanja izraza potpuno drugačiji i zavisi od suštine odnosa između veličina. Upravo iz tog razloga Ferari (Ferrari, 2006) smatra da učenici u procesu rešavanja problema prvo treba da koriste skraćenice koje su vezane za specifičan kontekst, dok se kasnije te skraćenice generalizuju izražavajući karakter nepoznate, koja se koristi u bilo kom problemu. U tom slučaju, isti autor tvrdi, da to zavisi od jezičkih kompetencija samog učenika, ali i njegovog potpunog uključivanja u ostvarivanje ciljeva aktivnosti.

Prema mišljenju Van Rejvika (Van Reeuwijk, 2001) u mnogim nastavnim programima algebarske teme se uvode veoma brzo, tako da učenici jedva imaju vremena da razviju pojmovno razumevanje algebarskih koncepata. Često ne postoji veza sa postojećim neformalnim znanjem učenika. Iz perspektive učenika, to implicira besmisleno predstavljanje i korišćenje simbola. Iz tih razloga nastavu algebre treba zasnovati na situacijama koje su bliske učenikovom iskustvu, njegovom neformalnom znanju, utkati ih u aritmetiku i u kontekst koji je blizak učeniku, kao i situaciji učenja koja za učenike imaju smisla. Sa druge strane, učenicima treba sugerisati da koriste različite oblike vizuelno-šematskih reprezentacija kada rešavaju probleme realnog konteksta, kako bi razumeli odnose i bili uspešni u njihovom predstavljanju (Milinković et al., 2022).

Realni kontekst se vidi kao situacija koja je iskustveno bliska učeniku, pri čemu predstavlja polaznu tačku koja će poslužiti učeniku za ponovno otkrivanje matematičkih istina. Na taj način, cilj realnog konteksta predstavlja stvaranje pogodne situacije koja će omogućiti formiranje formalnog matematičkog znanja učenika, u ovom slučaju korišćenje simbola kao oznake za nepoznatu i promenljivu. Osnova ovakvog pristupa u nastavi leži u tome da učenje matematike treba da ima karakteristike kognitivnog rasta, a ne da se pojmi isključivo kao proces slaganja delova matematičkog znanja. Stoga je proces učenja zasnovan na konstrukciji matematičkih znanja i modela na osnovu učeničkog iskustva iz realnog života.

Problem izražen u obliku realnog konteksta može biti značajan motivišući faktor kako bi se uticalo na razvijanje pojma promenljive, ali i uvođenje simbola za označavanje iste. Navedimo primer koji izdvaja Arkavi (Arcavi, 1994), u kojem se pred učenike postavlja problem u obliku fotografije, koja prikazuje vozilo na ulazu u tunel na čijem vrhu se nalazi znak: "2.90". Od učenika je zatraženo da protumače šta broj može značiti. Neki od predloga bili su da se broj odnosi na težinu vozila u prolazu, njegovu širinu, njegovu visinu, ali su takođe razmatrane i relevantne merne jedinice. Na kraju su se učenici složili da bi oznaka mogla biti samo „maksimalna visina za prolaz vozila", pri čemu su navodili moguće visine. Na kraju je od njih zatraženo da sve to zapišu i na matematički način. Ovakav način uvođenja simbola bio je u vezi sa razumevanjem konkretne situacije, ali i duboko vezan za značenje promenljive, što je jedan od najznačajnijih ciljeva – simbol ne može da postoji bez značenja, ali ni značenje bez simbola (Radford, 2004). Iste stavove iznose i Karaher i saradnici (Carraher et al., 2008), koji smatraju da algebru na mlađem školskom uzrastu treba učiti u pozadini konteksta realnog problema, postepeno uvodeći formalni zapis i kroz čvrsto povezivanje svih sadržaja matematike. Na primer, autori su pred decu postavili dve iste kutije bombona. Učenicima je predočeno da se u njima nalazi isti broj bombona. Na jednoj kutiji stajale su 3 bombone. Učenici su imali zadatak da broj bombona izraze simbolički. Veliki broj učenika pokušao je da reši problem tako što je nepoznatoj količini dao određenu vrednost, dok je drugi pak pokušao da reši koristeći se crtežom koji izražava situaciju, kako bi grafičkim prikazom ublažio apstraktnost problema. Algebarski postoji samo jedno tačno rešenje, ali logički gledano, tačno je svako rešenje koje izražava odnos gde u jednoj kutiji ima za 3 više bombona u odnosu na broj bombona u drugoj kutiji. Rezultati upućuju na to da će učenici možda moći da preusmere fokus sa pojedinačnih elemenata na skupove i njihove međusobne odnose. Stoga realni kontekst služi kao pozadina da bi se opisale veze između fizičkih veličina, pri čemu se na kraju formiraju generalizacije u obliku formalnih predstava, tako da „matematički predmet više nije pojedinačni slučaj ili vrednost, već odnos, odnosno funkcionalni odnos između dve promenljive" (Carraher et al., 2008: 247).

Svakodnevni jezik omogućava da se generalizaciji pruži smisao, prenese informacija i upotpuni kontekst koji nedostaje u algebarskoj notaciji. Iz navedenog razloga, polazište za uvođenje nepoznate i promenljive jeste prirodni jezik učenika, odnosno problemi izraženi kroz realne situacije svakodnevnog života, koji predstavljaju dobru osnovu za razumevanje ideja algebre i razvoj algebarskih sposobnosti.

Modelovanjem realističnih situacija ka konceptualnom razvijanju algebarske simbolike

Prvi korak na putu uvođenja i razvijanja algebarske simbolike jeste stvaranje realnog konteksta u kojem učenici mogu da prepoznaju problem, razmišljaju o njegovom rešavanju i grade osnov za postepen saznajni prelazak sa konkretnog na apstraktni, simbolički nivo. Realistične situacije deluju kao inicijalni pokretač, koji omogućava da se apstraktna ideja promenljive ili nepoznate pojavi iz iskustva učenika. Na ovaj način obezbeđuje se postepena matematizacija (Freudenthal, 1991), u kojoj se kroz više nivoa reprezentacije gradi most između svakodnevnog iskustva i formalnog algebarskog jezika. Okvir takvog učenja možemo predstaviti kroz sledeće faze:

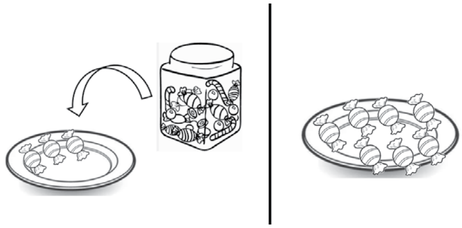
1) realistična situacija kojom se izražava algebarski problem;
2) modelovanje realistične situacije korišćenjem različitih načina reprezentacije problema;
3) prevođenje ideje na konkretno područje referentnog algebarskog jezika uz korišćenje algebarskih simbola i operacija (Milinković i Maričić, 2022).

Kako bismo ilustrovali navedene faze učenja, navešćemo nekoliko primera problema realnog konteksta, koji mogu predstavljati oslonac za uvođenje simbola kao oznake za promenljivu i nepoznatu u mlađim razredima osnovne škole.

Primer 1

Crtež 1

Koliko bombona je dodato na tanjir?



Na osnovu problema izraženog u vidu realne situacije (Crtež 1), učenik je u situaciji da zaključuje o odnosima i količinama bombona koje postoje u zadatku. Osnovni problem vezan je za uočavanje nepoznatog broja, tako da je učenik u situaciji da na osnovu očiglednog primera postepeno pređe na polje formalne algebarske notacije. Pošto je očigledna brojnost bombona u tanjiru pre i posle dodavanja, može se zaključiti da je broj dodatih bombona nepoznat. Formiranje pojma nepoznatog broja započinje bez simbolizacije, pa se ovaj problem u matematičkom obliku može predstaviti i kao jednakost sa „držačem mesta", koji u ovom smislu predstavlja ulogu nepoznatog sabirka:

$$3 + \quad = 7$$

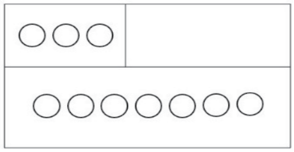
Ovim putem se problem svodi na matematički oblik koji bismo mogli izraziti u formi pitanja „Koji broj treba dodati broju 3 da bi se dobio broj 7?" Ovakav tip zadatka, u obliku realnog konteksta, poslužio bi kao oslonac u prelasku na jezik formalne algebre, a rešavanje ovog problema moglo bi da se svede na prethodno iskustvo u sabiranju brojeva do 10.

Navedeni način zapisivanja (3 + ____ = 7), omogućava učenicima da uoče strukturu izraza i pojam nepoznatog pre susreta sa apstraktnim simbolom (x), na šta posebno ukazuju Radford (Radford, 2000) i Kijeran (Kieran, 1981), ističući da ovi zadaci olakšavaju prelaz od aritmetičkog načina mišljenja ka algebarskom. Prema njihovom mišljenju, zadaci sa *držačima mesta* omogućavaju učenicima da razviju osećaj za nepoznatu veličinu u konkretnom kontekstu, pre formalnog prihvatanja simboličkog jezika algebre. Ovakvi zadaci predstavljaju korak ka postepenoj simbolizaciji, jer kroz *držače mesta* učenici prvo shvataju ideju nepoznate vrednosti, zatim uče da je označe slovnim simbolom, a kasnije da izvode operacije nad njom. Na taj način *držači mesta* funkcionišu kao most između aritmetike i algebre, uvodeći učenike u novi način mišljenja koji podrazumeva apstrakciju i generalizaciju.

Sledeći korak u razvijanju ideje nepoznate, ali i simbola za njegovo označavanje, može se zasnivati na slikama ili ideografima, koji predstavljaju nešto apstraktnije forme (Crtež 2).

Crtež 2

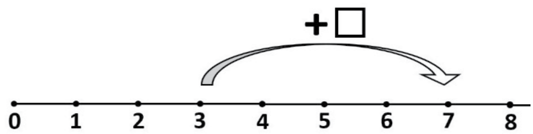
Modelovanje problema ideografom



Dalje udaljavanje od konkretnog je prelazak na modelovanje situacije na brojevnoj polupravoj (Crtež 3).

Crtež 3

Modelovanje problema na brojevnoj polupravoj



Korišćenje ovih modela u velikoj meri može ublažiti apstraktnost algebarske notacije, tako što stvara uporište u uvođenju pojma nepoznatog broja „h", kao broja čija se vrednost može utvrditi. Ovakvi grafički prikazi, implicirajući problem izražen u obliku realne situacije, mogu poslužiti kako bi se konačno zapisala jednačina u čisto algebarskoj formi:

$$3 + h = 7$$

Rešenje novog zadatka u algebarskom obliku bilo bi:

$$h = 4$$

Učenik na kraju zaključuje da je broj dodatih bombona 4. U procesu provere tačnosti rešenja jednačine, sledeći zadatak je ispitati da li je izraz koji se nalazi sa desne strane znaka jednakosti ekvivalentan broju koji se nalazi sa njegove leve strane, odnosno:

$$3 + 4 = 7$$

$7 = 7$, što znači da smo dobili tačno rešenje.

Primer 2

Bibliotekarka Milica za mesec dana rasporedi određeni broj knjiga na police. Bibliotekarka Milica rasporedi 120 knjiga manje od bibliotekara Petra. Napiši izraz koji pokazuje broj knjiga koje rasporedi bibliotekar Petar za mesec dana.

Proces rešavanja problema ponovo počinje od analize kontekstualno zasnovanog problema, koji dalje prelazi u polje apstraktne matematike. Učenik, razmišljajući o odnosima koji postoje u zadatku, može zaključiti sledeće: ako bibliotekarka Milica za mesec dana rasporedi 120 knjiga manje od bibliotekara Petra, to znači da bibliotekar Petar mesečno složi 120 knjiga više od bibliotekarke Milice. Uzimajući to u obzir i uvažavajući činjenicu da je broj knjiga koje bibliotekarka Milica složi za mesec dana označen sa h, taj broj knjiga koje složi bibliotekar Petar mora biti $h + 120$. U razmišljanju o odnosima, misao učenika je na polju apstraktne matematike potpuno izdvojena od realnog problema. Na ovaj način ostvarena je veza između značenja i simbola.

Primer 3

Miloš je na testu postigao određeni broj poena i položio. Na narednom testu postigao je 12 poena više nego na prvom. Da li to znači da je i drugi test položio? Zašto? Kako bi matematički zapisao broj poena koje je osvojio na drugom testu?

Na osnovu podataka koji su iskazani u zadatku, možemo reći da postoji promenljiva, odnosno vrednost koja nije tačno određena. U ovom slučaju to je broj poena koje je Miloš postigao na prvom testu. Analizirajući odnose između podataka koji su dati u zadatku, učenik može da zaključi da postoji određeni nepoznati broj poena na prvom testu, kao i da je na drugom testu Miloš postigao 12 poena više. Pored toga, razmišljajući o odnosima, da bi se moglo odgovoriti na pitanje da li je i drugi test položen, mora se imati u vidu još jedna nepoznata veličina, a to je broj poena dovoljnih za polaganje testa, tako da je nemoguće dati tačan odgovor na ovo pitanje. U razmišljanju o odnosima, poslednji korak u procesu matematizacije jeste simbolizacija u kojoj se izražavaju odonosi između nepoznatog broja na prvom i drugom testu, u ovom slučaju: broj Miloševih postignutih poena na prvom testu: h; broj Miloševih postignutih poena na drugom testu: $h + 12$.

Ovim putem smo želeli da na konkretnim primerima pokažemo na koji način je moguće uvesti slovne oznake kroz primere realnog konteksta. Povezivanjem sa problemima svakodnevnog života, kroz procese modelovanja, pokušava se prevazići problem razumevanja simbola koji se pojavljuju nerazdvojivo od svog značenja. Na taj način se osigurava da simbolizacija i značenje istovremeno postoje u jedinstvu realnog konteksta, podržavajući konstrukciju neke nove matematičke stvarnosti.

Na bazi ovakvih ideja smo želeli da istražimo da li prikazani pristup uvođenja slova u nastavu algebre doprinosi razvijanju sposobnosti pravilnog razumevanja simbola, kao što su oznake za promenljivu ili nepoznatu u sadržajima algebre na ranom školskom uzrastu.

Metodološki okvir

Uzorak istraživanja

Uzorak istraživanja (N = 257) odabran je iz populacije učenika koji su pohađali četvrti razred osnovne škole u Republici Srbiji. Istraživanje je sprovedeno primenom eksperimentalne metode sa paralelnim grupama. Eksperimentalnu grupu činili su učenici pet odeljenja iz jedne osnovne škole (N = 130), dok su kontrolnu grupu činili učenici pet odeljenja iz druge dve osnovne škole (N = 127). Grupe su bile ujednačene prema uspehu, socijalnom statusu i polu, a pored toga je ujednačenost kontrolisana i analizom kovarijanse.

Metod istaživanja

Tehnika korišćena u istraživanju bila je testiranje, a istraživački instrument bio je test. Za potrebe našeg istraživanja, i u skladu sa usvojenom metodom istraživanja, osmišljena su dva ekvivalentna testa: inicijalni test – za ispitivanje početnog nivoa razumevanja simbola, kao znaka za nepoznatu i promenljivu i finalni test – za utvrđivanje efekata eksperimentalnog programa. Problemi u testu su izražavali realne situacije, kao i probleme iz svakodnevnog života u kojima je upotreba simbola bila važna za razumevanje sadržaja algebre.

Pouzdanost testa je procenjena pomoću Kronbahovog alfa koeficijenta, kao indikatora pouzdanosti naših testova (inicijalnog i finalnog testa). Ovim testom se pokušava odrediti postojanje unutrašnje saglasnosti skale, odnosno stavki iz kojih se ona sastoji (Tabela 1).

Tabela 1
Vrednosti Kronbahovog alfa koeficijenta za testove

	Cronbach's Alpha	Cronbach's Alpha Based on Standardized Items	N of Items
Inicijalni test	.795	.885	10
Finalni test	.813	.926	10

Napomena: N – broj stavki; Kronbahov alfa – koeficijent pouzdanosti unutrašnje konzistentnosti; vrednosti veće od 0,7 ukazuju na prihvatljivu pouzdanost instrumenta.

Vrednosti Kronbah alfa koeficijenta (Tabela 1) za inicijalni i finalni test pokazuju rezultate veće od 0,7 što, prema mišljenju Palant (Pallant, 2017), predstavlja prihvatljive i poželjne vrednosti relijabilnosti.

Na početku eksperimenta izvršili smo inicijalno testiranje kako bismo utvrdili nivo razumevanja simbola kao znaka za nepoznatu i promenljivu u obe grupe. Nakon početnog testiranja, a u skladu sa kontekstualnim pristupom, uveli smo eksperimentalni program u rad eksperimentalne grupe. Oba testa su sadržala probleme koji su imali za cilj testiranje nivoa razumevanja simbola kao oznake za nepoznatu i promenljivu.

Postupak istraživanja

Nastava matematike u svim grupama se odvijala prema redovnom nastavnom planu i programu koji je propisalo Ministarstvo prosvete Republike Srbije. Eksperimentalni program je sproveden tokom redovne nastave matematike u periodu od tri meseca i realizovan je kroz 27 časova. Sadržaji koji su obuhvaćeni istraživanjem su: jednačine sa sabiranjem, oduzimanjem, množenjem i deljenjem; nejednačine sa sabiranjem i oduzimanjem; funkcionalna zavisnost između rezultata i komponenata računskih operacija sabiranja, oduzimanja, množenja i deljenja. Na ovaj način su obuhvaćeni svi sadržaji u kojima se simbol pojavljuje kao oznaka za nepoznatu, promenljivu ili slobodnu promenljivu.

Eksperimentalna grupa je radila prema programu koji su činili posebno pripremljeni časovi, gde su se sadržaji algebre koji se odnose na pravilno razumevanje simbola kao oznake za promenljivu i nepoznatu podučavali modelovanjem realnog konteksta. Nastava je započinjala postavljanjem problema u realnom kontekstu, gde su učenici bili podstaknuti da identifikuju nepoznate veličine i vežbaju predstavljanje istih korišćenjem algebarske notacije. Zatim su učenici, uz podršku učitelja, rešavali nekoliko primera zadataka primenom iste metodologije. Nakon zajedničkog rada, učenici su samostalno rešavali zadatke iz udžbenika, koristeći usvojenu terminologiju i algebarski zapis za rešavanje zadataka, uz proveru tačnosti dobijenih rezultata.

Program je bio direktno usmeren na: (a) korišćenje realnog konteksta za razvijanje algebarskih pojmova; (b) razumevanje veze koja postoji između verbalnih opisa i algebarskih izraza; (v) strukturno razumevanje slovnih izraza kao matematičkih objekata i (g) modelovanje stvarnih životnih situacija uz korišćenje različitih reprezentacija (verbalna reprezentacija, didaktička sredstva, dijagrami ili slike, simbolička reprezentacija). Učitelji koji su realizovali program u eksperimentalnoj grupi prošli su obuku u trajanju od pet časova pre početka eksperimentalne faze. Trening je organizovao istraživački tim i obuhvatao je metodološka uputstva za sprovođenje intervencije i korišćenje pripremljenih materijala. Tokom realizacije, nastavni sadržaji su izvođeni u tesnoj saradnji sa istraživačima, što je omogućilo veću doslednost postupaka i povećalo pouzdanost dobijenih rezultata.

Kontrolna grupa radila je u skladu sa programom matematike zasnovanom na nastavi u kojoj dominira frontalni oblik rada, a koji nije bio zasnovan na realističnim situacijama i modelovanju, već na okvirima koji su postavljani u udžbeniku. I eksperimentalna i kontrolna grupa koristile su iste udžbenike iz matematike. Tipičan čas je uključivao rad na primerima kroz objašnjavanje, rešavanje zadataka na tabli i učeničko individualno rešavanje zadataka. Naglasak je bio na proceduri rešavanja zadatka i pravilnoj simboličkoj manipulaciji, a ne na postupku modelovanja ili korišćenja višestrukih reprezentacija u procesu učenja i rešavanja zadataka. Upotreba didaktičkog materijala, dijagrama i zadataka koji zahtevaju prevođenje stvarnog konteksta u algebarske iskaze bila je ograničena ili izostala. Rad u kontrolnoj grupi bio je fokusiran pretežno na procedure rešavanja zadataka (npr. ispravno rešavanje jednačina) umesto na konceptualno razumevanje i transfer znanja. Učitelji u kontrolnoj grupi nisu imali dodatnu obuku osim redovnog školskog stručnog usavršavanja.

Analiza i obrada podataka

Rezultati do kojih smo došli u istaživanju su obrađeni kvantitavno i kvalitativno. U kvantitativnoj analizi dobijenih podataka koristili smo analizu varijanse i analizu kovarijanse, kako bi se utvrdile moguće statistički značajne razlike u postignuću između grupa. U kvalitativnoj analizi analizirana su rešenja finalnog testa, pri čemu smo posebno pažnju posvetili analizi postupka i načina rešavanja zadataka. U radu smo se posebno osvrnuli na način na koji učenik pristupa rešavanju zadataka i greške koje se u tom procesu javljaju. Kvalitativna analiza obuhvatila je rešenja svih učenika, bez obzira na grupu kojoj učenik pripada.

Rezultati i diskusija

Efekti kontektstualnog pristupa na razumevanje algebarske simbolike

Rezultati inicijalnog testiranja (Tabela 2) pokazuju je da je eksperimentalna grupa na inicijalnom merenju (M = 4.62; SD = 3.07) prosečno postigla približno isti broj bodova kao i kontrolna grupa (M = 4.56; SD = 2.94). Analiza varijanse ($F(1, 255) = .023$, $p = .881$) rezultata na inicijalnom merenju pokazuje da između eksperimentalne i kontrolne grupe na inicijalnom merenju ne postoje statistički značajne razlike u pravilnom razumevanju simbola u ranoj algebri (Tabela 3). Vrednosti Levenovog testa inicijalnog merenja ($F(1, 255) = .074$; $p = .786$) (Tabela 3) pokazuju da je ispunjena pretpostavka o jednakosti varijanse, usled čega se analiza varijanse ovih rezultata može smatrati pouzdanom.

Tabela 2
Postignuća učenika eksperimentalne i kontrolne grupe u sposobnosti pravilnog shvatanja simbola u algebri na inicijalnom i finalnom merenju

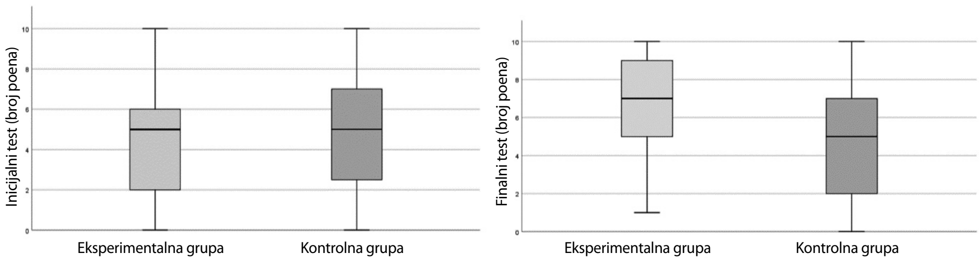
		N	*M*	*SD*
Inicijalni test	Eksperimentalna grupa	130	4.62	3.07
	Kontrolna grupa	127	4.56	2.94
Finalni test	Eksperimentalna grupa	130	6.89	2.41
	Kontrolna grupa	127	4.60	2.83

Napomena: *N* – broj ispitanika; *M* – aritmetička sredina; *SD* – standardna devijacija.

Rezultati finalnog merenja pokazuju da je eksperimentalna grupa u proseku postizala bolje rezultate (M = 6.89; SD = 2.41) u odnosu na kontrolnu grupu (M = 4.60; SD = 2.83) u merenju ove sposobnosti (Tabela 1). Za razliku od eksperimentalne grupe, kontrolna grupa je ostala na približno istom prosečnom broju poena kao na inicijalnom merenju (Grafikon 1).

Grafikon 1

Postignuće učenika eksperimentalne i kontrolne grupe u shvatanju simbola u algebri



Kako bismo utvrdili razlike u razvijenosti sposobnosti pravilnog shvatanja simbola u algebri, koristićemo analizu varijanse. Vrednosti Levenovog testa finalnog merenja (*F*(1, 255) = 4.796; *p* = .209) pokazuju da je i na finalnom merenju ispunjena pretpostavka o jednakosti varijanse, pa se analiza varijanse ovih rezultata takođe može smatrati pouzdanom.

Tabela 3

Analiza varijanse inicijalnog i finalnog merenja sposobnosti pravilnog shvatanja simbola u algebri

	Levene Statistic	df1	df2	Sig.
Inicijalni test	.074	1	255	.786
Finalni test	4.796	1	255	.209

		Sum of Squares	df	Mean Square	F	Sig.
Inicijalni test	Između grupa	.204	1	.204	.023	.881
	Unutar grupa	2306.076	255	9.043		
	Ukupno	2306.280	256			
Finalni test	Između grupa	338.031	1	338.031	48.837	.000
	Unutar grupa	1765.012	255	6.922		
	Ukupno	2103.043	256			

Napomena: Levene Statistic – vrednost Levenovog testa jednakosti varijansi; df – stepen slobode; Sig. – nivo značajnosti; F – vrednost Fišerovog testa; Between/Within Groups – vrednosti varijanse u ANOVA modelu.

Rezultati analize varijanse finalnog merenja (*F*(1, 255) = 48.837; *p* = .000) pokazuju da postoji statistički značajna razlika u postignuću kontrolne i eksperimentalne grupe. Dobijeni rezultati pokazuju da su učenici eksperimentalne grupe statistički značajno postigli bolje rezultate u odnosu na kontrolnu grupu kada se, pod uticajem eksperimentalnog programa, ispitivalo u kojoj je meri razvijena njihovasposobnost pravilnog razumevanja simbola u algebri.

Kako bismo bili sigurni da su dobijeni rezultati pouzdani i da na njih nije uticala eventualna neujednačenost grupa, sproveli smo analizu kovarijanse. Na ovaj način, analizom kovarijanse, želeli smo istražiti uticaj eksperimentalnog programa, zasnovanog na principima kontekstualnog pristupa u nastavi algebre, na sposobnost pravilnog razumevanja ideje simbola u algebri. Vrednost Levenovog testa $F(1, 255) = 1.059$; $p = .305$ pokazuje da je ispunjena pretpostavka o jednakosti varijansi, tako da je moguće uraditi analizu kovarijanse kako bi se utvrdile razlike (Tabela 4).

Tabela 4
Levenov test analize varijanse

Zavisna varijabla: Finalno merenje

F	df1	df2	Sig.
1.059	1	255	.305

Testira nultu hipotezu da je varijansa greške zavisne promenljive jednaka u svim grupama.
a. Model istraživanja: Konstanta + Simbol1 + Grupa

Napomena: F – vrednost Fišerovog testa; df_1 – broj stepeni slobode za faktor (grupe); df_2 – broj stepeni slobode za grešku; *Sig.* – nivo statističke značajnosti (p-vrednost).

Analizom kovarijanse u postupku uklanjanja kovarijata (rezultata inicijalnog merenja) utvrdili smo postojanje statistički značajnih razlika između eksperimentalne i kontrolne grupe u finalnom merenju sposobnosti razumevanja simbola u algebri ($F(1, 254) = 117.010$; $p = .000$) (Tabela 5). Ovakav rezultat pokazuje značajne razlike između eksperimentalne i kontrolne grupe u merenju ove sposobnosti. Veličina parcijalnog eta kvadrata iznosi .315, što predstavlja veliki uticaj. To znači da je 31.5% varijanse u finalnom merenju (sposobnosti razumevanja simbola) objašnjeno je nezavisnom promenljivom (grupom – načinom rada). Ako se u obzir uzme uticaj kovarijata na finalno merenje (rezultati finalnog merenja sposobnosti razumevanja simbola), kada se ukloni uticaj nezavisne promenljive (grupe) uočljive se takođe statistički značajne razlike ($F(1, 254) = 377.835$; $p = .000$). To znači da je kontekstualni pristup učenju značajno doprineo razvoju sposobnosti razumevanja simbola, nezavisno od početnog znanja učenika. S druge strane, kada se kontroliše uticaj nezavisne varijable (grupe), rezultati inicijalnog merenja – kao kovarijata – objašnjavaju čak 59.8% varijanse u finalnom postignuću ($F(1, 254) = 377.835$; $p = .000$). To pokazuje da je početni nivo znanja najjači pojedinačni prediktor kasnijeg razumevanja simbola u algebri. Drugim rečima, iako početno znanje ima dominantnu ulogu u predviđanju uspeha, način rada (kontekstualni pristup) takođe ima značajan uticaj na razvoj sposobnosti razumevanja algebarskih simbola.

Tabela 5

Analiza kovarijanse inicijalnog i finalnog merenja u sposobnosti pravilnog shvatanja simbola u algebri

Zavisna varijabla: Finalno merenje						
Izvor	*Type III Sum of Squares*	*Df*	*Mean Square*	*F*	*Sig.*	*Partial Eta Squared*
Korigovani model	1393,502[a]	2	696,751	249,421	0,000	0,663
Konstanta modela	536,250	1	536,250	191,966	0,000	0,430
Početno merenje	1055,471	1	1055,471	**377,835**	**0,000**	**0,598**
Grupa	326,864	1	326,864	**117,010**	**0,000**	**0,315**
Greška	709,541	254	2,793			
Ukupno	10626,000	257				
Korigovano ukupno	2103,043	256				

a. R Squared = 0,663 (Adjusted R Squared = 0,660)

Napomena: Type III Sum of Squares – mera varijanse; *F* – odnos srednjih kvadrata; *Sig.* – nivo značajnosti; *Partial Eta Squared* – mera veličine efekta; R^2 – koeficijent determinacije.

Na osnovu analize dobijenih rezultata možemo zaključiti da se može stimulisati pravilno razumevanje simbola kao oznake za promenljivu ili nepoznatu ukoliko je nastava algebre organizovana u skladu sa principima kontekstualnog pristupa i realnim situacijama svakodnevnog života. Nakon sprovedenog eksperimenta, rezultati su pokazali da postoje statistički značajne razlike između grupa, iz čega može proizići zaključak da se kontekstualnim pristupom može uticati na pravilno razumevanje simbola u algebri. Simbol predstavlja značajan pojam u algebri, pa je iz tog razloga veoma važno pravilno razvijanje ovog pojma od najranijeg uzrasta. Slovna oznaka (simbol) predstavlja značajan pojam rane algebre, koji na određeni način, kao oznaka, upućuje na algebru kao posebnu oblast matematike. Simbol ima moć da skrati zapis i izrazi kompleksne i raznovrsne zakonitosti na jednostavan način, a samim tim i olakša komunikaciju. Ovim istraživanjem smo dokazali da se putem nastave organizovane u skladu sa kontekstom realnih (životnih) situacija, može uticati pozitivno na pravilan razvoj ovog pojma kod učenika. U nastavi algebre na mlađem školskom uzrastu pojam simbola (slova za označavanje nepoznate i promenljive) suštinski je vezan sa drugim pojmovima kao što su jednačine, nejednačine, nizovi, formule i drugo. Kada je reč o nastavi matematike, učenici simbol često koriste kao oznaku u kojoj slovo etiketira predmet na koji se odnosi, što može doprineti ometanju pravilne interpretacije slova kao oznake u početnoj nastavi algebre (McNeil et al., 2010). Nasuprot ovim rezultatima, analiza rešenja učenika u ovom istraživanju pokazala je da određeni broj učenika jeste koristio slova za označavanje predmeta ili osoba, ali i da takva upotreba simbola nije ometala algebarske manipulacije i da su učenici uvek dolazili do tačnog rešenja. Nasuprot rezultatima prethodno navedenih istraživanja, u ovom istraživanju su učenici koristili različite interpretacije kao oznake za promenljivu ili nepoznatu, ali su njima manipulisali kao standardnim algebarskim simbolima.

Dobijeni rezultati, u skladu sa istraživanjem Stivensa i saradnika (Stephens et al., 2015) koji su u okviru istraživanja, pokazali su da se kroz adekvatan pristup i vežbanje algebarske sposobnosti na zadacima realnog konteksta mogu razvijati pojmovi nepoznate i promenljive, kao i simbola za njihovo označavanje. Prema tome, autori sugerišu da se algebarskim obrazovanjem u mlađim razredima osnovne škole mogu ublažiti neke od poteškoća koje učenici imaju sa učenjem algebre u starijim razredima.

Na osnovu dobijenih rezultata, možemo zaključiti da problem izražen u obliku konteksta svakodnevnog života može pomoći u interpretaciji i lakšem razumevanju suštinske ideje, zatim i uloge simbola u matematičkim jednakostima ili nejednakostima, jednačinama ili nejednačinama. U prilog ovim rezultatima navedimo i primer istraživanja Blantona i saradnika (Blanton et al., 2017) koje je pokazalo određene mogućnosti razumevanja slova kao oznaka za nepoznatu i promenljivu, i to već kod učenika prvog razreda osnovne škole. Autori smatraju da se i na ovako ranom uzrastu deca mogu naučiti da razmišljaju na sofisticiraniji način o promenljivim količinama, kao i da ih pri tom označavaju koristeći simbole. Ključnu ulogu u shvatanju ideje promenljive i simbola za njeno označavanje treba da imaju različite vrste nestandardnih oblika označavanja, što uključuje i prirodni jezik. Naše istraživanje pokazalo je slične rezultate, pri čemu se slažemo sa autorima da se kroz dugoročna iskustva sa simboličkim notacijama može graditi i pojam promenljive, a samim tim i preduprediti kasniji potencijalni problemi u shvatanju ovog pojma. Tako je istraživanje Risted i saradnika (Rystedt et al., 2016) pokazalo da su dvanaestogodišnji učenici sposobni da koriste bogatstvo različitih kontekstualnih resursa. Učenici su uspeli da koriste širok spektar interpretacija slova, kao simbola za oznaku promenljive ili nepoznate, ali da su u tom odabiru pokazali određenu neodlučnost. Njihovo istraživanje je pokazalo da je dijalog između dece predstavljao ključnu pomoć u prelasku sa primitivne na napredniju interpretaciju simbola, što potvrđuje značaj situacija svakodnevnog života, ali i moć reči i govora svakodnevnog života.

Do rezultata sličnih našim dolazi i Radford (Radford, 2022) u istraživanju u kojem su, umesto korišćenja zadataka koji uključuje otvorene aritmetičke jednakosti ili jednačine, korišćene priče-problemi kroz vizuelne interpretacije kako bi se formirao pojam jednačine, znaka jednakosti i pojma nepoznate. Ove priče bile su uokvirene u narative koji su omogućavali nastavniku i učenicima formiranje jednačina kroz kontekstualna značenja. Rezultati su pokazali bolje rezultate u relacionom razumevanju delova jednačine i dublje razumevanje matematičkih operacija koje igraju centralnu ulogu u uprošćavanju jednačina.

Naše istraživanje potvrdilo je rezultate drugih istraživača, kao što je istraživanje Van Ruvejka (Van Reeuwijk, 2001), u kojem su rezultati pokazali potrebu učenika za korišćenjem različitih veština i alata za manipulisanje u procesu rešavanja jednačina. Proces učenja započinje od problema realnog konteksta, u čijem rešavanju učenici imaju potrebu da samostalno razvijaju strategije kojima će pomoći učeniku da savlada formalnu algebarsku notaciju, samim tim i pravilnu upotrebu simbola u algebri.

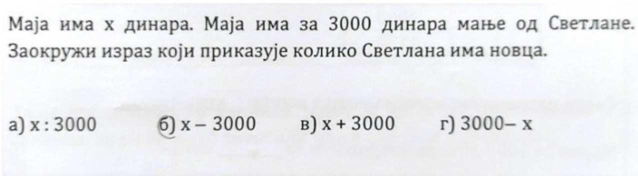
Karakteristične greške učenika u razumevanju algebarske simbolike

Analizom radova učenika želeli smo da sagledamo i identifikujemo tipične greške koje su učenici pravili na zadacima iz testa, i tako proniknemo u suštinu dečjeg razumevanja slova kao oznake za nepoznatu i promenljivu u algebarskom izrazu. Ako se u obzir uzme način na koji učenik shvata odnose između promenljivih, analiza radova je pokazala da je najveći broj grešaka učenika vezan za upotrebu slova kao oznake za promenljivu u algebarskom izrazu, što ćemo ilustrovati kroz analizu Primera 1.

Primer 1. *Maja ima h dinara. Maja ima za 3000 dinara manje od Svetlane. Zaokruži izraz koji prikazuje koliko Svetlana ima novca.*

a) $x : 3000$ b) $x - 3000$ c) $x + 3000$ d) $3000 - x$

Slika 1
Rešenje zadatka sa Majom i Svetlanom – inicijalni test


Маја има х динара. Маја има за 3000 динара мање од Светлане. Заокружи израз који приказује колико Светлана има новца.

а) x : 3000 б) x – 3000 в) x + 3000 г) 3000– x

Određeni broj učenika se u rešavanju navedenog zadatka odlučio za rešenje pod b) (Slika 1). Razlog tome možemo pronaći u nepravilnom shvatanju odnosa između suma novca koje imaju Maja i Svetlana. Ovde se radi o takozvanoj „grešci preokreta", u kojoj se doslovno prebacuje značenje koje simbol kao promenljiva ima u problemu. Do sličnih rezultata u svom istraživanju dolaze i Veinberg i saradnici (Weinberg et al., 2016), koji izvor *greške preokreta* pronalaze u nepotpunom razumevanju simbola, odnosno nerazumevanju pojma promenljive ili nepoznate i znaka jednakosti. Ova vrsta greške nastaje usled nepotpunog razumevanja simbola, nepoznate veličine ili znaka jednakosti. Ukoliko se zadaci predstave u realnom kontekstu bližem iskustvu učenika, veća je verovatnoća da će oni uspeti da izgrade smisaono razumevanje simbola i da izbegnu ovakve tipične greške. Stoga, razmatranje „greške preokreta" služi kao pokazatelj kako uvođenje realnih situacija može ublažiti ili prevazići teškoće u simboličkom mišljenju.Neke od zahteva koji se očekuju kod učenika mlađeg školskog uzrasta i prelasku na algebarsku notaciju izdvaja Ferari (Ferrari, 2006), koji smatra da jezičke kompetencije učenika treba da omoguće učenicima da u međusobnoj komunikaciji postanu svesni prelaska na algebarsku notaciju. Značajno je potpuno aktivno učešće učenika i u nematematičkim aktivnostima, dok je uloga učitelja od presudnog značaja u određivanju učeničkih ciljeva i aktivnosti u tako organizovanoj nastavi. Rezultati istraživanja koje je sprovela Van Amerom (Van Amerom, 2002) pokazali su da se simbolizacija teško razvija u učenju i nastavi algebre na mlađem školskom uzrastu. Tačnije, rezultati pokazuju da su neki od učenika, pod uticajem realističnih problema, sposobni da rasuđuju o nepoznatom, ali da će zapis i dalje ostati aritmetički. Isto istraživanje je pokazalo da čak i matematički nadareni učenici, koji postižu uspeh u algebarskom rezonovanju, ipak imaju slabo razvijenu sposobnost simbolizacije. Sa druge strane, istraživanja pokazuju da su učenici starijih razreda (9. razred) sposobni da razumeju i koriste algebarske vizuelne reprezentacije kada se to od njih zahteva, ali da se ipak u objašnjavanju jednačina i nejednačina oslanjaju na standardne algoritme, što je u vezi sa boljim algebarskim zaključivanjem (Ünal et al., 2023).Analiza učeničkih grešaka u korišćenju simbola kao znakova za nepoznatu ili promenljivu u skladu je sa ciljem istraživanja, jer upravo one pružaju uvid u nivo razumevanja i način na koji učenici interpretiraju algebarske simbole. Greške nisu samo propusti, već dragoceni pokazatelji procesa mišljenja, kao i tipičnih teškoća sa kojima se učenici susreću u ranoj algebri. U tom smislu, analiza ovih grešaka omogućava da se jasnije sagledaju razlike između kontrolne i eksperimentalne grupe, kao i efekti primenjenog pristupa u razvijanju simboličke kompetencije.

Analiziraćemo neke od primera grešaka karakterističnih za upotrebu simbola kao oznake za promenljivu ili nepoznatu, koje su se pojavile kod dela učenika kontrolne grupe u finalnom testiranju. U Primeru 2 smo prikazali nekoliko najčešćih grešaka u rešavanju zadataka koji su bili karakteristični kod učenika u upotrebi simbola kao oznake za nepoznatu ili promenljivu. Učenici su rešavali sledeći zadatak (Primer 2).

Primer 2. *Bojana i Marko imaju istu količinu novca svako u svojoj kasici. Bojana u ruci ima još 200 dinara.*

Izrazi koliko novca ima Marko. ________________

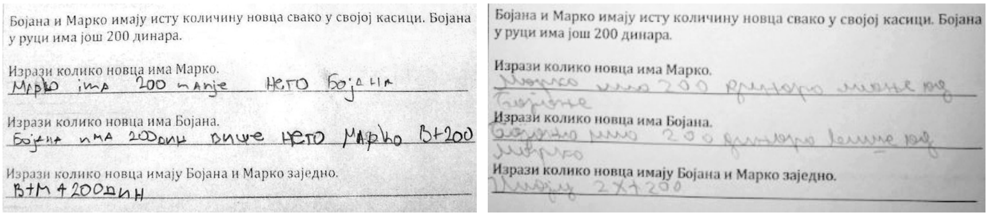
Izrazi koliko novca ima Bojana. ________________

*Izrazi koliko novc imaju Bojana i Marko zajedno.*________________

Na slici 2 prikazane su neke tipične greške koje su pravili učenici prilikom rešavanja zadatka.

Slika 2

Rešenje zadatka sa Bojanom i Markom – finalni test

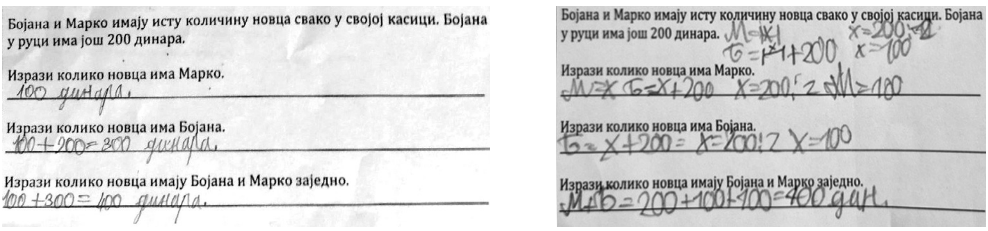


U navedenim rešenjima Primera 2 može se uočiti da učenik nije sposoban da na pravi način iskoristi simbol kao oznaku za nepoznatu količinu novca Marka i Bojane. U ovim primerima rešenja učenici su koristili reči da bi izrazili količinu novca svakog od dvoje dece. Dakle, učenici su bili u situaciji da promenljive zanemaruju ili odbace, već umesto toga koriste reči kojima pokušavaju utvrditi i izraziti odnose. Na potrebu za izražavanjem generalizacija i matematičkih istina korišćenjem reči, pre uvođenja algebarske notacije ukazali su Rasel i saradnici (Russell et al., 2011). Tek u primeru koji se odnosio na zajedničku količinu novca, učenici koriste simbole kao oznaku za nepoznatu količinu novca. Slične greške u rezultatima učenika su pokazalo je istraživanje Stivensa i saradnika (Stephens et al., 2015) koje je imalo sličan karakter kao i naše.

Pored toga, određeni broj učenika je imao potrebu da u rešavanju ovog zadatka potpuno izbegne nepoznatu, tako da je umesto simbola nepoznatim davao tačno određenu vrednost (Slika 3).

Slika 3

Rešenje zadatka sa Bojanom i Markom – finalni test



U nemogućnosti da shvate ideju nepoznate, učenici izbegavaju zapisivanje algebarskom notacijom. Nasuprot tome, deo njih se odlučuje da je bolje dati proizvoljnu određenu vrednost, kako bi se ispunili zahtevi u zadatku. Generalno govoreći, zadatak rešen na ovaj način daje rešenje koje je empirijski tačno, ali izraženo konkretnim brojem bez generalizacija. Ista pojava zapažena je u istraživanju koje su sproveli Karaher i saradnici (Carraher et al., 2008), gde je deo učenika u rešenju izbegavao da koristi promenljive, već je umesto njih nepoznatoj količini davao proizvoljnu vrednost. Neka od istraživanja, kao što je i istraživanje Stejsija i Mekgregora (Stacey & MacGregor, 1999), pokazuju da se čak i učenici starijeg uzrasta pre orijentišu na aritmetički način rešavanja problema, nego na rešavanje korišćenjem simbola (algebarski zapis – jednačina). Do sličnih rezultata u svom istraživanju dolazi i Zeljić (Zeljić, 2014), koja navodi da su učenici imali tendenciju da promenljivoj dodeljuju numeričku vrednost, da zanemaruju ili ignorišu promenljivu, da slovo tretiraju kao konkretan objekat ili da pogrešno razumeju matematičku strukturu problema na koji se promenljiva odnosi. Na sličan način i Ozgeldi (Özgeldi, 2013) u svom istraživanju dolazi do zaključka da neki učenici nisu sposobni da predstavljaju generalizacije sve dok ne postanu sposobni da manipulišu upotrebom promenljive. Kako bi se podržalo razumevanje generalizacija, od pomoći mogu biti različite koncepcije promenljivih. U njih spada i prazno mesto u zapisu ili crtež i skica koja ima ulogu promenljive ili nepoznate u prvom susretu učenika sa idejom nepoznate ili promenljive. Korišćenjem reprezentacija, učenici podstiču svoje matematičko mišljenje konstruisanjem apstraktne ideje, kao što su algebarski pojmovi u konkretne ideje, koristeći se pri tom logičkim mišljenjem (Goldin, 2020).

Tako Radford (Radford, 2018) u istraživanju dolazi do zaključka da prirodni jezik sa svojim širokim sistemom mogućnosti može ponuditi kvalitetan semiotički materijal za stvaranje kontekstualnih generalizacija, ali da se isti mora vremenom povući u pozadinu kako bi njegovo mesto zauzeo novi kognitivni oblik simbolizacije. Ova kontekstualna uopštavanja mogu se razumeti kao ikonički oblik označavanja koji će vremenom prerasti u čist algebarski simbolizam. Rezultati ovog istraživanja potvrdili su prethodno rečeno, jer su učenici u prelasku na simboličku notaciju često koristili različite vrste ikoničkih reprezentacija u obliku skica ili slika, kako bi pojednostavili odnose i problemsku situaciju približili sopstvenom mišljenju.

Zaključak

Na osnovu dobijenih rezultata, može se zaključiti da primenjeni kontekstualni pristup, koji je zasnovan na realnim životnim situacijama, može pozitivno uticati na sposobnost učenika da pravilno razumeju simbole kao oznake za promenljivu ili nepoznatu. Ovaj zaključak je posebno važan ako uzmemo u obzir činjenicu da je simbol jedan od najznačajnijih pojmova u algebri, kao i da je značajan za kasnije usvajanje apstraktnijih algebarskih sadržaja. Vrednost jednog takvog pristupa, koji karakteriše visok stepen očiglednosti, kroz procese prelaska sa nižeg na viši nivo apstraktnog mišljenja modeliranjem situacija, prvenstveno se ogleda u njegovoj efikasnosti. Posebno je važno naglasiti da usvajanje takvog pristupa u matematičkom obrazovanju može dovesti do razvoja složenijih matematičkih pojmova, gde simbolička notacija postaje osnova za stvaranje generalizacija, kao i za druge grane matematike. U Republici Srbiji je program matematike tako osmišljen da podržava ideju uvođenja algebarskih sadržaja uz oslanjanje na aritmetiku od prvog razreda osnovne škole. Imajući u vidu tu činjenicu, kao i činjenicu da smo inicijalnim testiranjem pokazali određene nedostatke u razumevanju sadržaja algebre na uzrastu učenika četvrtog razreda, možemo reći da može i treba primenjivati model učenja zasnovan na situacijama realnog konteksta i procesa matematizacije. Za razvoj algebarskog mišljenja, od presudnog značaja je razumevanje i upotreba slova kao oznake za nepoznatu ili promenljivu, a put do razumevanja njegovog značenja može biti kroz pažljivo odabrane sadržaje oblikovane u realnom kontekstu. Tako Sfard i Linčevski (Sfard & Linchevski, 1994) smatraju da učenik, dok se suočava sa jednačinama i nejednačinama, mora biti u stanju da se kreće između operativnog pristupa kada se njegova misao usredsređuje na procese (predstavljene algebarskim izrazima) i strukturalni pristup, kada se fokusiraju na apstraktne objekte koji se kriju iza simbola (Sfard & Linchcvski, 1994). Upravo u ovom odnosu se i ogleda značaj kontekstualnog pristupa koji omogućava učeniku da u svakom trenutku njegova misao putuje između algebarske notacije i realnog konteksta, pri čemu mu na tom putu može pomoći model koji učenik u postupku učenja sam gradi.

Podaci dobijeni u istraživanju Jupri i saradnika iz TIMCS testiranja (Jupri et al., 2014), pokazali su značajne probleme kod učenika u učenju iz oblasti algebre u mnogim državama širom sveta. Najveće poteškoće se odnose na razumevanje osnovne algebarske forme, ali posebno funkciju nepoznate i promenljive sadržane u algebarskom izrazu. Ako se ova činjenica ima u vidu, posebno se ističe značaj dobijenih rezultata u našem istraživanju.

Na osnovu dobijenih rezultata možemo zaključiti da je značajna prednost kontekstualnog pristupa u tome što on u nastavni proces uvodi sve faze modelovanja – razumevanje realne situacije, transformaciju situacionog modela u matematički model i postepeni prelaz na formalnu algebarsku notaciju. Upravo izostajanje ovih faza u radu sa učenicima kontrolne grupe može objasniti slabije rezultate koje su postigli u pravilnom razumevanju simbola. Za razliku od njih, učenici iz eksperimentalne grupe imali su priliku da problem sagledaju kroz realističan kontekst, da ga modeluju različitim predstavama, a zatim da ga prevedu u simboličku formu. Ovakvo postepeno prelaženje iz realnog u simboličko doprinelo je dubljem razumevanju funkcije simbola u algebri.

Ograničenje ovog istraživanja odnosi se na činjenicu da su u eksperimentalnoj i kontrolnoj grupi nastavu realizovali različiti nastavnici, što je moglo uticati na ishode nezavisno od primenjenog pristupa. Dizajn istraživanja bio bi metodološki snažniji ukoliko bi u svakoj školi bilo obuhvaćeno više odeljenja koja bi služila kao eksperimentalne i kontrolne grupe, čime bi se umanjio uticaj individualnih stilova nastavnika. U budućim istraživanjima moglo bi se razmotriti kako obezbediti da sve grupe učenika budu u što ravnopravnijem položaju, odnosno kako nakon istraživanja omogućiti i kontrolnoj grupi pristup istim benefitima koji su pruženi eksperimentalnoj.

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A Route to Understanding Symbols in Algebra: from Real-life Situations to Symbolic Language

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Abstract *In this paper, the authors present a learning framework for introducing letters as symbols for the unknown and variable in mathematics, a framework that is based on real-life situations, their modeling and translation into algebraic language. In the empirical part of the paper, we conducted a research with the aim of investigating the effects of such an approach on the development of algebraic symbolic thinking ability of the students in the early elementary school grades. The research was conducted on a sample of 257 fourth-grade students of primary school in the Republic of Serbia, by using an experimental method with parallel groups. The research results show positive effects of the implemented approach on a better understanding of symbolic notation and a greater degree of algebraic generalization. These results confirm the hypothesis that the introduction of algebraic symbols through real-life problems and context-related activities can significantly improve students' ability to use algebraic notation and interpret it correctly.*
In this paper, the authors present a learning framework for introducing letters as symbols for the unknown and variable in mathematics, a framework that is based on real-life situations, their modeling and translation into algebraic language. In the empirical part of the paper, we conducted a research with the aim of investigating the effects of such an approach on the development of algebraic symbolic thinking ability of the students in the early elementary school grades. The research was conducted on a sample of 257 fourth-grade students of primary school in the Republic of Serbia, by using an experimental method with parallel groups. The research results show positive effects of the implemented approach on a better understanding of symbolic notation and a greater degree of algebraic generalization. These results confirm the hypothesis that the introduction of algebraic symbols through real-life problems and context-related activities can significantly improve students' ability to use algebraic notation and

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interpret it correctly. In this paper, the authors present a learning framework for introducing letters as symbols for the unknown and variable in mathematics, a framework that is based on real-life situations, their modeling and translation into algebraic language. In the empirical part of the paper, we conducted a research with the aim of investigating the effects of such an approach on the development of algebraic symbolic thinking ability of the students in the early elementary school grades. The research was conducted on a sample of 257 fourth-grade students of primary school in the Republic of Serbia, by using an experimental method with parallel groups. The research results show positive effects of the implemented approach on a better understanding of symbolic notation and a greater degree of algebraic generalization. These results confirm the hypothesis that the introduction of algebraic symbols through real-life problems and context-related activities can significantly improve students' ability to use algebraic notation and interpret it correctly.

Keywords: *algebra, contextual approach, mathematics, unknown, variable.*

Introduction

Students' first encounter with symbols and the language of mathematics occurs in the early stages of mathematics education and arithmetic learning. By forming the concept of natural number and its symbolic representation, children begin to learn about mathematical symbols, translate real-life situations into mathematical notations and solve problems. At this age, introducing algebraic symbolic notation and entering into the field of algebra represent a particular challenge for students. It is exactly this first encounter with algebra that is considered a "critical moment for most people who decide that they do not have a knack for mathematics and that they do not understand such mathematical ideas" (Milinković & Maričić, 2022, p. 245). In arithmetic, children operate with numbers, while in algebra they demonstrate their ability to manipulate numerical and symbolic representations, that is, to use and understand the role of letters in mathematical notations. Introducing symbolic algebraic notation represents a great challenge not only for practitioners, but also for theoreticians and researchers of mathematics education who are curious as to how and when mathematical symbols should be introduced (Brizuela et al. 2015; Carraher et al., 2008; Dabić Boričić, 2019; Milinković & Maričić, 2022, Radford, 2011).

From Arithmetic to Algebra in the Development of Symbolic Language

The research by Christou et al. (2005) shows that the students' experience of letter signifiers in algebra is strongly influenced by their experience with numbers in arithmetic learning. In fact, a successful transition to algebra, which is based on solid arithmetic foundations, may also be useful in the problem-solving process (Akkan et al., 2011). Similar results were obtained by Sun et al. (2023), whose research shows that it is necessary to identify the elementary arithmetic content in the curricula on the basis of which algebraic symbolism is to be introduced. The development of students' early algebraic thinking follows a route from "arithmetic thinking" over "generalized algebraic thinking" to "symbolic algebraic thinking". Therefore, it can be said that "while […] mathematically algebraic symbolization is a generalization of arithmetic, cognitively it is more accurate to say [that] algebra symbolization is the articulation of arithmetic" (Heffernan & Koedinger, 2022, p. 484).

However, it must be noted that one of the reasons for misinterpreting the concepts of unknown and variable, as well as symbols used to represent them, lies in the fact that letters are also used in arithmetic, although in a completely different manner. Letter signifiers in arithmetic are mostly abbreviations of the names of objects, directly referring to the concepts themselves, while the role of letters in algebra is entirely different. Such understanding and such use of letters on the students' part can lead to errors and a deeper misunderstanding of symbols as signs for variables. When it comes to solving algebraic problems and developing algebraic thinking, two important stages can be identified: 1) the transition from verbal to symbolic notation, 2) the transition from arithmetic to algebra. Each one of them is conditioned by specific relations that exist among the students' thinking, the teaching process and the characteristics of algebra (Milinković i Maričić, 2022).

When the meaning of a symbol is taken into consideration, Kieran (1981) believes that the process is directly related to the formation of the concept of equation. That way, starting from arithmetic and Bruner's modes of representation (enactive, iconic and symbolic), students can intuitively understand the meaning of equivalence and then gradually transform it into the understanding of a proper form of equation. This way, students' understanding of algebra would become anchored in arithmetic. This process would start from arithmetic equivalence where the first step would involve using a finger to hide one of the numbers and then replacing the finger with one of the placeholders, so that the whole process would end in introducing a symbol as a sign for the unknown. This way, the letter hiding the number becomes the unknown – the term which closely corresponds to the idea of the hidden number which can be determined.

If the teaching of algebra is taken into consideration in addition to all of the above, the following question can be asked: *Which method should be used in introducing letters and algebraic notation in early mathematics education and how to make sure that the process follows a natural route?* To answer this question, one must take the complex and abstract nature of algebraic concepts into account, as well as the fact that students in the earlier grades of elementary school have difficulty in understanding the concepts of unknown and variable (Akgün & Özdemir, 2006; Carraher et al., 2008; Stephens et al., 2015) and also in using symbols to represent them (Booth, 1988; MacGregor & Stacey, 1997; McNeil et al., 2010; Specht, 2005).

The problem students at this age are asked to solve should be related to the development of the concept of symbol in algebraic notation in the form of:

- the unknown (the hidden number, the number whose value is to be recovered, as an unknown, but fixed value – equations);
- the variable (a set of numbers which meet specific requirements for functional dependency – inequalities, functions, sequences);

- "the free variable (the symbol representing any number in a given set of numbers – generalization)" (Milinković i Maričić, 2022, p. 251).

Real-world Context as a Starting Point for Introducing Algebraic Symbolism

The process of forming the concepts of variable and unknown, as well as symbols used to represent them, lies in the unity of meaning and sign, i.e. the unity of the concept and the symbol that stands for the concept. In that process, the context as a carrier of the concept meaning plays a decisive role. This context, expressed by means of ordinary language, helps make sense of generalization, thus clarifying such an abstract concept to the students. For this reason, the basis for the development of the concept of symbol and its meaning through the role of the unknown and variable must be set in real-life contexts as well as problems related to what students experience on a daily basis. Therefore, the importance of understanding letters in an algebraic expression is directly related to the context of the problem. Rystedt et al. (2016) believe that using diverse contexts and conversations about them can facilitate the transition from primitive to more advanced interpretations of symbols, where the context plays an important role in expressing the relations among quantities. On the other hand, according to Radford (2002), verbal language allows people to create texts which correspond to the described actions and are equipped with a range of possibilities for clarifying and adequately representing the meanings, which is not the case with algebraic notation. Constructing a symbolic notation for a textual problem requires a different approach, since the problem situation develops in the way it is being read, i.e., from left to right, while the starting point in a symbolic notation does not have a fixed position. In a symbolic notation, the order of writing an expression is completely different and depends on the very nature of the relations among quantities. For that reason, Ferrari (2006) believes that, in the problem-solving process, students should use context-specific abbreviations first, while these same abbreviations are later generalized, thus expressing the character of the unknown used in any problem. According to the same author, in that case, it depends both on the language competences of the students and their full engagement in achieving the activity goals.

According to Van Reeuwijk (2001), in many curricula, algebra topics are introduced very quickly, so that the students barely have time to develop an understanding of algebraic concepts. There is often no connection made with the existing non-formal knowledge of the students. For students, it is a meaningless representation and use of symbols. For these reasons, teaching of algebra should be based on the situations which are closely related to the students' personal experience and their non-formal education. These situations should be woven into arithmetic and into the context that is experientially close to the students and meaningful learning situations. On the other hand, students should be advised to use a variety of forms of visual and schematic representations in solving real-context problems, so as to be able to understand the relations and succeed in representing them (Milinković et al., 2022).

A real context is viewed as a situation that is experientially close to the students, thus being the starting point for the students in rediscovering mathematical truths. Accordingly, the goal of using real contexts is to create appropriate situations, which will enable the construction of the students' formal mathematical knowledge, that is, the use of symbols as signs for the unknown and variable. The basis for using such an approach in teaching lies in the fact that learning mathematics should have the characteristics of learners' cognitive growth. That is, it should not be viewed solely as a process of putting pieces of mathematical knowledge together. This way, the learning process is based on constructing mathematical knowledge and models on the basis of students' real-life experiences.

The problem expressed in the form of a real context may be an important motivating factor not only in developing the concept of variable, but also in introducing a symbol that represents it. Consider, for instance, the example given by Arcavi (1994), in which the students were presented with a problem in the form of a photograph showing a vehicle entering a tunnel with number "2.90" written on top of the entrance. The students were asked to interpret the meaning of the number. Some of the guesses were that the number refers to the weight of a passing vehicle, its width and height. Additionally, relevant measurement units were also discussed. The students eventually agreed that the sign could only refer to the "maximum height allowed for a vehicle to pass", listing allowable vehicle heights in addition. In the end, the students were asked to write that all down in a mathematical way. Introducing symbols in that way was related to one's understanding of a concrete situation, but on a deeper level it was also related to the meaning of the variable, which is one of the most important goals of this activity, since the signifier cannot exist without its' meaning nor can a meaning exist without a signifier (Radford, 2004).

This opinion is shared by Carraher et al. (2008) who believe that algebra classes in the early grades of primary school should be built on background contexts of the mathematical problems, gradually introducing formal notation and bringing all types of mathematical content closer together. For example, the authors placed two identical boxes of candies in front of the children. The students were told that the boxes contain the same number of candies. There were additional three candies placed on top of one of the boxes. The students were then assigned a task where they had to express the number of candies using symbols. A large number of the students tried to solve the problem by assigning a particular value to the unknown quantity, whereas the others tried to solve the problem by making use of drawings in order to mitigate the abstractness of the problem. From the perspective of algebra, there is only one correct solution to the problem, although, from a logical perspective, each solution expressing the relation where one of the boxes contains 3 candies more than the other box is correct. The results suggest that the students might be able to shift the focus from individual elements to sets of the elements and their interrelations. This way, a real context serves as the background for describing the relations among different physical quantities, which eventually leads to making generalizations in the form of formal representations, so that "the mathematical object is no longer the single case or value but more likely the relation, i.e. the functional relationship between two variables" (Carraher et al., 2008, p. 247).

Ordinary language makes it possible to make sense of generalization, convey information and supplement the context that is missing in algebraic notation. This is why the students' natural language and the problems expressed through real-life situations, which represent a good basis for understanding algebraic concepts and developing algebraic abilities, are the starting point for introducing the unknown and variable.

Modeling Realistic Situations as a Pathway to the Conceptual Development of Algebraic Symbolism

The first step in introducing and developing algebraic symbolization is the creation of a real-world context in which students can identify the problem, think about its solution and build a foundation for the gradual cognitive transition from concrete instances to an abstract, symbolic level. Realistic situations function as an initial stimulus that allows the abstract idea of a variable or an unknown to emerge from a student's experience. Consequently, this method ensures gradual mathematization (Freudenthal, 1991), in which a bridge is built through multiple levels of representation between everyday experience and the formal algebraic language. The framework for such learning can be represented through the following stages:

(1) a real-life situation expressing an algebraic problem;

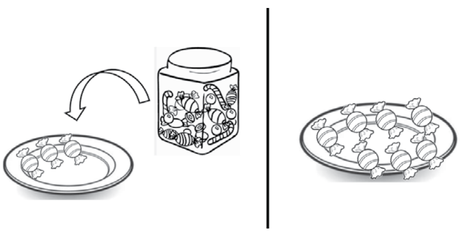
(2) modeling of a real-life situation by using different ways of representing a problem;

(3) translating a concept into the concrete field of referential algebraic language by using algebraic symbols and operations (Milinković & Maričić, 2022).

To illustrate the above mentioned learning stages, we will give a few examples of real-context problems, which may serve as a basis for introducing symbols as signs for the unknown and variable to the students in the early grades of elementary school.

Example 1

Picture 1
Example of a real-context problem
How many candies have been added to the plate?



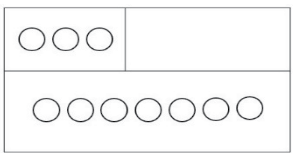
Based on a problem expressed in the form of a real-life situation (Picture 1), the students can draw conclusions about the relations and amounts of candies in the problem. The main problem is related to being aware of the existence of an unknown number, so that the students are now in a position where they can make gradual transition to the field of formal algebraic notation by relying on an obvious example. Since the number of candies on the plate before and after adding candies on top of the box is obvious, it can be concluded that the number of candies added is not known. The formation of the concept of unknown number begins without using any symbols, so that it can also be mathematically represented as an equivalent to a "placeholder", which in this sense represents an unknown addend:

$$3 + \quad = 7$$

This way, the problem is reduced to the mathematical form which can be expressed through the following question: Which number should be added to 3 in order to obtain 7? Thus, a real-context problem serves as a basis for the transition to the language of formal algebra, while the process of solving the problem can be reduced to the students' former experience in summing numbers to 10. The mentioned way of representation (3 + ____ = 7) enables students to perceive the structure of the expression and the concept of the unknown before the abstract symbol (x) is introduced, as particularly emphasized by Radford (2000) and Kieran (1981), who point out that these tasks facilitate the transition from arithmetic to algebraic thinking. According to them, *placeholder* tasks enable students to develop a sense of the unknown quantity within a concrete context, before formally adopting the symbolic language of algebra. Such tasks represent a step towards gradual symbolization, since, through the use of placeholders, students first understand the idea of an unknown value, then learn how to represent it using a letter symbol, and later to perform an operation with it. This way, placeholders function as a bridge between arithmetic and algebra, thus introducing students to a new way of thinking that involves abstraction and generalization.

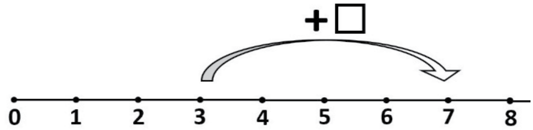
The next step in the formation of the concept of unknown, as well as of the symbol used to represent it, can be based on pictures and ideographs, which depict somewhat more abstract forms (Picture 2).

Picture 2
Modeling of a problem through an ideograph



Further distancing from the concrete situation marks the transition to the modeling of the situation on a number line (Picture 3).

Picture 3
Modeling of a problem on a number line



Using these models can largely mitigate the abstractness of algebraic notation, thus creating a basis for introducing the concept of an unknown number "x", as the number whose value can be determined. The problem expressed in the form of real-life situations and graphic representations, such as the one above, may serve as a basis for writing an equation in an exclusively algebraic form:

$$3 + h = 7$$

The solution to the new problem is following:

$$h = 4$$

Finally, the students conclude that the number of the candies added is 4. When it comes to verifying if the solution to the equation is correct, the task is to investigate whether the expression on the left-hand side is equal to the number on the right-hand side of the equation, that is:

$$3 + 4 = 7$$

$7 = 7$, which means that the solution is correct.

Example 2

Milica, the librarian, organizes a certain number of books on the shelves within a month. Milica organizes 120 books less than the librarian Petar. Write an expression which shows the number of books Petar organizes within a month.

Again, the problem-solving process starts with the analysis of a context-based problem and then enters into the field of abstract mathematics. The students, thinking about the relations that exist in the problem, can conclude that: if the librarian Milica organizes 120 books less than the librarian Petar within a month, it also means that the librarian Petar organizes 120 books more than the librarian Milica within a month. Based on this, as well as the fact that the number of books Milica organizes within a month is represented by an *x*, it can be concluded that the number of books Petar organizes within a month is h + 120. In thinking about mathematical relations, the students' thoughts are completely within the domain of abstract mathematics, separated from the real-world problem. This way, the relation between the meaning and the symbol used to represent it is established.

Example 3

Miloš scored a certain number of points in the test and passed it. On the second test, he scored 12 points more than in the first one. Does it necessarily mean that he also passed the second test? Why? How would you write the number of points Miloš scored in the second test in a mathematical way?

Given the data in this mathematical problem, we can say that there is a variable, that is, a value which is not precisely determined. In this case, it is the number of points Miloš scored in the first test. By analyzing the relations among the data given in the problem, the students can conclude that there is an unknown number of points in the first test and also that Miloš scored 12 points more than that in the second test. In addition, to be able to answer the question whether Miloš passed the second test, when thinking about relations, the students must also take into consideration the other unknown quantity, that being the number of points necessary to pass the test, so that it is impossible to give the correct answer to this question. When it comes to thinking about relations, the final step in the mathematization process is symbolization in which the relations between the unknown number of points in the first and second tests are expressed using symbols (in this case, the number of points Miloš scored in the first test: x; the number of points Miloš scored in the second test: $x + 12$).

Through these concrete examples we wanted to present a method in which letter symbols can be introduced using real-context problems. By relying on real-life problems and the modeling processes, we tried to overcome the problem of understanding symbols which are inseparable from their meanings. This way, it is ensured that symbols and their meanings exist as a unity in the real context, thus supporting the construction of a new mathematical reality.

On the basis of these ideas, we wanted to investigate whether the above mentioned approach to introducing letters in algebra teaching contributes to the development of the ability of the correct understanding of symbols as signifiers for the unknown or variable in algebra content in the lower grades of primary school.

Methodological Framework

Research Sample

The research sample (N = 257) was drawn from the population of students in the fourth grade of primary school in the Republic of Serbia. In the research, we used an experimental method with parallel groups. The experimental group comprised the students from five classes of one primary school (N = 130), while the control group consisted of the students from five classes of two other schools (N = 127). The groups were homogenized in terms of the students' school achievement, social status, sex, and the homogeneity of groups was controlled by using an analysis of covariance.

Research Instruments

The technique used in the research was testing, and the research instrument was a test. For the purposes of our research, and according to the adopted research method, two equivalent test forms were designed: an initial test – to investigate the initial level of understanding of symbols as signs for the unknown and variable – and a final test – to determine the effects of the experimental program. The problems in the test expressed real situations or everyday life problems in which using symbols was important for understanding algebra content.

Test reliability was assessed by means of Cronbach's alpha as an indicator of the reliability of our tests (the initial and final tests). It attempts to measure the internal consistency of the scale, that is, of its constituent items (Table 1).

Table 1
Values of Cronbach's alpha coefficient for the tests

	Cronbach's Alpha	Cronbach's Alpha Based on Standardized Items	N of Items
Initial test	.795	.885	10
Final test	.813	.926	10

Note. N – number of items; Cronbach's Alpha – internal consistency reliability coefficient; values above 0.7 indicate acceptable reliability.

The values of Cronbach's alpha coefficient (Table 1) for the initial and final tests were found to be greater than 0.7, which is considered an acceptable and desirable level of reliability by Pallant (2017).

At the beginning of the experiment, we performed initial testing to determine the level of understanding of symbols as signifiers for unknowns and variables in both groups. Following the initial testing, and in accordance with the contextual approach, we introduced the experimental program into the work of the experimental group. Both tests contained the problems aimed to test the level of understanding of symbols as signs for the unknown and variable.

Procedure

Mathematics lessons in all groups were conducted in accordance with the regular curriculum prescribed by the Ministry of Education of the Republic of Serbia. The experimental program was implemented during regular mathematics classes over a period of three months and it consisted of 27 classes. The content covered by the research included: equations with addition, subtraction, multiplication and division; inequalities with addition and subtraction; and the functional dependence between the results and the elements of the arithmetic operations of addition, subtraction, multiplication and division. This method includes all types of content in which a symbol represents an unknown, a variable, or a free variable.

The experimental group followed the program containing specially designed lessons in which, by modeling real contexts, the students were taught algebra content referring to the correct understanding of symbols as signs for unknowns and variables.. The lesson began by situating the problem in a real-life context, where the students were encouraged to identify the unknown quantities and practice representing those quantities by means of algebraic notation. The students then tried to solve a few such problems with the help of the teacher, using the same methodology. After working on the problem together, the students tried to solve the problems from the textbook on their own, by using the adopted terminology and algebraic notation for solving the problem, eventually verifying the correctness of the results.

The program was directly focused on: (a) using real-world contexts for developing algebraic concepts; (b) understanding the relationship between verbal representations and algebraic expressions; (c) structural understanding of letter expressions as mathematical objects; and (d) modeling real-life situations through the use of various representations (verbal, didactic tools, diagrams or pictures, symbolic). The teachers, who were responsible for implementing the program in the experimental group, completed a five-hour training program before the experimental phase started. The training program was organized by the research team and included methodological instructions necessary for the implementation and use of the prepared materials. During the implementation, the teaching content was delivered in close collaboration with the researchers, which enabled greater consistency of procedures and also increased the reliability of the obtained results.

The control group followed the mathematics curriculum that is based on teacher-centered instruction, which was not based on real-life situations and modeling, but instead, on the frameworks provided in the textbook. Both the experimental and control groups used the same mathematics textbooks. A typical lesson involved working on examples through explanations, solving problems on the blackboard and individual student problem-solving. The emphasis was on the procedure of problem-solving and correct symbolic manipulation, rather than on the procedure of modeling or using multiple representations in the learning and problem-solving process. The use of the didactic materials, diagrams and tasks, which require translating real-world contexts into algebraic expressions, was either limited or absent. The work in the control group primarily focused on problem-solving procedures (e.g., correctly solving equations) rather than on conceptual understanding and knowledge transfer. The teachers in the control group did not receive any additional training apart from their regular professional development in school.

Data Analysis and Processing

The obtained results were analyzed both quantitatively and qualitatively. In the quantitative analysis of the collected data, we used an analysis of variance and analysis of covariance in order to determine possible statistically significant differences between the achievements of each group. In the qualitative analysis, the solutions from the final test were analyzed, with particular attention given to the analysis of the procedures and methods used to solve the problems. The study is focused especially on the way the students approached problem-solving and the errors they made during the process. The qualitative analysis included the solutions of all students, regardless of the group they belonged to.

Results and Discussion

The Effects of a Contextual Approach on the Understanding of Algebraic Symbolism

The results of the initial testing (Table 2) show that the experimental group (M = 4.62; SD = 3.07) has scored almost the same number of points as the control group (M = 4.56; SD = 2.94). The analysis of variance ($F(1, 255) = .023$, $p = .881$) of the results in the initial testing shows that there are no statistically significant differences between the experimental and control groups in the correct understanding of symbols in early algebra (Table 3). The values of the Levene's test for the initial testing ($F(1, 255) = .074$; $p = .786$) (Table 3) show that the assumption of the equality of variances is not violated. Therefore, the analysis of these results' variance can be considered reliable.

Table 2
Achievements of the students from the experimental and control groups regarding the ability to correctly understand the use of symbols in algebra in the initial and final testing

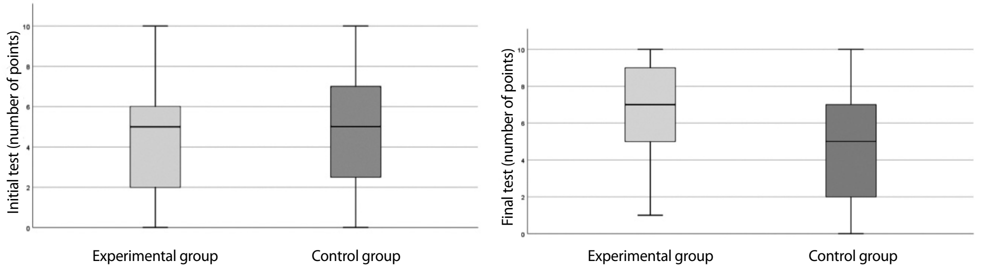
		N	*M*	*SD*
Initial test	Experimental group	130	4.62	3.07
	Control group	127	4.56	2.94
Final test	Experimental group	130	6.89	2.41
	Control group	127	4.60	2.83

Note. *N* – number of participants; *M* – arithmetic mean; *SD* – standard deviation.

The results of the final testing show that the experimental group has achieved better results (M = 6.89; SD = 2.41) than the control group (M = 4.60; SD = 2.83) , on average, with regard to the measured ability (Table 1). In contrast to the experimental group, the control group has scored almost the same average number of points as in the initial testing (Figure 1).

Figure 1

Achievements of the students from the experimental and control groups in the understanding of symbols in algebra



To determine the differences in the development of the ability to correctly understand symbols in algebra, we performed an analysis of variance. The values of Levene's test for the final testing ($F(1, 255) = 4.796$; $p = .209$) show that the assumption of the equality of variances is not violated either. Therefore, the analysis of these results' variance can be considered reliable.

Table 3

Analysis of variance in the initial and final testing of the ability to correctly understand symbols in algebra

	Levene Statistic	df1	df2	Sig.
Initial test	.074	1	255	.786
Final test	4.796	1	255	.209

		Sum of Squares	df	Mean Square	F	Sig.
Initial test	Between Groups	.204	1	.204	.023	.881
	Within Groups	2306.076	255	9.043		
	Total	2306.280	256			
Final test	Between Groups	338.031	1	338.031	48.837	.000
	Within Groups	1765.012	255	6.922		
	Total	2103.043	256			

Note. Levene Statistic – value of Levene's test for equality of variances; df – degrees of freedom; Sig. – significance level; F – Fisher's test value; Between/Within Groups – variance sources in ANOVA model.

The results of the analysis of variance in the final testing ($F(1, 255) = 48.837$; $p = .000$) show that there is a statistically significant difference between the control group's and the experimental group's achievements. The obtained results demonstrate that, statistically, under the influence of the experimental program, the students from the experimental group have significantly better results compared to those from the control group, considering the level of the ability of understanding the use of symbols in algebra correctly.

To make sure that the obtained results are reliable and not influenced by the possible non-homogeneity between the groups, we performed an analysis of covariance. This way, we wanted to investigate the effect of the experimental program, which is based on the principles of the contextual approach in teaching algebra, on the students' ability to understand the concept of symbol in algebra correctly. The value of the Levene's test (F(1, 255) = 1.059; p = .305) shows that the assumption of the equality of variances is not violated. Consequently, we can perform an analysis of covariance to determine the differences (Table 3).

Table 4
Levene's test of analysis of variance

Dependet Variable: Final measurement

F	*df1*	*df2*	*Sig.*
1.059	1	255	.305

Note. F – value of the Fisher's test; df_1 – degrees of freedom for the factor (between groups); df_2 – degrees of freedom for the error; *Sig.* – level of statistical significance (p-value).

By analyzing covariance in the process of eliminating covariates (the results of the initial testing), we found statistically significant differences between the experimental and control groups in the final testing of the students' ability to understand symbols in algebra (F(1, 254) = 117.010; p = .000) (Table 4). This result indicates significant differences between the experimental and control groups in testing this ability. Partial eta squared value is .315, which indicates a large effect. This further implies that 31.5% of the variance in the final testing (of the ability to understand symbols) can be accounted for by the independent variable (i.e., the group – the implemented teaching method). If the effect of the covariates on the final testing is considered (the results of the final testing of the ability to understand symbols) and after the effect of the independent variable (i.e., the group) is eliminated, it can be concluded that there are also statistically significant differences (F(1, 254) = 377.835; p = .000). This implies that a contextual approach to learning contributed significantly to the development of the ability to understand symbols, irrespective of the student's prior knowledge. On the other hand, when controlling the effect of the independent variable (group), the results of the initial measurement – as a covariate – account for as much as 59.8% of the variance in the final achievement (F(1,254) = 377,835; p = ,000). This suggests that the initial level of knowledge is the strongest individual predictor of subsequent understanding of symbols in algebra. Therefore, although prior knowledge plays a dominant role in predicting success, a mode of instruction (contextual approach) also has a significant effect on the development of the ability to understand algebraic symbols.

Table 5
Analysis of covariance in the initial and final testing of the ability to correctly understand symbols in algebra

Dependet Variable: Final measurement						
Source	*Type III Sum of Squares*	*Df*	*Mean Square*	*F*	*Sig.*	*Partial Eta Squared*
Corrected Model	1393,502[a]	2	696,751	249,421	0,000	0,663
Intercept	536,250	1	536,250	191,966	0,000	0,430
Initial measurement	1055,471	1	1055,471	**377,835**	**0,000**	**0,598**
Group	326,864	1	326,864	**117,010**	**0,000**	**0,315**
Error	709,541	254	2,793			
Total	10626,000	257				
Corrected Total	2103,043	256				

Note. *Type III Sum of Squares* – measure of variance; *F* – ratio of mean squares; *Sig*. – significance level; *Partial Eta Squared* – measure of effect size; R^2 – coefficient of determination.

Based on the analysis of the obtained results, we can conclude that teaching algebra in accordance with the principles of the contextual approach and real-life situations can positively influence the development of the students' ability to correctly understand symbols as signs for the unknown or variable. The results show that there are statistically significant differences between the two groups after the experiment is conducted, hence, it can be concluded that the contextual approach positively influences the correct understanding of symbols in algebra. A symbol represents an important concept in algebra and for that reason it is extremely important that this concept is correctly understood from the earliest age. Letter sign (symbol) is an important concept in early algebra which, in some way, as a sign, refers to algebra as a special branch of mathematics. A symbol has the power to reduce a signification and express complex and different laws in a simple way, thus facilitating communication. Through the present research, we showed that the teaching based on real-life contexts can have a positive effect on the students' correct understanding of the concept of symbols in algebra. In teaching algebra in the early grades of primary school, the concept of symbol (the letter used for representing the unknown and variable) is essentially related to other concepts such as equations, inequalities, sequences, formulae, etc. Regarding the teaching of mathematics, students often use symbols as signs in which the letter signifies the object it refers to, which may hinder the correct interpretation of letters used as symbols in early mathematics education (McNeil et al., 2010).

Contrary to these results, the analysis of the students' solutions in the present research shows that a certain number of the students used letters to represent objects or persons, but that such use of symbols did not hinder algebraic manipulations and that the students always managed to produce the correct solution to the problem. Contrary to the results of the aforementioned studies in our research, the students used various interpretations as symbols for the unknown or variable, but they also manipulated them as standard algebraic symbols.

The obtained results are compatible with the findings of Stephens et al. (2015) who showed that the concepts of unknown and variable, as well as the symbols used to represent them, can be developed through an adequate approach and practice of algebraic skills using real-context problems. Therefore, the authors suggest that algebra education in the lower grades of primary school can alleviate the difficulties the students encounter in learning algebra in the higher grades of primary school. Based on the obtained data, we can conclude that the problem expressed in the form of a real-life context can facilitate the interpretation and understanding of the key concept and role of a symbol in mathematical equivalence or non-equivalence, equations or inequalities. In favor of these findings, the study of Blanton et al. (2017) even indicated some examples of the possibility for students in the first grade of primary school to be able to understand letters as symbols for the unknown and variable. The authors believe that even at such an early age, children can be taught to think in a more sophisticated way about variable quantities and to represent them using symbols. A key role in understanding the concept of variable and a symbol used to represent it should be played by different types of non-standard forms of representations, including the natural language. Our research resulted in similar findings, and we agree with these authors that long-term experience in using symbolic notation can also help develop the concept of variable, thus preventing possible problems related to the understanding of this concept. The study conducted by Rystedt et al. (2016) showed that 12-year-old students were capable of using a wealth of different contextual resources. The students managed to use a wide range of letter interpretations as symbols for the unknown or variable, but they were also somewhat uncertain in their choice. The research also showed that the dialogue among children was most helpful in the transition from a primitive to a more advanced interpretation of symbols, which confirms the significance of real-life situations, as well as the power of everyday life words and speech.

Our results are similar to those obtained by Radford (2022) in his research where, instead of using problems involving open arithmetic equivalence or equations, story-problems were used through visual interpretations, to form the concepts of equation, equal sign and variable. The stories were framed in narratives, allowing the teacher and the students to create equations using contextual meanings. The research indicated better results in the relational understanding of the equated parts, as well as a deeper understanding of mathematical operations, which play a key role in simplifying equations.

Our research has confirmed the results of other studies, such as the one by Van Reeuwijk (2001) where the results indicated the students' need for using various skills and manipulating tools in the process of solving equations. The learning process begins with the real-context problem in solving of which the students feel the need to independently develop strategies that will help them master formal algebraic notation and consequently the correct use of symbols in algebra.

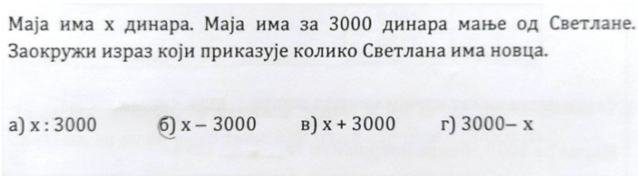
Typical Student Errors in Understanding Algebraic Symbolism

Through the analysis of the students' works, we aimed to consider and identify the typical errors the students made in solving the problems on the test and to gain deeper insight into the essence of children's understanding of a letter as a signifier for the unknown and the variable in an algebraic expression. If the way in which the student understands the relationships between variables is taken into consideration, the analysis of the students' works showed that the majority of the errors were related to using a letter as a sign for the variable in an algebraic expression, which will be illustrated through the analysis of Example 1 below.

Example 1 *Maja has x dinars, which is 3000 dinars less than Svetlana. Circle the expression that correctly expresses the amount of money Svetalana has.*

a) x : 3000 b) x – 3000 c) x + 3000 d) 3000 – x

Figure 1.
Solution to the problem with Maja and Svetlana


Маја има х динара. Маја има за 3000 динара мање од Светлане.
Заокружи израз који приказује колико Светлана има новца.

а) х : 3000 б) х – 3000 в) х + 3000 г) 3000– х

A certain number of the students chose the solution under b) (Figure 1). The reason for this may be the students' misunderstanding of the relation between the amounts of money Maja and Svetlana have. Namely, it concerns the so-called "reversal error", where the meaning of a symbol used as a variable in the problem is literally reversed. Similar results are obtained by Weinberg et al. (2016), who showed that the *reversal error* is caused by the incomplete understanding of symbols and the lack of understanding of the concepts of variable or unknown and the equal sign.

This type of error occurs due to an incomplete understanding of symbols, the unknown quantity or the equals sign. When the problems are presented in a real-world context which is closer to students' experience, it is more likely that they will develop a meaningful understanding of symbols and avoid making these typical errors. Thus, examination of the "reversal error" serves as an indicator of how an introduction of real-life situations can actually help to mitigate or overcome difficulties in symbolic reasoning.

Ferrari (2006), who believes that students' language competences should enable them to become aware of the transition to algebraic notation in their communication with other students, identified some of the requirements the students should meet in the early grades of primary school and in the transition to algebraic notation. It is important that students fully and actively participate in non-mathematical activities as well, while the role of the teacher is of key importance in setting the students' goals and activities in the teaching process organized in such a way. The results obtained by Van Amerom (2002) showed that symbolization is difficult to develop in the learning and teaching of algebra in the early grades of primary school. Specifically, her results showed that, although some of the students, influenced by problems based on real life, were capable of reasoning about the unknowns, their notation remained at the level of arithmetic. The same study showed that even mathematically gifted students who are very successful in algebraic reasoning have a poorly developed ability of symbolization. On the other hand, studies also show that the students in the higher grades of primary school (the ninth grade) are able to understand and use algebraic visual representations when asked to do so, but that in explaining equations and inequalities they rely on the standard algorithms, which is associated with advanced algebraic reasoning (Ünal et al., 2023).

The analysis of students' errors in using symbols as representations of the unknown or the variable aligns with the aim of the study, since such errors offer valuable insight into the level of understanding and in the ways in which students interpret algebraic symbols. Errors are not perceived merely as mistakes, but they serve as important indicators of the thinking process and the typical difficulties students encounter in early algebra. In this sense, the analysis of these errors provides a clearer view of the differences between the control and experimental groups, as well as of the effects of the applied approach on the development of symbolic competence.

We will also analyze some examples of errors characteristic of the use of symbols as signs for the unknown or variable that were recorded in the solutions produced by some of the students from the control group in the final testing. In Example 2 below, we show a few most common errors in the students' solutions to the problems which asked them to use symbols for the unknown or variable. The students were given the following problem to solve (Example 2).

Example 2. *Bojana and Marko have the same amount of money, each in their own piggy bank. Bojana has another 200 dinars in her hand.*

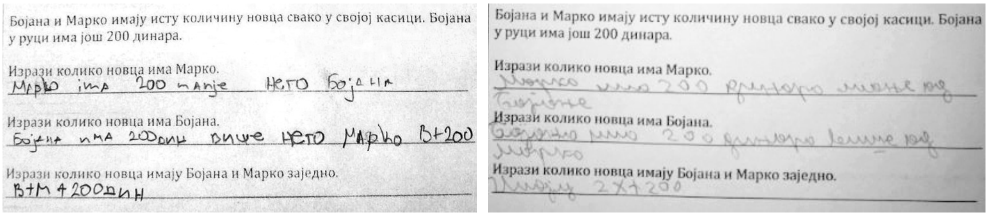
Write an expression showing how much money Marko has. ________________

Write an expression showing how much money Bojana has. ________________

Write an expression showing how much money Bojana and Marko have. ________________

Figure 2 shows some of the typical errors made by the students in solving the problem.

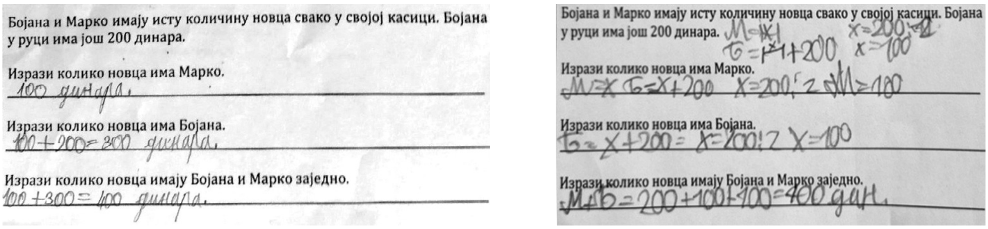
Figure 2.
Solution to the problem with Bojana and Marko



In the above mentioned solutions to Pictures 2, it can be observed that the student is not able to use a symbol as a sign for Marko's and Bojana's unknown amounts of money. In these examples, the students used words to express the amount of money each child has. Therefore, the students were in a situation where they either ignored or discarded variables. Instead, they used words by means of which they tried to determine and express the relations existing in the problem. The need for expressing generalizations and mathematical truths through words, prior to the introduction of algebraic notation, was also highlighted by Russell et al. (2011). The students used symbols as signs for the unknown amount of money only in the example referring to the amount of money Bojana and Marko have together. Similar errors in the students' solutions were also reported in the study by Stephens et al. (2015), which was similar in character to our research.

In addition, a certain number of the students felt the need to completely avoid using the unknown in the process of solving the problem, and, instead of using symbols, assigned particular values to the unknowns (Figure 3).

Figure 3.
Solution to the problem with Bojana and Marko



Being unable to understand the concept of unknown, the students avoided using algebraic notation in writing expressions. Contrary to that, some students thought it was better to provide an arbitrary, but known value in order to meet the problem's requirements. Generally speaking, the problem solved in this way results in the solution that is empirically correct, but is expressed by means of concrete numbers, without any generalizations. Similarly, in the study by Carraher et al. (2008), a number of the students avoided using variables in the solution, by instead assigning an arbitrary value to the unknown amount. Some research studies, such as the one by Stacey & MacGregor (1999), show that even students in the higher grades tend to solve the problem in an arithmetic way, rather than by using symbols (algebraic notation – equation). Similar results were also obtained by Zeljić (2014), who reports that the students tend to assign a numerical value to the variable, to ignore the variable, to treat a letter as a physical object, or to misunderstand the mathematical structure of the problem to which the variable refers to. In a similar manner, Özgeldi (2013) concluded that some of the students were not able to represent generalizations until they learned how to use the variable. Various conceptions of the variables can contribute to the understanding of generalizations. They include an empty space in the notation, a drawing or a scheme representing the variable or unknown in the students' first encounter with the concept of unknown or variable. Through representations, the students can improve their mathematical thinking by translating abstract concepts such as algebraic concepts into concrete ideas, using logical thinking (Goldin, 2020).

In his research, Radford (2018) concludes that the natural language, with its wide range of possibilities, can offer the quality semiotic material for producing contextual generalizations, but that it also has to recede into the background to allow a new cognitive form – symbolization – to take up its space. These contextual generalizations may be understood as an exemplary form of signifying which transforms into pure algebraic symbolization over time. The results of this research study have confirmed the aforementioned, since the students often used different types of iconic representations in the form of schemes or pictures in the transition to symbolic notation in order to simplify the observed relations and bring the problem situation closer to their own thinking.

Conclusion

On the basis of the results obtained herein, it can be concluded that the implemented contextual approach, which is based on real-life situations, can positively influence the students' ability to correctly understand symbols as signs for the variable or unknown. This fact is especially important if we consider the fact that symbol is one of the most significant concepts in algebra and also its significance for the acquisition of more abstract algebraic content later on. The value of one such approach, characterized by a high degree of obviousness, through the processes of the transition from lower to higher level of abstract thinking by modeling the situations, is primarily reflected in its efficiency. It is especially important to stress that adopting such an approach in mathematics education can lead to the development of more complex mathematical concepts, where symbolic notation becomes the basis for producing generalizations as well as for other branches of mathematics. In the Republic of Serbia, the curriculum for mathematics is designed in such a way as to support the integration of the algebraic content from the first grade of primary school by building on arithmetic foundations. Taking this fact into account, as well as the fact that the initial measurement indicated certain deficiencies in fourth-grade students' understanding of algebraic content, we can conclude that the learning model based on real-life situations and the mathematization process can be applied and should be applied.

For the development of algebraic thinking, it is of crucial importance that students understand and use letters as symbols for the unknown or variable, and the route to understanding their meanings can be through carefully selected content modeled in real-life contexts. Sfard and Linchevski (1994) therefore believe that the students, in their encounter with equations and inequalities, must be able to move between an operational approach, in which they focus on processes (represented by algebraic expressions) and a structural approach, in which they focus on abstract objects behind the symbols (Sfard & Linche vski, 1994). It is exactly this relation that reflects the significance of the contextual approach which allows the students' thoughts to travel between algebraic notation and real-life contexts at all times, in the process of which they can benefit from the model they build on their own during the learning process.

The data gathered from TIMSS assessments in the study by Jupri et al. (2014) indicated considerable difficulties in the students' algebra learning in many countries across the world. Major difficulties included the understanding of basic algebraic forms, and especially the function of the unknown and variable in algebraic expressions. In view of this fact, the results obtained in our research are all the more significant.

Based on the obtained results, it can be concluded that a significant advantage of the contextual approach lies in the fact that the contextual approach incorporates all modeling phases into the teaching process – understanding the real-life situation, transforming the situational model into a mathematical model, as well as gradually transitioning to formal algebraic notation. It is exactly the absence of these phases in the work with the students from the control group that explains the weaker results they achieved in proper understanding of the symbols. Unlike them, the students from the experimental group had the opportunity to approach the problem through a real-world context, to model it using various representations, and then to translate it into a symbolic form. This gradual transition from the real to the symbolic has contributed to a deeper understanding of the role of symbols in algebra.

A limitation of the present research refers to the fact that different teachers delivered instruction in the experimental and control groups, which might have influenced the outcomes regardless of the instructional approach applied. The research design would be methodologically stronger if multiple classes, functioning as experimental and control groups, were included in each school, thus reducing the influence of individual teaching styles. Future research could consider how to ensure that all student groups are treated more equally, that is, how to ensure that the control group enjoys the same benefits provided to the experimental group.

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