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Izvođenje transportnih jednačina trodimenzionalnog neizotermnog turbulentnog strujanja u cilindričnim koordinatama

Originalni naučni rad

U radu su dati osnovni izrazi u krivolinijskim koordinatama, a zatim su u nastavku rada redom izvedene u cilindričnim koordinatama: jednačina entalpije (temperature), Navije-Stoksove i Rejnoldsove jednačine turbulentnog strujanja, transportne jednačine Rejnoldsovih (turbulentnih) napona, transportna jednačina kinetičke energije turbulencije i na kraju transportna jednačina brzine disipacije kinetičke energije turbulencije. Ovim radom je pokušano popuniti prazninu u našoj i stranoj literaturi u oblasti turbulentnih transportnih procesa u vezi sa izvođenjem turbulentnih transportnih jednačina u cilindričnim koordinatama. Ovaj rad može biti od koristi studentima diplomskih i doktorskih studija, kao i inženjerima i istraživačima u praksi pri rešavanju inženjerskih problema i modeliranju turbulentnih strujanja koja su tako česta u termotehnici i energetici.
Ključne reči: *izvođenje transportnih jednačina, turbulentno strujanje, temperatura, modeliranje turbulencije, cilindrične koordinate*

Osnovni izrazi u krivolinijskim koordinatama i jednačina kontinuiteta u cilindričnim koordinatama

Prelazak sa Dekartovih pravouglih koordinata x, y, z na u opštem slučaju ortogonalne krivolinijske koordinate radi se kad je potrebno opisati strujni prostor specifičnog oblika. Da bi prešli sa Dekartovih pravouglih koordinata na krivolinijske koordinate uvodimo pretpostavku da su pravougle koordinate x, y, z neprekidne funkcije krivolinijskih ortogonalnih koordinata q_1, q_2, q_3, odnosno:

$$x = x\ \ q_1, q_2, q_3 \ , \quad y = y\ \ q_1, q_2, q_3 \quad \text{и} \quad z = z\ \ q_1, q_2, q_3 \tag{1}$$

Na slici 1 (*a*) prikazane su koordinatne površi $q_i = const., i = 1,\ 2,\ 3,$ koordinatne linije q_i i jedinični vektori $\vec{e_i}$ usmereni po tangentama na koordinatne linije u tački M u stranu porasta odgovarajućih promenljivih q_i. Sa prihvaćenim oznakama [1] i izrazom za

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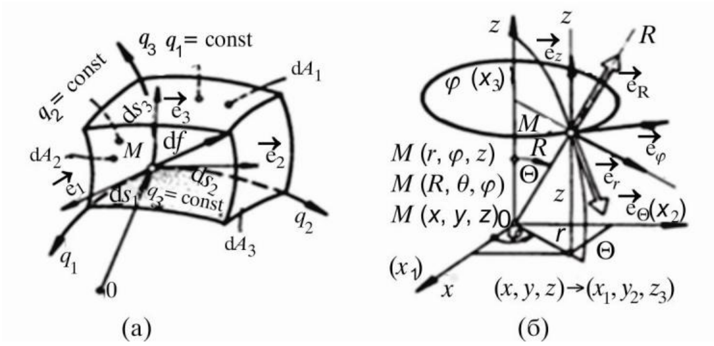
radijus-vektor $\vec{r} = \vec{r}\,(q_1, q_2, q_3)$ slede relacije:

$$\frac{\partial \vec{r}}{\partial q_i} = H_i \vec{e}_i, \quad \mathrm{d}\vec{r} = \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial q_i} \mathrm{d}q_i, \quad \mathrm{d}s_i = H_i \mathrm{d}q_i, \quad \mathrm{d}s^2 \equiv (\mathrm{d}\vec{r})^2 = \sum_{i=1}^{3} H_i^{\,2} \mathrm{d}q_i^{\,2},$$
$$\mathrm{d}A_i = \mathrm{d}s_j \mathrm{d}s_k = H_j H_k \mathrm{d}q_j \mathrm{d}q_k, \quad \mathrm{d}V = \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 = H_1 H_2 H_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3, \tag{2}$$

gde su $\mathrm{d}A_i$ – površina stranica elementarnog paralelopipeda, dV – zapremina elementarnog paralelopipeda, čije su dužine stranica $\mathrm{d}s_i$, $\mathrm{d}s^2$ kvadrat dužine luka, tj. elementa $\mathrm{d}\vec{r}$, $i, j, k = 1, 2, 3$ – čine cikličnu permutaciju brojeva 1, 2 и 3.



Slika 1. Ortogonalne krivolinijske koordinate

Veličine H_i označavaju Lameove koeficijente (mere promenu vektora položaja od tačke do tačke u prostoru) određene izrazima:

$$H_i = \sqrt{(\partial x / \partial q_i)^2 + (\partial y / \partial q_i)^2 + (\partial z / \partial q_i)^2}, \qquad i = 1, 2, 3, \tag{3}$$

$$\text{jer je } H_i = |\partial \vec{r} / \partial q_i| = \left| (\partial x / \partial q_i)\vec{i} + (\partial y / \partial q_i)\vec{j} + (\partial z / \partial q_i)\vec{\mathrm{k}} \right|, \quad |\vec{\mathrm{e}}_i| = 1 \text{ i } (\vec{\mathrm{e}}_i, \vec{\mathrm{e}}_k) = 0 \text{ за } \mathrm{i} \neq \mathrm{k}$$

Saglasno izrazima (3) i oznakama na slici 1. Lameovi koeficijenti u cilindričnim koordinatama imaju vrednosti:

$$H_1 = H_r = \sqrt{\left(\frac{\partial (r\cos\varphi)}{\partial r}\right)^2 + \left(\frac{\partial (r\sin\varphi)}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = \sqrt{\cos^2\varphi + \sin^2\varphi + 0} = 1$$
$$H_2 = H_\varphi = \sqrt{\left(\frac{\partial (r\cos\varphi)}{\partial \varphi}\right)^2 + \left(\frac{\partial (r\sin\varphi)}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = \sqrt{r^2\sin^2\varphi + r^2\cos^2\varphi + 0} = r$$
$$H_3 = H_z = \sqrt{\left(\frac{\partial (r\cos\varphi)}{\partial z}\right)^2 + \left(\frac{\partial (r\sin\varphi)}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = \sqrt{0 + 0 + 1} = 1 \tag{4}$$

Gradijent skalarnog polja $U = U(q_1, q_2, q_3)$ u krivolinijskim ortogonalnim koordinatama definisan je izrazom:

$$\mathrm{grad}\, U = \lim_{\substack{A \to 0 \\ V \to 0}} \frac{1}{V} \oint_A U \vec{n} \mathrm{d}A = \sum_{i=1}^{3} \frac{1}{H_i} \frac{\partial U}{\partial q_i} \vec{e}_i \tag{5}$$

gde su značenja za A, V и $\vec{n}$ data pri analizi formule (6).

Divergencija vektorskog polja $\vec{u}=\vec{u}\;q_1,q_2,q_3$ u tački M, saglasno njenoj definiciji, dobija se posmatranjem ukupnog fluksa (protoka) vektora $\vec{u}=u_i\vec{e}_i$ kroz površ A koja ograničava zapreminu beskonačno malog krivolinijskog paralelopipeda v. sl. 1(a), i u krivolinijskom sistemu koordinata ima oblik:

$$\operatorname{div}\vec{u}=\lim_{\substack{V\to 0\\ V\to M}}\frac{1}{V}\oint_A \vec{n},\vec{u}\;\mathrm{d}A=\frac{1}{H_1H_2H_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(\frac{H_1H_2H_3u_i}{H_i}\right) \tag{6}$$

u kome $\vec{n}$ označava jedinični vektor spoljašnje normale površi A, a $\mathrm{d}A_i$ и V, tj. $\mathrm{d}V$ su definisane formulama (2).

Na osnovu definicije rotora vektorskog polja $\vec{u}=u_i\vec{e}_i$ dobija se izraz za $\operatorname{rot}\vec{u}$ u krivolinijskim ortogonalnim koordinatama, koji prikazan pomoću determinante glasi:

$$\operatorname{rot}\vec{u}=\lim_{V\to 0}\frac{1}{V}\oint_A \vec{n},\vec{u}\;\mathrm{d}A=\begin{vmatrix}\frac{1}{H_2H_3}\vec{e}_1 & \frac{1}{H_1H_3}\vec{e}_2 & \frac{1}{H_1H_2}\vec{e}_3\\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3}\\ H_1u_1 & H_2u_2 & H_3u_3\end{vmatrix} \tag{7}$$

Shodno obrascu (6) dobija se sledeći diferencijalni oblik jednačine kontinuiteta u ortogonalnim krivolinijskim koordinatama:

$$\frac{\partial\rho}{\partial t}+\frac{1}{H_1H_2H_3}\sum_{k=1}^{3}\frac{\partial}{\partial q_k}\left(\frac{H_1H_2H_3\rho u_k}{H_k}\right)=0 \tag{8}$$

Jednačina kontinuiteta u cilindričnim koordinatama $q_1=r,\,q_2=\varphi,\,q_3=z$ dobija se uvrštavanjem relacija (4) u izraz (8), tako da sledi:

$$\frac{\partial\rho}{\partial t}+\frac{1}{1r1}\frac{\partial}{\partial r}\left(\frac{1r1\rho u_r}{1}\right)+\frac{1}{1r1}\frac{\partial}{\partial\varphi}\left(\frac{1r1\rho u_\varphi}{r}\right)+\frac{1}{1r1}\frac{\partial}{\partial z}\left(\frac{1r1\rho u_z}{1}\right)=0 \tag{9}$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\;r\rho u_r\;+\frac{1}{r}\frac{\partial}{\partial\varphi}\;\rho u_\varphi\;+\frac{\partial}{\partial z}\;\rho u_z\;=0$$

Jednačina entalpije (temperature)

Jednačina entalpije u opštem obliku glasi:

$$\rho\frac{\partial h}{\partial t}+\rho\;\vec{u},\nabla\;h=-p\cdot\nabla\cdot\vec{u}-\nabla\cdot\vec{q}+\underline{\underline{\mathrm{T}}}^{\text{напона}}:\nabla\vec{u}+\vec{u}\cdot\nabla p+G_e \tag{10}$$

pri čemu je $h=u_e+p/\rho=c_pT$ i $\vec{q}=-k_p\cdot\nabla T$ dok tenzor napona u cilindričnim koordinatama ima sledeći oblik:

$$\underline{\underline{\mathrm{T}}}=\begin{bmatrix} 2\mu\dfrac{\partial u_r}{\partial r} & \mu\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial\varphi}+\dfrac{\partial u_\varphi}{\partial r}-\dfrac{u_\varphi}{r}\right) & \mu\left(\dfrac{\partial u_z}{\partial r}+\dfrac{\partial u_r}{\partial z}\right) \\ \mu\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial\varphi}+\dfrac{\partial u_\varphi}{\partial r}-\dfrac{u_\varphi}{r}\right) & 2\mu\left(\dfrac{u_r}{r}+\dfrac{1}{r}\dfrac{\partial u_\varphi}{\partial\varphi}\right) & \mu\left(\dfrac{\partial u_\varphi}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial\varphi}\right) \\ \mu\left(\dfrac{\partial u_z}{\partial r}+\dfrac{\partial u_r}{\partial z}\right) & \mu\left(\dfrac{\partial u_\varphi}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial\varphi}\right) & 2\mu\dfrac{\partial u_z}{\partial z} \end{bmatrix} \tag{11}$$

gde je sa u_e označena unutrašnja energija, a sa G_e generisanje toplote u jedinici zapremine.

Gradijent brzine u cilindričnom koordinatnom sistemu [2] ima sledeći oblik:

$$\nabla\vec{u}=\begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\varphi}{\partial r} & \dfrac{\partial u_z}{\partial r} \\ \dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial\varphi}-u_\varphi\right) & \dfrac{1}{r}\left(\dfrac{\partial u_\varphi}{\partial\varphi}+u_r\right) & \dfrac{1}{r}\dfrac{\partial u_z}{\partial\varphi} \\ \dfrac{\partial u_r}{\partial z} & \dfrac{\partial u_\varphi}{\partial z} & \dfrac{\partial u_z}{\partial z} \end{bmatrix} \tag{12}$$

Uvrštavanjem izraza (11) i (12) u (10) dobija se sledeći oblik entalpijske jednačine u cilindričnim koordinatama:

$$\rho c_p\left[\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\varphi}{r}\frac{\partial T}{\partial\varphi}+u_z\frac{\partial T}{\partial z}\right]=\underbrace{-p\frac{\partial u_r}{\partial r}-\frac{p}{r}\frac{\partial u_\varphi}{\partial\varphi}-p\frac{\partial u_z}{\partial z}}_{\text{zapreminski rad za nest. fluid jednak je nuli}}+$$

$$+k_p\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial\varphi^2}+\frac{\partial^2 T}{\partial z^2}\right]+\underline{\underline{\underline{\mathrm{T}}}^{напона}:\nabla\vec{u}}+u_r\frac{\partial p}{\partial r}+\frac{u_\varphi}{r}\frac{\partial p}{\partial\varphi}+u_z\frac{\partial p}{\partial z}+G_e \tag{13}$$

Podvučeni član u prethodnoj jednačini predstavlja deformacioni rad [3], tj. deo rada površinskih sila koji dovodi do disipacije, kojom se mehanička energija nepovratno pretvara u toplotu i time dovodi do promene temperature. Podvučeni disipacioni član u jednačini (13) se preuređuje na sledeći način:

$$\begin{aligned}
\Phi &= \underline{\underline{\mathrm{T}}}^{напона}:\nabla\vec{u}= \\
&=\tau_{rr}\left(\frac{\partial u_r}{\partial r}\right)+\tau_{\varphi\varphi}\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r}{r}\right)+\tau_{zz}\left(\frac{\partial u_z}{\partial z}\right)+\tau_{r\varphi}\left(\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\varphi}\right)+ \\
&+\tau_{\varphi z}\left(\frac{1}{r}\frac{\partial u_z}{\partial\varphi}+\frac{\partial u_\varphi}{\partial z}\right)+\tau_{rz}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)= \\
&=2\mu\frac{\partial u_r}{\partial r}\left(\frac{\partial u_r}{\partial r}\right)+2\mu\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}\right)\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r}{r}\right)+2\mu\frac{\partial u_z}{\partial z}\left(\frac{\partial u_z}{\partial z}\right)+ \\
&+\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}\right)\left(\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\varphi}\right)+\mu\left(\frac{\partial u_\varphi}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\varphi}\right)\left(\frac{1}{r}\frac{\partial u_z}{\partial\varphi}+\frac{\partial u_\varphi}{\partial z}\right)+ \\
&+\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)=2\mu\left(\frac{\partial u_r}{\partial r}\right)^2+2\mu\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}\right)^2+2\mu\left(\frac{\partial u_z}{\partial z}\right)^2+ \\
&+\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\varphi}\right]^2+\mu\left(\frac{\partial u_\varphi}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\varphi}\right)^2+\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)^2
\end{aligned} \tag{14}$$

Konačan oblik entalpije (temperature) u cilindričnom koordinatnom sistemu za nestišljiv fluid konstantne toplotne provodljivosti glasi:

$$\rho c_p\left(\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\varphi}{r}\frac{\partial T}{\partial \varphi}+u_z\frac{\partial T}{\partial z}\right)=k_p\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}+\frac{\partial^2 T}{\partial z^2}\right]+$$
$$+2\mu\left(\frac{\partial u_r}{\partial r}\right)^2+2\mu\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi}\right)^2+2\mu\left(\frac{\partial u_z}{\partial z}\right)^2+\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \varphi}\right]^2+$$
$$+\mu\left(\frac{\partial u_\varphi}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \varphi}\right)^2+\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)^2+u_r\frac{\partial p}{\partial r}+\frac{u_\varphi}{r}\frac{\partial p}{\partial \varphi}+u_z\frac{\partial u_z}{\partial z}+G_e \qquad (15)$$

Navije-Stoksove jednačine

Uopštena Njutnova hipoteza o naponima (prema kojoj je tenzor napona linearna funkcija tenzora brzine deformisanja) se piše u obliku:

$$p_{ik}=-p\delta_{ik}+2\mu\dot{S}_{ik}-\frac{2}{3}\mu\delta_{ik}\operatorname{div}\vec{u} \qquad (16)$$

gde su: p_{ik} – komponente tenzora napona; p – pritisak; δ_{ik} – Kronekerov simbol; μ – dinamička viskoznost fluida; $\dot{S}_{ik}$ – komponente tenzora brzine deformisanja, a $\operatorname{div}\vec{u}$ – divergencija brzine definisana formulom (6).

U kinematici fluida izvode se komponente $\dot{S}_{ik}$ u krivolinijskom ortogonalnom sistemu koordinata [1]. U ovom slučaju se navode dve od njih:

$$\dot{S}_{11}=\frac{1}{H_1}\frac{\partial u_1}{\partial q_1}+\frac{u_2}{H_1H_2}\frac{\partial H_1}{\partial q_2}+u_3\frac{\partial H_1}{\partial q_3}$$
$$\dot{S}_{23}=\frac{1}{2}\left(\frac{1}{H_3}\frac{\partial u_2}{\partial q_3}+\frac{1}{H_2}\frac{\partial u_3}{\partial q_2}-\frac{u_2}{H_2H_3}\frac{\partial H_2}{\partial q_3}-\frac{u_3}{H_2H_3}\frac{\partial H_3}{\partial q_2}\right) \qquad (17)$$

Ostale komponente $\dot{S}_{22}$, $\dot{S}_{33}$, $\dot{S}_{13}$ и $\dot{S}_{12}$ se lako dobijaju iz prethodne formule cikličnom permutacijom indeksa 1, 2, 3. Tenzor brzine deformisanja je simetričan $\dot{S}_{ij}=\dot{S}_{ji},\ i\neq j$ tako da je među devet komponenata $\dot{S}_{ik}$ samo šest različitih. To isto, saglasno izrazu (16) važi i za tenzor napona, tj. $p_{ik}=p_{ki}$ za $i\neq k$.

Normalni naponi $i=k$ u viskoznom nestišljivom fluidu u krivolinijski ortogonalnim koordinatama mogu da se saglasno (16) napišu u sledećem obliku [1]:

$$p_{kk}=-p+2\mu\left[\sum_{m=1}^{3}\frac{u_m}{H_mH_k}\frac{\partial H_k}{\partial q_m}+\frac{\partial}{\partial q_k}\left(\frac{u_k}{H_k}\right)\right]\quad \text{(ne sumira se po } k\text{)}. \qquad (18)$$

Za $i\neq k$ iz (16) i (17) slede formule za tangentne napone:

$$p_{ik}=\mu\left[\frac{H_i}{H_k}\frac{\partial}{\partial q_k}\left(\frac{u_i}{H_i}\right)+\frac{H_k}{H_i}\frac{\partial}{\partial q_i}\left(\frac{u_k}{H_k}\right)\right],\ \text{gde se po } i \text{ i } k \text{ ne sumira!} \qquad (19)$$

Kod cilindričnih koordinata važe relacije:

$$q_1=r,\ q_2=\varphi,\ q_3=z;\quad H_1=H_r=1,\ H_2=H_\varphi=r,\ H_3=H_z=1;$$
$$u_1=u_r,\ u_2=u_\varphi,\ u_3=u_z;\quad p_{11}=p_{rr},...;\ p_{12}=p_{r\varphi},...; \qquad (20)$$

Kada se prethodne oznake uvrste u (18) i (19) i obave naznačena diferenciranja dobijaju se sledeće formule za normalne i tangentne napone [4] u cilindričnim koordinatama:

$$
\begin{aligned}
p_{rr} &= -p + 2\mu\left[\frac{u_r}{1\cdot 1}\frac{\partial}{\partial r}\frac{1}{} + \frac{u_\varphi}{r\cdot 1}\frac{\partial}{\partial \varphi}\frac{1}{} + \frac{u_z}{1\cdot 1}\frac{\partial}{\partial z}\frac{1}{} + \frac{\partial}{\partial r}\left(\frac{u_r}{1}\right)\right] = -p + 2\mu\frac{\partial u_r}{\partial r}\\
p_{\varphi\varphi} &= -p + 2\mu\left[\frac{u_r}{1\cdot r}\frac{\partial r}{\partial r} + \frac{u_\varphi}{r\cdot r}\frac{\partial r}{\partial \varphi} + \frac{u_z}{1\cdot r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \varphi}\left(\frac{u_\varphi}{r}\right)\right] = -p + 2\mu\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi}\right)\\
p_{zz} &= -p + 2\mu\left[\frac{u_r}{1\cdot z}\frac{\partial z}{\partial r} + \frac{u_\varphi}{r\cdot z}\frac{\partial z}{\partial \varphi} + \frac{u_z}{1\cdot z}\frac{\partial z}{\partial z} + \frac{\partial}{\partial z}\left(\frac{u_z}{1}\right)\right] = -p + 2\mu\frac{\partial u_z}{\partial z}\\
p_{r\varphi} &= \mu\left[\frac{1}{r}\frac{\partial}{\partial \varphi}\left(\frac{u_r}{1}\right) + \frac{r}{1}\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right] = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right)\\
p_{\varphi z} &= \mu\left[\frac{r}{1}\frac{\partial}{\partial z}\left(\frac{u_\varphi}{r}\right) + \frac{1}{r}\frac{\partial}{\partial \varphi}\left(\frac{u_z}{1}\right)\right] = \mu\left(\frac{\partial u_\varphi}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \varphi}\right)\\
p_{zr} &= \mu\left[\frac{1}{1}\frac{\partial}{\partial z}\left(\frac{u_r}{1}\right) + \frac{1}{1}\frac{\partial}{\partial r}\left(\frac{u_z}{1}\right)\right] = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)
\end{aligned} \tag{21}
$$

Diferencijalne jednačine količine kretanja u krivolinijskim ortogonalnim koordinatama u skraćenoj tenzorskoj formi glase:

$$
\begin{aligned}
&\rho\left[\frac{\partial u_i}{\partial t} + \sum_{k=1}^{3}\left(\frac{u_k}{H_k}\frac{\partial u_i}{\partial q_k} - \frac{u_k^2}{H_i H_k}\frac{\partial H_k}{\partial q_i} + \frac{u_k u_i}{H_k H_i}\frac{\partial H_i}{\partial q_k}\right)\right] =\\
&= \rho F_i + \frac{1}{H_i}\sum_{k=1}^{3}\left[\frac{1}{H_1 H_2 H_3}\frac{\partial}{\partial q_k}\left(\frac{H_1 H_2 H_3 H_i}{H_k}p_{ik}\right) - \frac{p_{kk}}{H_k}\frac{\partial H_k}{\partial q_i}\right]
\end{aligned} \tag{22}
$$

pri čemu se ne vrši sumiranje po indeksu i $i = 1, 2, 3$.

Za cilindrične koordinate se posle izvesnog računa i sređivanja dobijaju sledeće jednačine:

$$
\begin{aligned}
&\rho\left[\frac{\partial u_r}{\partial t} + \frac{u_r}{1}\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi} + \frac{u_z}{1}\frac{\partial u_r}{\partial z} - \frac{u_r^{\,2}}{1}\frac{\partial 1}{\partial r} - \frac{u_\varphi^{\,2}}{r}\frac{\partial r}{\partial r} - \frac{u_z^{\,2}}{1}\frac{\partial 1}{\partial r} + \frac{u_r u_r}{1}\frac{\partial 1}{\partial r} + \frac{u_r u_\varphi}{1}\frac{\partial 1}{\partial \varphi} + \frac{u_r u_z}{1}\frac{\partial 1}{\partial z}\right] =\\
&= \rho F_r + \frac{\partial p_{rr}}{\partial r} + \frac{p_{rr}}{r} + \frac{1}{r}\frac{\partial p_{r\varphi}}{\partial \varphi} + \frac{\partial p_{rz}}{\partial z} - \frac{p_{rr}}{1}\frac{\partial 1}{\partial r} - \frac{p_{\varphi\varphi}}{r}\frac{\partial r}{\partial r} - \frac{p_{zz}}{1}\frac{\partial 1}{\partial r}, \quad \text{odnosno:}\\
&\rho\left[\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\varphi^2}{r}\right] = \rho F_r + \frac{\partial p_{rr}}{\partial r} + \frac{1}{r}\frac{\partial p_{r\varphi}}{\partial \varphi} + \frac{\partial p_{rz}}{\partial z} + \frac{p_{rr} - p_{\varphi\varphi}}{r}
\end{aligned} \tag{23}
$$

Istim postupkom i za preostale dve koordinate [5] dobijamo sledeće izraze:

$$
\rho\left[\frac{\partial u_\varphi}{\partial t} + u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial \varphi} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{u_r u_\varphi}{r}\right] = \rho F_\varphi + \frac{\partial p_{\varphi r}}{\partial r} + \frac{1}{r}\frac{\partial p_{\varphi\varphi}}{\partial \varphi} + \frac{\partial p_{\varphi z}}{\partial z} + \frac{2p_{r\varphi}}{r}
$$

$$
\rho\left[\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_z}{\partial \varphi} + u_z\frac{\partial u_z}{\partial z}\right] = \rho F_z + \frac{\partial p_{rz}}{\partial r} + \frac{1}{r}\frac{\partial p_{z\varphi}}{\partial \varphi} + \frac{\partial p_{zz}}{\partial z} + \frac{p_{rz}}{r}
$$

Kada se izrazi za napone (21) uvrste u jednačine (23) smatrajući pri tome da je $\mu = \text{const.}$, dobijaju se Navije-Stoksove jednačine u cilindričnim koordinatama za strujanje nestišljivog fluida konstantne viskoznosti:

$$
\begin{aligned}
&\rho\left[\frac{\partial u_r}{\partial t} + \ \vec{u},\nabla\ \ u_r - \frac{u_\varphi^2}{r}\right] =\\
&= \rho F_r + \frac{\partial(-p + 2\mu\frac{\partial u_r}{\partial r})}{\partial r} + \frac{1}{r}\frac{\partial\left[\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right)\right]}{\partial \varphi} + \frac{\partial\left[\mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)\right]}{\partial z} +
\end{aligned}
$$

$$+\frac{-p+2\mu\frac{\partial u_r}{\partial r}+p-2\mu\frac{u_r}{r}-2\mu\frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi}}{r}=\rho F_r-\frac{\partial p}{\partial r}+\mu\Delta u_r-\mu\frac{u_r}{r^2}-\mu\frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}$$

$$\rho\left[\frac{\partial u_r}{\partial t}+\ \vec{u},\nabla\ u_r-\frac{u_\varphi^2}{r}\right]=\rho F_r-\frac{\partial p}{\partial r}+\mu\left(\Delta u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}\right) \tag{24}$$

Na isti način se dobijaju jednačine [5] i za preostale dve koordinate:

$$\rho\left[\frac{\partial u_\varphi}{\partial t}+\ \vec{u},\nabla\ u_\varphi+\frac{u_r u_\varphi}{r}\right]=\rho F_\varphi-\frac{1}{r}\frac{\partial p}{\partial\varphi}+\mu\left(\Delta u_\varphi-\frac{u_\varphi}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\varphi}\right) \tag{25}$$

$$\rho\left[\frac{\partial u_z}{\partial t}+\ \vec{u},\nabla\ u_z\right]=\rho F_z-\frac{\partial p}{\partial z}+\mu\Delta u_z \tag{26}$$

pri čemu je izraz za Hamiltonov operator ∇ u cilindričnim koordinatama:

$$\nabla=\vec{i}_r\frac{\partial}{\partial r}+\vec{i}_\varphi\frac{1}{r}\frac{\partial}{\partial\varphi}+\vec{i}_z\frac{\partial}{\partial z} \tag{27}$$

dok Laplasov operator Δ u cilindričnim koordinatama ima sledeće oblik:

$$\Delta\equiv\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2} \tag{28}$$

Rejnoldsove jednačine

Navije-Stoksove jednačine za $r,\ \varphi$ и z koordinatu glase:

$$\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\varphi}{r}\frac{\partial u_r}{\partial\varphi}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\varphi^2}{r}\right]=$$
$$=\rho F_r-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\varphi^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}\right) \tag{29}$$

$$\rho\left[\frac{\partial u_\varphi}{\partial t}+u_r\frac{\partial u_\varphi}{\partial r}+\frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial\varphi}+u_z\frac{\partial u_\varphi}{\partial z}+\frac{u_r u_\varphi}{r}\right]=$$
$$=\rho F_\varphi-\frac{1}{r}\frac{\partial p}{\partial\varphi}+\mu\left(\frac{\partial^2 u_\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial u_\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\varphi}{\partial\varphi^2}+\frac{\partial^2 u_\varphi}{\partial z^2}-\frac{u_\varphi}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\varphi}\right) \tag{30}$$

$$\rho\left[\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\varphi}{r}\frac{\partial u_z}{\partial\varphi}+u_z\frac{\partial u_z}{\partial z}\right]=$$
$$=\rho F_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\varphi^2}+\frac{\partial^2 u_z}{\partial z^2}\right) \tag{31}$$

Primena Rejnoldsove statistike osrednjavanja u vremenu definiše se sledećim izrazima:

$$\overline{f} = \lim_{T \to 0} \frac{1}{T} \int_0^T f\ \ x, y, z, t\ \ \mathrm{d}T, \quad f = \overline{f} + f', \quad \overline{f'} \equiv 0, \quad \overline{f_1 + f_2} = \overline{f_1} + \overline{f_2},$$
$$\overline{\overline{f}} = \overline{f}, \quad \overline{\overline{f_1} f_2} = \overline{f_1}\,\overline{f_2}, \quad \overline{\overline{f_1} f_2^{'}} = 0, \quad \overline{\frac{\partial f}{\partial t}} = \frac{\partial \overline{f}}{\partial t}, \quad \overline{\frac{\partial f}{\partial x}} = \frac{\partial \overline{f}}{\partial x}, \quad \overline{f_1 f_2} = \overline{f_1}\ \overline{f_2} + \overline{f_1^{'} f_2^{'}} \tag{32}$$

gde su: f – trenutna vrednost fizičke veličine, $\overline{f}$ – prosečna, vremenska osrednja vrednost, f' – fluktuacija (odstupanje od prosečne vrednosti) veličine, i T – vreme osrednjavanja.

Vremenskim osrednjavanjem N-S jednačina dobijamo Rejnoldsove jednačine.

Trenutna vrednost promenljive se dekomponuje na usrednjeni deo po vremenu i fluktuacioni deo, tako da je:

$$u_r = \overline{u_r} + u_r^{'},\ u_\varphi = \overline{u_\varphi} + u_\varphi^{'},\ u_z = \overline{u_z} + u_z^{'},\ F = \overline{F} + F',\ p = \overline{p} + p' ...$$

Izvodi se Rejnoldsova jednačina za r koordinatu, vremenskim osrednjavanjem jednačine (29).

Vremenskim osrednjavanjem izraza sa leve strane jednačine (29) dobija se:

$$\rho \left[\overline{ \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r} \frac{\partial u_r}{\partial \varphi} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\varphi^2}{r} } \right] =$$
$$= \rho \left[\overline{ \frac{\partial\ \overline{u_r} + u_r'}{\partial t} + \ \overline{u_r} + u_r'\ \frac{\partial\ \overline{u_r} + u_r'}{\partial r} + \frac{\overline{u_\varphi} + u_\varphi'}{r} \frac{\partial\ \overline{u_r} + u_r'}{\partial \varphi} + \ \overline{u_z} + u_z'\ \frac{\partial\ \overline{u_r} + u_r'}{\partial z} - \frac{\overline{u_\varphi} + u_\varphi'\ ^2}{r} } \right] =$$
$$= \rho \left[\overline{\frac{\partial \overline{u_r}}{\partial t}} + \overline{\frac{\partial u_r'}{\partial t}} + \overline{\overline{u_r} \frac{\partial \overline{u_r}}{\partial r}} + \overline{\overline{u_r} \frac{\partial u_r'}{\partial r}} + \overline{u_r' \frac{\partial \overline{u_r}}{\partial r}} + \overline{u_r' \frac{\partial u_r'}{\partial r}} + \frac{1}{r} \left(\overline{\overline{u_\varphi} \frac{\partial \overline{u_r}}{\partial \varphi}} + \overline{\overline{u_\varphi} \frac{\partial u_r'}{\partial \varphi}} + \overline{u_\varphi' \frac{\partial \overline{u_r}}{\partial r}} + \overline{u_\varphi' \frac{\partial u_r'}{\partial \varphi}} \right) + \right.$$
$$\left. + \overline{\overline{u_z} \frac{\partial \overline{u_r}}{\partial z}} + \overline{\overline{u_z} \frac{\partial u_r'}{\partial z}} + \overline{u_z' \frac{\partial \overline{u_r}}{\partial z}} + \overline{u_z' \frac{\partial u_r'}{\partial z}} - \frac{\overline{\overline{u_\varphi}^2}}{r} - \frac{\overline{2 \overline{u_\varphi} u_\varphi'}}{r} - \frac{\overline{u_\varphi'^2}}{r} \right] =$$
$$= \rho \left[\frac{\partial \overline{u_r}}{\partial t} + \overline{u_r} \frac{\partial \overline{u_r}}{\partial r} + \frac{1}{r} \overline{u_\varphi} \frac{\partial \overline{u_r}}{\partial \varphi} - \frac{\overline{u_\varphi}^2}{r} + \overline{u_z} \frac{\partial \overline{u_z}}{\partial z} \right] + \rho \left[\frac{1}{r} \frac{\partial}{\partial r} r \left(\overline{u_r' u_r'} \right) + \frac{1}{r} \frac{\partial}{\partial \varphi} \left(\overline{u_\varphi' u_r'} \right) + \frac{\partial}{\partial z} \left(\overline{u_z' u_r'} \right) - \frac{1}{r} \left(\overline{u_\varphi' u_\varphi'} \right) \right]$$

Vremenskim osrednjavanjem izraza sa desne strane jednačine (29) dobija se:

$$\overline{ \rho F_r - \frac{\partial p}{\partial r} + \mu \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \varphi^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\varphi}{\partial \varphi} \right) } =$$
$$= \rho\ \overline{\overline{F_r} + F_r'}\ - \frac{\partial\ \overline{\overline{p} + p'}}{\partial r} + \mu \left(\frac{\partial^2\ \overline{\overline{u_r} + u_r'}}{\partial r^2} + \frac{1}{r} \frac{\partial\ \overline{\overline{u_r} + u_r'}}{\partial r} + \frac{1}{r^2} \frac{\partial^2\ \overline{\overline{u_r} + u_r'}}{\partial \varphi^2} + \frac{\partial^2\ \overline{\overline{u_r} + u_r'}}{\partial z^2} - \frac{\overline{\overline{u_r} + u_r'}}{r^2} - \frac{2}{r^2} \frac{\partial\ \overline{\overline{u_\varphi} + u_\varphi'}}{\partial \varphi} \right) =$$
$$= \rho \overline{F_r} - \frac{\partial \overline{p}}{\partial r} + \mu \left(\frac{\partial^2 \overline{u_r}}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{u_r}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \overline{u_r}}{\partial \varphi^2} + \frac{\partial^2 \overline{u_r}}{\partial z^2} - \frac{\overline{u_r}}{r^2} - \frac{2}{r^2} \frac{\partial \overline{u_\varphi}}{\partial \varphi} \right)$$

Izjednačavanjem leve i desne strane vremenski osrednjene N-S jednačine (29) dobija se Rejnoldsova jednačina za r koordinatu:

$$\rho \left[\frac{\partial \overline{u_r}}{\partial t} + \overline{u_r} \frac{\partial \overline{u_r}}{\partial r} + \frac{1}{r} \overline{u_\varphi} \frac{\partial \overline{u_r}}{\partial \varphi} - \frac{\overline{u_\varphi}^2}{r} + \overline{u_z} \frac{\partial \overline{u_z}}{\partial z} \right] = \rho \overline{F_r} - \frac{\partial \overline{p}}{\partial r} +$$
$$+ \mu \left(\frac{\partial^2 \overline{u_r}}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{u_r}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \overline{u_r}}{\partial \varphi^2} + \frac{\partial^2 \overline{u_r}}{\partial z^2} - \frac{\overline{u_r}}{r^2} - \frac{2}{r^2} \frac{\partial \overline{u_\varphi}}{\partial \varphi} \right) + \frac{1}{r} \frac{\partial}{\partial r}\ r \tau_{rr}^t\ + \frac{1}{r} \frac{\partial}{\partial \varphi} \tau_{r\varphi}^t + \frac{\partial}{\partial z} \tau_{rz}^t - \frac{1}{r} \tau_{\varphi\varphi}^t \tag{33}$$

Istim postupkom se dobijaju i jednačine [5] za preostale dve koordinate:

$$\rho\left[\frac{\partial \overline{u_\varphi}}{\partial t}+\overline{u_r}\frac{\partial \overline{u_\varphi}}{\partial r}+\frac{1}{r}\overline{u_\varphi}\frac{\partial \overline{u_\varphi}}{\partial \varphi}+\frac{\overline{u_r}\,\overline{u_\varphi}}{r}+\overline{u_z}\frac{\partial \overline{u_\varphi}}{\partial z}\right]=\rho\overline{F_\varphi}-\frac{1}{r}\frac{\partial \overline{p}}{\partial \varphi}+$$
$$+\mu\left(\frac{\partial^2 \overline{u_\varphi}}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{u_\varphi}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \overline{u_\varphi}}{\partial \varphi^2}+\frac{\partial^2 \overline{u_\varphi}}{\partial z^2}-\frac{\overline{u_\varphi}}{r^2}+\frac{2}{r^2}\frac{\partial \overline{u_r}}{\partial \varphi}\right)+\frac{\partial}{\partial r}\tau^t_{r\varphi}+\frac{1}{r}\frac{\partial}{\partial \varphi}\tau^t_{\varphi\varphi}+\frac{\partial}{\partial z}\tau^t_{z\varphi}+\frac{2}{r}\tau^t_{r\varphi} \tag{34}$$

$$\rho\left[\frac{\partial \overline{u_z}}{\partial t}+\overline{u_r}\frac{\partial \overline{u_z}}{\partial r}+\frac{1}{r}\overline{u_\varphi}\frac{\partial \overline{u_z}}{\partial \varphi}+\overline{u_z}\frac{\partial \overline{u_z}}{\partial z}\right]=\rho\overline{F_z}-\frac{\partial \overline{p}}{\partial z}+$$
$$+\mu\left(\frac{\partial^2 \overline{u_z}}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{u_z}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \overline{u_z}}{\partial \varphi^2}+\frac{\partial^2 \overline{u_z}}{\partial z^2}\right)+\frac{1}{r}\frac{\partial}{\partial r}r\tau^t_{rz}+\frac{1}{r}\frac{\partial}{\partial \varphi}\tau^t_{\varphi z}+\frac{\partial}{\partial z}\tau^t_{zz} \tag{35}$$

gde su $\tau^t_{ij}=\tau^t_{ji}=-\rho\overline{u_i'u_j'}$ u jednačinama (33), (34) i (35) označene komponente tenzora turbulentnih napona, za $i, j = r,\ \varphi,\ z$.

Treba napomeniti da jednačina kontinuiteta važi i za vremenski osrednjene (prosečne) i za fluktuacione brzine:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho\overline{u_r}\right)+\frac{1}{r}\frac{\partial}{\partial \varphi}\left(\rho\overline{u_\varphi}\right)+\frac{\partial}{\partial z}\left(\rho\overline{u_z}\right)=0 \tag{36}$$

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho\overline{u_r'}\right)+\frac{1}{r}\frac{\partial}{\partial \varphi}\left(\rho\overline{u_\varphi'}\right)+\frac{\partial}{\partial z}\left(\rho\overline{u_z'}\right)=0 \tag{37}$$

Transportne jednačine Rejnoldsovih (turbulentnih) napona

U indeksnom zapisu Navije-Stoksova jednačina za nestišljiv fluid ima sledeći oblik:

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=F_i-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_k \partial x_k} \tag{38}$$

Transportne jednačina za turbulentne napone dobijaju se množenjem opšteg oblika Navije-Stoksove jednačine za trenutnu brzinu u_i komponentom u_j zatim izmenom indeksa *i*, *j* pri čemu se dobija druga jednačina, i sabiranjem ovih dveju jednačina. Na kraju se izvrši osrednjavanje po vremenu.

Polazi se od N-S jednačine (29) za *r* koordinatu:

$$\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial t}+\left(\overline{u_r}+u_r'\right)\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial r}+\frac{\overline{u_\varphi}+u_\varphi'}{r}\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial \varphi}+\left(\overline{u_z}+u_z'\right)\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial z}-\frac{\left(\overline{u_\varphi}+u_\varphi'\right)^2}{r}=$$
$$=\overline{F_r}+F_r'-\frac{\partial\left(\overline{p}+p'\right)}{\rho\partial r}+\nu\left(\frac{\partial^2\left(\overline{u_r}+u_r'\right)}{\partial r^2}+\frac{1}{r}\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\left(\overline{u_r}+u_r'\right)}{\partial \varphi^2}+\frac{\partial^2\left(\overline{u_r}+u_r'\right)}{\partial z^2}-\frac{\overline{u_r}+u_r'}{r^2}-\frac{2}{r^2}\frac{\partial\left(\overline{u_\varphi}+u_\varphi'\right)}{\partial \varphi}\right)$$

gde je sa ν označena kinematska viskoznost.

Prethodna jednačina se pomnoži sa u_r':

$$u_r'\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial t}+u_r'\left(\overline{u_r}+u_r'\right)\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial r}+u_r'\frac{\overline{u_\varphi}+u_\varphi'}{r}\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial \varphi}+u_r'\left(\overline{u_z}+u_z'\right)\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial z}-u_r'\frac{\left(\overline{u_\varphi}+u_\varphi'\right)^2}{r}=$$
$$=u_r'\overline{F_r}+u_r'F_r'-u_r'\frac{\partial\left(\overline{p}+p'\right)}{\rho\partial r}+\nu u_r'\left(\frac{\partial^2\left(\overline{u_r}+u_r'\right)}{\partial r^2}+\frac{1}{r}\frac{\partial\left(\overline{u_r}+u_r'\right)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\left(\overline{u_r}+u_r'\right)}{\partial \varphi^2}+\frac{\partial^2\left(\overline{u_r}+u_r'\right)}{\partial z^2}-\frac{\overline{u_r}+u_r'}{r^2}-\frac{2}{r^2}\frac{\partial\left(\overline{u_\varphi}+u_\varphi'\right)}{\partial \varphi}\right)$$

Zbog lakšeg uvida u dalji račun uvodi se smena: $\overline{u_r}=U_r$, $\overline{u_\varphi}=U_\varphi$ i $\overline{u_z}=U_z$!

Zatim se izvrši množenje i vremensko osrednjavanje prethodne jednačine:

$$\underbrace{\overline{u_r'}\frac{\partial U_r}{\partial t}}_{0}+\overline{u_r'\frac{\partial u_r'}{\partial t}}+\underbrace{\overline{u_r'}U_r\frac{\partial U_r}{\partial r}}_{0}+\overline{u_r'u_r'}\frac{\partial U_r}{\partial r}+\overline{u_r'U_r\frac{\partial u_r'}{\partial r}}+\overline{u_r'u_r'\frac{\partial u_r'}{\partial r}}+\frac{1}{r}\left(\underbrace{\overline{u_r'}U_\varphi\frac{\partial U_r}{\partial \varphi}}_{0}+\overline{u_r'u_\varphi'}\frac{\partial U_r}{\partial \varphi}+\overline{u_r'U_\varphi\frac{\partial u_r'}{\partial \varphi}}+\overline{u_r'u_\varphi'\frac{\partial u_r'}{\partial \varphi}}\right)+$$

$$+\underbrace{\overline{u_r'}U_z\frac{\partial U_r}{\partial z}}_{0}+\overline{u_r'u_z'}\frac{\partial U_r}{\partial z}+\overline{u_r'U_z\frac{\partial u_r'}{\partial z}}+\overline{u_r'u_z'\frac{\partial u_r'}{\partial z}}-\frac{1}{r}\left(\underbrace{\overline{u_r'}U_\varphi^2}_{0}+2\overline{u_r'U_\varphi u_\varphi'}+\overline{u_r'u_\varphi'u_\varphi'}\right)=$$

$$=\underbrace{\overline{u_r'}F_r}_{0}+\overline{u_r'F_r'}-\underbrace{\frac{\overline{u_r'}}{\rho}\frac{\partial P}{\partial r}}_{0}-\frac{\overline{u_r'\partial p'}}{\rho\,\partial r}+\nu\left(\underbrace{\overline{u_r'}\frac{\partial^2U_r}{\partial r^2}}_{0}+\overline{u_r'\frac{\partial^2u_r'}{\partial r^2}}+\frac{1}{r}\underbrace{\overline{u_r'}\frac{\partial^2U_r}{\partial r}}_{0}+\frac{1}{r}\overline{u_r'\frac{\partial^2u_r'}{\partial r}}+\frac{1}{r^2}\underbrace{\overline{u_r'}\frac{\partial^2U_r}{\partial \varphi^2}}_{0}+\frac{1}{r^2}\overline{u_r'\frac{\partial^2u_r'}{\partial \varphi^2}}+\underbrace{\overline{u_r'}\frac{\partial^2U_r}{\partial z^2}}_{0}+\overline{u_r'\frac{\partial^2u_r'}{\partial z^2}}\right)+$$

$$+\nu\left(-\underbrace{\overline{u_r'}\frac{U_r}{r^2}}_{0}-\frac{\overline{u_r'u_r'}}{r^2}-\frac{2}{r^2}\underbrace{\overline{u_r'}\frac{\partial U_\varphi}{\partial \varphi}}_{0}-\frac{2}{r^2}\overline{u_r'\frac{\partial u_\varphi'}{\partial \varphi}}\right)$$, tako da ostaje

$$\overline{u_r'\frac{\partial u_r'}{\partial t}}+\overline{u_r'u_r'}\frac{\partial U_r}{\partial r}+\overline{u_r'U_r\frac{\partial u_r'}{\partial r}}+\overline{u_r'u_r'\frac{\partial u_r'}{\partial r}}+\frac{1}{r}\left(\overline{u_r'u_\varphi'}\frac{\partial U_r}{\partial \varphi}+\overline{u_r'U_\varphi\frac{\partial u_r'}{\partial \varphi}}+\overline{u_r'u_\varphi'\frac{\partial u_r'}{\partial \varphi}}\right)+$$

$$+\overline{u_r'u_z'}\frac{\partial U_r}{\partial z}+\overline{u_r'U_z\frac{\partial u_r'}{\partial z}}+\overline{u_r'u_z'\frac{\partial u_r'}{\partial z}}-\frac{1}{r}\left(2\overline{u_r'U_\varphi u_\varphi'}+\overline{u_r'u_\varphi'u_\varphi'}\right)=$$

$$=\overline{u_r'F_r'}-\frac{\overline{u_r'\partial p'}}{\rho\,\partial r}+\nu\left(\overline{u_r'\frac{\partial^2u_r'}{\partial r^2}}+\frac{1}{r}\overline{u_r'\frac{\partial^2u_r'}{\partial r}}+\frac{1}{r^2}\overline{u_r'\frac{\partial^2u_r'}{\partial \varphi^2}}+\overline{u_r'\frac{\partial^2u_r'}{\partial z^2}}-\frac{\overline{u_r'u_r'}}{r^2}-\frac{2}{r^2}\overline{u_r'\frac{\partial u_\varphi'}{\partial \varphi}}\right) \tag{39}$$

Takođe ponavlja se ceo postupak tako što se pomnoži jednačina (29) sa u_r' i izvrši vremensko osrednjavanje, pa jednostavno prethodnu jednačinu treba pomnožiti sa 2.

Viskozni član se može napisati u obliku:

$$\overline{u_i'\frac{\partial^2u_j'}{\partial x_k^2}}+\overline{u_j'\frac{\partial^2u_i'}{\partial x_k^2}}=\frac{\partial^2\,\overline{u_iu_j}}{\partial x_k^2}-2\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{40}$$

korelacije fluktuacionog pritiska se mogu predstaviti sa:

$$\overline{u_j'\frac{\partial p'}{\partial x_i}}+\overline{u_i'\frac{\partial p'}{\partial x_j}}=\frac{\partial\,\overline{p'u_j'}}{\partial x_i}-\overline{p'\frac{\partial u_j'}{\partial x_i}}+\frac{\partial\,\overline{p'u_i'}}{\partial x_j}-\overline{p'\frac{\partial u_i'}{\partial x_j}} \tag{41}$$

dok se difuzioni član može napisati u obliku:

$$\overline{u_j'u_k'\frac{\partial u_i'}{\partial x_k}}+\overline{u_i'u_k'\frac{\partial u_j'}{\partial x_k}}=\frac{\partial\,\overline{u_i'u_j'u_k'}}{\partial x_k}-\overline{u_i'u_j'\frac{\partial u_k'}{\partial x_k}}=\frac{\partial\,\overline{u_i'u_j'u_k'}}{\partial x_k} \tag{42}$$

Množenjem jednačine (39) sa 2 sledi konačan oblik transportne jednačine za $\overline{u_r'u_r'}$ komponentu tenzora turbulentnog napona:

$$
\begin{aligned}
&\underbrace{\frac{\partial\,\overline{u_r'u_r'}}{\partial t}}_{1}+\underbrace{U_r\frac{\partial\,\overline{u_r'u_r'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\,\overline{u_r'u_r'}}{\partial\varphi}+U_z\frac{\partial\,\overline{u_r'u_r'}}{\partial z}-2\frac{U_\varphi}{r}\overline{u_r'u_\varphi'}}_{2}=\\
&\underbrace{-2\left(\overline{u_r'u_r'}\frac{\partial U_r}{\partial r}+\frac{1}{r}\overline{u_r'u_\varphi'}\frac{\partial U_r}{\partial\varphi}+\overline{u_r'u_z'}\frac{\partial U_r}{\partial z}\right)+2\frac{U_\varphi}{r}\overline{u_r'u_\varphi'}}_{3}+\underbrace{2\overline{u_r'F_r'}}_{4}+\\
&+\underbrace{2\frac{\overline{p'}}{\rho}\frac{\overline{\partial u_r'}}{\partial r}}_{5}\underbrace{-\frac{2}{\rho}\frac{\partial\,\overline{(p'u_r')}}{\partial r}}_{6a}\underbrace{-\left[\frac{1}{r}\frac{\partial\,\overline{(ru_r'u_r'^2)}}{\partial r}+\frac{1}{r}\frac{\partial\,\overline{(u_\varphi'u_r'^2)}}{\partial\varphi}+\frac{\partial\,\overline{(u_z'u_r'^2)}}{\partial z}\right]+\frac{2}{r}\overline{u_r'u_\varphi'u_\varphi'}}_{6б}+\\
&+\underbrace{\nu\left[\frac{\overline{\partial^2 u_r'^2}}{\partial r^2}+\frac{1}{r}\frac{\overline{\partial u_r'^2}}{\partial r}+\frac{1}{r^2}\frac{\overline{\partial^2 u_r'^2}}{\partial\varphi^2}+\frac{\overline{\partial^2 u_r'^2}}{\partial z^2}-\frac{2}{r^2}\left(u_r'^2-u_\varphi'^2\right)-\frac{4}{r^2}\frac{\partial\,\overline{(u_\varphi'u_r')}}{\partial\varphi}\right]}_{6в}-\\
&\underbrace{-2\nu\left[\overline{\left(\frac{\partial u_r'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_r'}{\partial\varphi}\right)^2}+\overline{\left(\frac{\partial u_r'}{\partial z}\right)^2}-2\frac{\overline{u_\varphi'}}{r^2}\frac{\overline{\partial u_r'}}{\partial\varphi}+\frac{\overline{u_\varphi'^2}}{r^2}\right]}_{7}
\end{aligned}
\tag{43}
$$

gde podvučeni članovi imaju sledeće značenje: (1) – lokalna promena turbulentnih napona, (2) – konvektivna promena turbulentnih napona, (3) – produkcija turbulentnih napona usled deformisanja usrednjenog toka, (4) – produkcija turb. napona usled delovanja fluktuirajućih zaprminskih sila, (5) – redistribucija među pojedinim komponentama turbulentnih napona usled dejstva fluktuirajućeg pritiska, (6) – difuzioni transport turbulentnih napona usled fluktuirajućeg pritiska (član 6a), brzine (član 6b) i molekularnog transporta (član 6v), i (7) – viskoza disipacija turbulentnih napona.

Pomoću istog postupka izvode se i ostalih pet jednačina za turbulentne napone [5] sa istim značenjima podvučenih članova kao u jedn. (43). U nastavku rada su dati konačni oblici transportnih jednačina za ostale komponente tenzora turbulentnih napona u cilindričnim koordinatama.

$$
\begin{aligned}
&\underbrace{\frac{\partial\,\overline{u_r'u_\varphi'}}{\partial t}}_{1}+\underbrace{U_r\frac{\partial\,\overline{u_r'u_\varphi'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\,\overline{u_r'u_\varphi'}}{\partial\varphi}+U_z\frac{\partial\,\overline{u_r'u_\varphi'}}{\partial z}+\frac{U_\varphi}{r}\,\overline{u_r'^2-u_\varphi'^2}}_{2}=\\
&\underbrace{-\left(\overline{u_\varphi'u_r'}\frac{\partial U_r}{\partial r}+\frac{1}{r}\overline{u_\varphi'u_\varphi'}\frac{\partial U_r}{\partial\varphi}+\overline{u_\varphi'u_z'}\frac{\partial U_r}{\partial z}\right)-\left(\overline{u_r'u_r'}\frac{\partial U_\varphi}{\partial r}+\frac{1}{r}\overline{u_r'u_\varphi'}\frac{\partial U_\varphi}{\partial\varphi}+\overline{u_r'u_z'}\frac{\partial U_\varphi}{\partial z}\right)}_{3}-\\
&\underbrace{-\overline{u_r'u_\varphi'}\frac{U_r}{r}+\overline{u_\varphi'^2}\frac{U_\varphi}{r}}_{3}+\underbrace{\overline{u_r'F_\varphi'}+\overline{u_\varphi'F_r'}}_{4}+\underbrace{\frac{1}{\rho}\left(\overline{p'\frac{\partial u_\varphi'}{\partial r}}+\overline{\frac{p'}{r}\frac{\partial u_r'}{\partial\varphi}}-\overline{p'\frac{u_\varphi'}{r}}\right)}_{5}\underbrace{-\frac{1}{\rho}\left[\frac{\partial\,\overline{(p'u_\varphi')}}{\partial r}+\frac{1}{r}\frac{\partial\,\overline{(p'u_r')}}{\partial\varphi}-\frac{\overline{p'u_\varphi'}}{r}\right]}_{6a}-\\
&\underbrace{-\left[\frac{1}{r}\frac{\partial}{\partial r}\overline{(ru_r'^2u_\varphi')}+\frac{1}{r}\frac{\partial\,\overline{(u_r'u_\varphi'^2)}}{\partial\varphi}+\frac{\partial\,\overline{(u_r'u_\varphi'u_z')}}{\partial z}\right]-\frac{\overline{u_\varphi'u_r'^2}}{r}+\frac{\overline{u_\varphi'^3}}{r}}_{6б}-\\
&+\underbrace{\nu\left(\frac{\partial^2\,\overline{(u_r'u_\varphi')}}{\partial r^2}+\frac{1}{r}\frac{\partial\,\overline{(u_r'u_\varphi')}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\,\overline{(u_r'u_\varphi')}}{\partial\varphi^2}+\frac{\partial^2\,\overline{(u_r'u_\varphi')}}{\partial z^2}-\frac{4}{r^2}\overline{u_\varphi'u_r'}+\frac{2}{r^2}\frac{\partial\,\overline{(u_r'^2-u_\varphi'^2)}}{\partial\varphi}\right)}_{6в}+\\
&\underbrace{-2\nu\left\{\left[\overline{\left(\frac{\partial u_r'}{\partial r}\right)}+\overline{\left(\frac{1}{r}\frac{\partial u_r'}{\partial\varphi}\right)}+\overline{\left(\frac{\partial u_r'}{\partial z}\right)}\right]\left[\overline{\left(\frac{\partial u_\varphi'}{\partial r}\right)}+\overline{\left(\frac{1}{r}\frac{\partial u_\varphi'}{\partial\varphi}\right)}+\overline{\left(\frac{\partial u_\varphi'}{\partial z}\right)}\right]+\frac{\overline{u_r'}}{r^2}\frac{\overline{\partial u_r'}}{\partial\varphi}-\frac{\overline{u_\varphi'}}{r^2}\frac{\overline{\partial u_\varphi'}}{\partial\varphi}-\frac{\overline{u_r'u_\varphi'}}{r^2}\right\}}_{7}
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
&\underbrace{\frac{\partial \overline{u_r'u_z'}}{\partial t}}_{1}+\underbrace{U_r\frac{\partial \overline{u_r'u_z'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial \overline{u_r'u_z'}}{\partial \varphi}+U_z\frac{\partial \overline{u_r'u_z'}}{\partial z}-\frac{U_\varphi}{r}\overline{u_\varphi'u_z'}}_{2}=\\
&\underbrace{-\left(\overline{u_r'u_r'}\frac{\partial U_z}{\partial r}+\frac{1}{r}\overline{u_r'u_\varphi'}\frac{\partial U_z}{\partial \varphi}+\overline{u_r'u_z'}\frac{\partial U_z}{\partial z}\right)-\left(\overline{u_z'u_r'}\frac{\partial U_r}{\partial r}+\frac{1}{r}\overline{u_z'u_\varphi'}\frac{\partial U_r}{\partial \varphi}+\overline{u_z'u_z'}\frac{\partial U_r}{\partial z}\right)}_{3}+\\
&\underbrace{+\overline{u_z'u_\varphi'}\frac{U_\varphi}{r}}_{3}+\underbrace{\overline{u_r'F_z'}+\overline{u_z'F_r'}}_{4}+\underbrace{\frac{1}{\rho}\left(\overline{p'\frac{\partial u_z'}{\partial r}}+\overline{p'\frac{\partial u_r'}{\partial z}}\right)}_{5}\underbrace{-\frac{1}{\rho}\left[\frac{\partial \overline{(p'u_z')}}{\partial r}+\frac{\partial \overline{(p'u_r')}}{\partial z}\right]}_{6a}-\\
&\underbrace{-\left[\frac{1}{r}\frac{\partial \overline{(r u_r'^2 u_z')}}{\partial r}+\frac{1}{r}\frac{\partial \overline{(u_r'u_\varphi'u_z')}}{\partial \varphi}+\frac{\partial \overline{(u_r'u_z'^2)}}{\partial z}\right]+\frac{\overline{u_z'u_\varphi'^2}}{r}}_{6б}+\\
&\underbrace{+\nu\left(\frac{\partial^2 \overline{(u_r'u_z')}}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{(u_r'u_z')}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \overline{(u_r'u_z')}}{\partial \varphi^2}+\frac{\partial^2 \overline{(u_r'u_z')}}{\partial z^2}-\frac{2}{r^2}\frac{\partial \overline{(u_\varphi'u_z')}}{\partial \varphi}-\frac{\overline{u_r'u_z'}}{r^2}\right)}_{6в}-\\
&\underbrace{-2\nu\left\{\left[\overline{\left(\frac{\partial u_r'}{\partial r}\right)}+\overline{\left(\frac{1}{r}\frac{\partial u_r'}{\partial \varphi}\right)}+\overline{\left(\frac{\partial u_r'}{\partial z}\right)}\right]\left[\overline{\left(\frac{\partial u_z'}{\partial r}\right)}+\overline{\left(\frac{1}{r}\frac{\partial u_z'}{\partial \varphi}\right)}+\overline{\left(\frac{\partial u_z'}{\partial z}\right)}\right]-\overline{\frac{u_\varphi'}{r^2}\frac{\partial u_z'}{\partial \varphi}}\right\}}_{7}
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&\underbrace{\frac{\partial \overline{u_\varphi'u_\varphi'}}{\partial t}}_{1}+\underbrace{U_r\frac{\partial \overline{u_\varphi'u_\varphi'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial \overline{u_\varphi'u_\varphi'}}{\partial \varphi}+U_z\frac{\partial \overline{u_\varphi'u_\varphi'}}{\partial z}+2\frac{U_\varphi}{r}\overline{u_\varphi'u_r'}}_{2}=\\
&\underbrace{-2\left(\overline{u_\varphi'u_r'}\frac{\partial U_\varphi}{\partial r}+\frac{1}{r}\overline{u_\varphi'u_\varphi'}\frac{\partial U_\varphi}{\partial \varphi}+\overline{u_\varphi'u_z'}\frac{\partial U_\varphi}{\partial z}\right)+2\frac{U_r}{r}\overline{u_\varphi'^2}}_{3}+\\
&\underbrace{+2\overline{u_\varphi'F_\varphi'}}_{4}+\underbrace{2\overline{\frac{p'}{r\rho}\frac{\partial u_\varphi'}{\partial \varphi}}+2\overline{\frac{p'}{\rho}\frac{u_r'}{r}}}_{5}-\\
&\underbrace{-\frac{2}{r\rho}\frac{\partial \overline{(p'u_\varphi')}}{\partial \varphi}-\frac{2}{\rho}\frac{\overline{p'u_r'}}{r}}_{6a}\underbrace{-\left[\frac{1}{r}\frac{\partial \overline{(r u_r'u_\varphi'^2)}}{\partial r}+\frac{1}{r}\frac{\partial \overline{u_\varphi'^3}}{\partial \varphi}+\frac{\partial \overline{(u_z'u_\varphi'^2)}}{\partial z}\right]-\frac{2}{r}\overline{u_r'u_\varphi'u_\varphi'}}_{6б}+\\
&\underbrace{+\nu\left[\frac{\partial^2 \overline{u_\varphi'^2}}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{u_\varphi'^2}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \overline{u_\varphi'^2}}{\partial \varphi^2}+\frac{\partial^2 \overline{u_\varphi'^2}}{\partial z^2}+\frac{2}{r^2}\overline{(u_r'^2-u_\varphi'^2)}+\frac{4}{r^2}\frac{\partial \overline{(u_r'u_\varphi')}}{\partial \varphi}\right]}_{6в}-\\
&\underbrace{-2\nu\left[\overline{\left(\frac{\partial u_\varphi'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_\varphi'}{\partial \varphi}\right)^2}+\overline{\left(\frac{\partial u_\varphi'}{\partial z}\right)^2}+2\overline{\frac{u_r'}{r^2}\frac{\partial u_\varphi'}{\partial \varphi}}+\frac{\overline{u_r'^2}}{r^2}\right]}_{7}
\end{aligned}
\tag{46}
$$

$$\underbrace{\frac{\partial \overline{u_\varphi' u_z'}}{\partial t}}_{1}+\underbrace{U_r\frac{\partial \overline{u_\varphi' u_z'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial \overline{u_\varphi' u_z'}}{\partial \varphi}+U_z\frac{\partial \overline{u_\varphi' u_z'}}{\partial z}+\frac{U_\varphi}{r}\overline{u_r' u_z'}}_{2}=$$

$$\underbrace{-\left(\overline{u_\varphi' u_r'}\frac{\partial U_z}{\partial r}+\frac{1}{r}\overline{u_\varphi' u_\varphi'}\frac{\partial U_z}{\partial \varphi}+\overline{u_\varphi' u_z'}\frac{\partial U_z}{\partial z}\right)-\left(\overline{u_z' u_r'}\frac{\partial U_\varphi}{\partial r}+\frac{1}{r}\overline{u_z' u_\varphi'}\frac{\partial U_\varphi}{\partial \varphi}+\overline{u_z' u_z'}\frac{\partial U_\varphi}{\partial z}\right)}_{3}+$$

$$+\underbrace{\overline{u_z' u_\varphi'}\frac{U_r}{r}}_{3}+\underbrace{\overline{u_\varphi' F_z'}+\overline{u_z' F_\varphi'}}_{4}+\underbrace{\overline{\frac{p'}{r\rho}\frac{\partial u_z'}{\partial \varphi}}+\overline{\frac{p'}{\rho}\frac{\partial u_\varphi'}{\partial z}}}_{5}-\underbrace{\frac{1}{r\rho}\frac{\partial\left(\overline{p' u_z'}\right)}{\partial \varphi}-\frac{1}{\rho}\frac{\partial\left(\overline{p' u_\varphi'}\right)}{\partial z}}_{6a}-$$

$$\underbrace{-\left(\frac{1}{r}\frac{\partial\left(\overline{u_r' u_\varphi' u_z'}\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\overline{u_\varphi'^2 u_z'}\right)}{\partial \varphi}+\frac{\partial\left(\overline{u_\varphi' u_z'^2}\right)}{\partial z}\right)-\frac{\overline{u_r' u_\varphi' u_z'}}{r}}_{6б}+$$

$$\underbrace{+\nu\left(\frac{\partial^2\left(\overline{u_z' u_\varphi'}\right)}{\partial r^2}+\frac{1}{r}\frac{\partial\left(\overline{u_z' u_\varphi'}\right)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\left(\overline{u_z' u_\varphi'}\right)}{\partial \varphi^2}+\frac{\partial^2\left(\overline{u_z' u_\varphi'}\right)}{\partial z^2}+\frac{2}{r^2}\frac{\partial\left(\overline{u_r' u_z'}\right)}{\partial \varphi}-\frac{\overline{u_\varphi' u_z'}}{r^2}\right)}_{6в}-$$

$$\underbrace{-2\nu\left\{\overline{\left[\left(\frac{\partial u_\varphi'}{\partial r}\right)+\left(\frac{1}{r}\frac{\partial u_\varphi'}{\partial \varphi}\right)+\left(\frac{\partial u_\varphi'}{\partial z}\right)\right]\left[\left(\frac{\partial u_z'}{\partial r}\right)+\left(\frac{1}{r}\frac{\partial u_z'}{\partial \varphi}\right)+\left(\frac{\partial u_z'}{\partial z}\right)\right]}+\overline{\frac{u_r'}{r^2}\frac{\partial u_z'}{\partial \varphi}}\right\}}_{7} \tag{47}$$

$$\underbrace{\frac{\partial \overline{u_z' u_z'}}{\partial t}}_{1}+\underbrace{U_r\frac{\partial \overline{u_z' u_z'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial \overline{u_z' u_z'}}{\partial \varphi}+U_z\frac{\partial \overline{u_z' u_z'}}{\partial z}}_{2}=$$

$$\underbrace{-2\left(\overline{u_z' u_r'}\frac{\partial U_z}{\partial r}+\frac{1}{r}\overline{u_z' u_\varphi'}\frac{\partial U_z}{\partial \varphi}+\overline{u_z' u_z'}\frac{\partial U_z}{\partial z}\right)}_{3}+\underbrace{2\overline{u_z' F_z'}}_{4}+\underbrace{2\overline{\frac{p'}{\rho}\frac{\partial u_z'}{\partial z}}}_{5}-$$

$$\underbrace{-\frac{2}{\rho}\frac{\partial \overline{p' u_z'}}{\partial z}}_{6a}-\underbrace{\left(\frac{1}{r}\frac{\partial\left(\overline{u_r' u_z'^2}\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\overline{u_\varphi' u_z'^2}\right)}{\partial \varphi}+\frac{\partial \overline{u_z'^3}}{\partial z}\right)}_{6б}+$$

$$\underbrace{+\nu\left(\frac{\partial^2 \overline{u_z'^2}}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{u_z'^2}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \overline{u_z'^2}}{\partial \varphi^2}+\frac{\partial^2 \overline{u_z'^2}}{\partial z^2}\right)}_{6в}-\underbrace{2\nu\overline{\left[\left(\frac{\partial u_z'}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_z'}{\partial \varphi}\right)^2+\left(\frac{\partial u_z'}{\partial z}\right)^2\right]}}_{7} \tag{48}$$

Transportna jednačina kinetičke energije turbulencije

Sabiranjem jednačina za normalne turbulentne napone za sve tri koordinate podeljene sa 2, definišući pri tome kinetičku energiju turbulencije kao $k=0,5\left(\overline{u_r'^2}+\overline{u_\varphi'^2}+\overline{u_z'^2}\right)$, dolazi se do konačne forme egzaktne transportne jednačine za kinetičku energiju turbulencije.

Sabiranjem jednačina (29 + 30 + 31) sledi:

$$
\begin{aligned}
&\frac{\partial\, \overline{u_r' u_r'}}{\partial t}+U_r\frac{\partial\, \overline{u_r' u_r'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\, \overline{u_r' u_r'}}{\partial \varphi}+U_z\frac{\partial\, \overline{u_r' u_r'}}{\partial z}-2\frac{U_\varphi}{r}\overline{u_r' u_\varphi'}+\\
&\frac{\partial\, \overline{u_\varphi' u_\varphi'}}{\partial t}+U_r\frac{\partial\, \overline{u_\varphi' u_\varphi'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\, \overline{u_\varphi' u_\varphi'}}{\partial \varphi}+U_z\frac{\partial\, \overline{u_\varphi' u_\varphi'}}{\partial z}+2\frac{U_\varphi}{r}\overline{u_\varphi' u_r'}+\\
&\frac{\partial\, \overline{u_z' u_z'}}{\partial t}+U_r\frac{\partial\, \overline{u_z' u_z'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\, \overline{u_z' u_z'}}{\partial \varphi}+U_z\frac{\partial\, \overline{u_z' u_z'}}{\partial z}=\\
&-2\left(\overline{u_r' u_r'}\frac{\partial U_r}{\partial r}+\frac{1}{r}\overline{u_r' u_\varphi'}\frac{\partial U_r}{\partial \varphi}+\overline{u_r' u_z'}\frac{\partial U_r}{\partial z}\right)+2\frac{U_\varphi}{r}\overline{u_r' u_\varphi'}-\\
&-2\left(\overline{u_\varphi' u_r'}\frac{\partial U_\varphi}{\partial r}+\frac{1}{r}\overline{u_\varphi' u_\varphi'}\frac{\partial U_\varphi}{\partial \varphi}+\overline{u_\varphi' u_z'}\frac{\partial U_\varphi}{\partial z}\right)+2\frac{U_r}{r}\overline{u_\varphi'^2}-\\
&-2\left(\overline{u_z' u_r'}\frac{\partial U_z}{\partial r}+\frac{1}{r}\overline{u_z' u_\varphi'}\frac{\partial U_z}{\partial \varphi}+\overline{u_z' u_z'}\frac{\partial U_z}{\partial z}\right)+2\overline{u_r' F_r'}+2\overline{u_\varphi' F_\varphi'}+2\overline{u_z' F_z'}+\\
&+2\frac{\overline{p'}}{\rho}\frac{\overline{\partial u_r'}}{\partial r}+2\frac{\overline{p'}}{r\rho}\frac{\overline{\partial u_\varphi'}}{\partial \varphi}+2\frac{\overline{p'}}{\rho}\frac{\overline{u_r'}}{r}+2\frac{\overline{p'}}{\rho}\frac{\overline{\partial u_z'}}{\partial z}-\frac{2}{\rho}\frac{\partial\left(\overline{p' u_r'}\right)}{\partial r}-\frac{2}{r\rho}\frac{\partial\left(\overline{p' u_\varphi'}\right)}{\partial \varphi}-\frac{2}{\rho}\frac{\overline{p' u_r'}}{r}-\frac{2}{\rho}\frac{\partial\left(\overline{p' u_z'}\right)}{\partial z}-\\
&-\left[\frac{1}{r}\frac{\partial\left(\overline{r u_r' u_r'^2}\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\overline{u_\varphi' u_r'^2}\right)}{\partial \varphi}+\frac{\partial\left(\overline{u_z' u_r'^2}\right)}{\partial z}\right]+\frac{2}{r}\overline{u_r' u_\varphi' u_\varphi'}-\\
&-\left[\frac{1}{r}\frac{\partial\left(\overline{r u_r' u_\varphi'^2}\right)}{\partial r}+\frac{1}{r}\frac{\overline{\partial u_\varphi'^3}}{\partial \varphi}+\frac{\partial\left(\overline{u_z' u_\varphi'^2}\right)}{\partial z}\right]-\frac{2}{r}\overline{u_r' u_\varphi' u_\varphi'}-\left[\frac{1}{r}\frac{\partial\left(\overline{r u_r' u_z'^2}\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\overline{u_\varphi' u_z'^2}\right)}{\partial \varphi}+\frac{\overline{\partial u_z'^3}}{\partial z}\right]+\\
&+\nu\left[\frac{\overline{\partial^2 u_r'^2}}{\partial r^2}+\frac{1}{r}\frac{\overline{\partial u_r'^2}}{\partial r}+\frac{1}{r^2}\frac{\overline{\partial^2 u_r'^2}}{\partial \varphi^2}+\frac{\overline{\partial^2 u_r'^2}}{\partial z^2}-\frac{2}{r^2}\left(\overline{u_r'^2-u_\varphi'^2}\right)-\frac{4}{r^2}\frac{\partial\left(\overline{u_\varphi' u_r'}\right)}{\partial \varphi}\right]+\\
&+\nu\left(\frac{\overline{\partial^2 u_\varphi'^2}}{\partial r^2}+\frac{1}{r}\frac{\overline{\partial u_\varphi'^2}}{\partial r}+\frac{1}{r^2}\frac{\overline{\partial^2 u_\varphi'^2}}{\partial \varphi^2}+\frac{\overline{\partial^2 u_\varphi'^2}}{\partial z^2}+\frac{2}{r^2}\left(\overline{u_r'^2-u_\varphi'^2}\right)+\frac{4}{r^2}\frac{\partial\left(\overline{u_r' u_\varphi'}\right)}{\partial \varphi}\right)+\\
&+\nu\left(\frac{\overline{\partial^2 u_z'^2}}{\partial r^2}+\frac{1}{r}\frac{\overline{\partial u_z'^2}}{\partial r}+\frac{1}{r^2}\frac{\overline{\partial^2 u_z'^2}}{\partial \varphi^2}+\frac{\overline{\partial^2 u_z'^2}}{\partial z^2}\right)-\\
&-2\nu\left(\overline{\left(\frac{\partial u_r'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_r'}{\partial \varphi}\right)^2}+\overline{\left(\frac{\partial u_r'}{\partial z}\right)^2}-2\overline{\frac{u_\varphi'}{r^2}\frac{\partial u_r'}{\partial \varphi}}+\frac{\overline{u_\varphi'^2}}{r^2}\right)-\\
&2\nu\left(\overline{\left(\frac{\partial u_\varphi'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_\varphi'}{\partial \varphi}\right)^2}+\overline{\left(\frac{\partial u_\varphi'}{\partial z}\right)^2}+2\overline{\frac{u_r'}{r^2}\frac{\partial u_\varphi'}{\partial \varphi}}+\frac{\overline{u_r'^2}}{r^2}\right)-2\nu\left(\overline{\left(\frac{\partial u_z'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_z'}{\partial \varphi}\right)^2}+\overline{\left(\frac{\partial u_z'}{\partial z}\right)^2}\right)
\end{aligned}
\tag{49}
$$

Preuređuju se prvo lokalni članovi (članovi vezani za promenu neke fizičke veličine u vremenu) na levoj strani jednačine (49):

$$
\frac{1}{2}\left[\frac{\partial\, \overline{u_r' u_r'}}{\partial t}+\frac{\partial\, \overline{u_\varphi' u_\varphi'}}{\partial t}+\frac{\partial\, \overline{u_z' u_z'}}{\partial t}\right]=\frac{\partial \overline{\frac{1}{2}\left(u_r'^2+u_\varphi'^2+u_z'^2\right)}}{\partial t}=\frac{\partial k}{\partial t}
\tag{50}
$$

koji predstavljaju lokalnu promenu kinetičke energije turbulencije.

Konvektivna promena kinetičke energije turbulencije ima sledeći oblik:

$$\frac{1}{2}\left[U_r\frac{\partial\,\overline{u_r'u_r'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\,\overline{u_r'u_r'}}{\partial\varphi}+U_z\frac{\partial\,\overline{u_r'u_r'}}{\partial z}-2\frac{U_\varphi}{r}\overline{u_r'u_\varphi'}\right]+$$
$$+\frac{1}{2}\left[U_r\frac{\partial\,\overline{u_\varphi'u_\varphi'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\,\overline{u_\varphi'u_\varphi'}}{\partial\varphi}+U_z\frac{\partial\,\overline{u_\varphi'u_\varphi'}}{\partial z}+2\frac{U_\varphi}{r}\overline{u_\varphi'u_r'}\right]+$$
$$+\frac{1}{2}\left[U_r\frac{\partial\,\overline{u_z'u_z'}}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\,\overline{u_z'u_z'}}{\partial\varphi}+U_z\frac{\partial\,\overline{u_z'u_z'}}{\partial z}\right]=U_r\frac{\partial\frac{1}{2}\overline{u_r'^2+u_\varphi'^2+u_z'^2}}{\partial r}+$$
$$+\frac{U_\varphi}{r}\frac{\partial\frac{1}{2}\left(\overline{u_r'^2+u_\varphi'^2+u_z'^2}\right)}{\partial\varphi}+U_z\frac{\partial\frac{1}{2}\left(\overline{u_r'^2+u_\varphi'^2+u_z'^2}\right)}{\partial z}=U_r\frac{\partial k}{\partial r}+\frac{U_\varphi}{r}\frac{\partial k}{\partial\varphi}+U_z\frac{\partial k}{\partial z} \tag{51}$$

Rad fluktuacionih zapreminskih sila ima sledeći oblik:

$$\overline{u_r'F_r'}+\overline{u_\varphi'F_\varphi'}+\overline{u_z'F_z'} \tag{52}$$

Produkcija kinetičke energije turbulencije (energija prenesena iz glavnog toka tangencijalnim turbulentnim naponima) ima sledeću formu:

$$\frac{1}{2}\left[-2\left(\overline{u_r'u_r'}\frac{\partial U_r}{\partial r}+\frac{1}{r}\overline{u_r'u_\varphi'}\frac{\partial U_r}{\partial\varphi}+\overline{u_r'u_z'}\frac{\partial U_r}{\partial z}\right)+2\frac{U_\varphi}{r}\overline{u_r'u_\varphi'}\right]$$
$$\frac{1}{2}\left[-2\left(\overline{u_\varphi'u_r'}\frac{\partial U_\varphi}{\partial r}+\frac{1}{r}\overline{u_\varphi'u_\varphi'}\frac{\partial U_\varphi}{\partial\varphi}+\overline{u_\varphi'u_z'}\frac{\partial U_\varphi}{\partial z}\right)+2\frac{U_r}{r}\overline{u_\varphi'^2}\right]$$
$$\frac{1}{2}\left[-2\left(\overline{u_z'u_r'}\frac{\partial U_z}{\partial r}+\frac{1}{r}\overline{u_z'u_\varphi'}\frac{\partial U_z}{\partial\varphi}+\overline{u_z'u_z'}\frac{\partial U_z}{\partial z}\right)\right]=$$
$$=-\left[\overline{u_r'u_r'}\frac{\partial U_r}{\partial r}+\overline{u_\varphi'u_\varphi'}\left(\frac{1}{r}\frac{\partial U_\varphi}{\partial\varphi}+\frac{U_r}{r}\right)+\overline{u_z'u_z'}\frac{\partial U_z}{\partial z}\right]-$$
$$-\left[\overline{u_r'u_\varphi'}\left(\frac{1}{r}\frac{\partial U_r}{\partial\varphi}+\frac{U_\varphi}{r}+\frac{\partial U_\varphi}{\partial r}\right)+\overline{u_\varphi'u_z'}\left(\frac{\partial U_\varphi}{\partial z}+\frac{1}{r}\frac{\partial U_z}{\partial\varphi}\right)+\overline{u_r'u_z'}\left(\frac{\partial U_r}{\partial z}+\frac{\partial U_z}{\partial r}\right)\right] \tag{53}$$

Turbulentna difuzija kinetičke energije turbulencije ima sledeći oblik:

$$\frac{1}{2}\left[\underbrace{2\frac{p'}{\rho}\overline{\frac{\partial u_r'}{\partial r}}+2\overline{\frac{p'}{r\rho}\frac{\partial u_\varphi'}{\partial\varphi}}+2\overline{\frac{p'}{\rho}\frac{u_r'}{r}}+2\overline{\frac{p'}{\rho}\frac{\partial u_z'}{\partial z}}}_{\text{Једнако нули због једначине континуитета}}-\frac{2}{\rho}\frac{\partial\,\overline{p'u_r'}}{\partial r}-\frac{2}{r\rho}\frac{\partial\,\overline{p'u_\varphi'}}{\partial\varphi}-\frac{2}{\rho}\frac{\overline{\partial p'u_r'}}{r}-\frac{2}{\rho}\frac{\partial\,\overline{p'u_z'}}{\partial z}\right]+$$
$$+\frac{1}{2}\left[-\left(\frac{1}{r}\frac{\partial\,\overline{ru_r'u_r'^2}}{\partial r}+\frac{1}{r}\frac{\partial\,\overline{u_\varphi'u_r'^2}}{\partial\varphi}+\frac{\partial\,\overline{u_z'u_r'^2}}{\partial z}\right)+\frac{2}{r}\overline{u_r'u_\varphi'u_\varphi'}\right]+$$
$$+\frac{1}{2}\left[-\left(\frac{1}{r}\frac{\partial\left(\overline{ru_r'u_\varphi'^2}\right)}{\partial r}+\frac{1}{r}\frac{\overline{\partial u_\varphi'^3}}{\partial\varphi}+\frac{\partial\left(\overline{u_z'u_\varphi'^2}\right)}{\partial z}\right)-\frac{2}{r}\overline{u_r'u_\varphi'u_\varphi'}\right]+$$
$$+\frac{1}{2}\left[-\left(\frac{1}{r}\frac{\partial\left(\overline{ru_r'u_z'^2}\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\overline{u_\varphi'u_z'^2}\right)}{\partial\varphi}+\frac{\overline{\partial u_z'^3}}{\partial z}\right)\right]=$$
$$-\frac{1}{\rho}\frac{\partial\left(\overline{p'u_r'}\right)}{\partial r}-\frac{1}{r\rho}\frac{\partial\left(\overline{p'u_\varphi'}\right)}{\partial\varphi}-\frac{1}{\rho}\frac{\overline{\partial p'u_r'}}{r}-\frac{1}{\rho}\frac{\partial\left(\overline{p'u_z'}\right)}{\partial z}+$$
$$-\frac{1}{r}\frac{\partial}{\partial r}\left(\overline{ru_r'\frac{u_r'^2+u_\varphi'^2+u_z'^2}{2}}\right)-\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\overline{u_\varphi'\frac{u_r'^2+u_\varphi'^2+u_z'^2}{2}}\right)-\frac{\partial}{\partial z}\left(\overline{u_z'\frac{u_r'^2+u_\varphi'^2+u_z'^2}{2}}\right)=$$
$$=-\frac{1}{r}\frac{\partial}{\partial r}\left[\overline{ru_r'\left(\frac{p'}{\rho}+k'\right)}\right]-\frac{1}{r}\frac{\partial}{\partial\varphi}\left[\overline{u_\varphi'\left(\frac{p'}{\rho}+k'\right)}\right]-\frac{\partial}{\partial z}\left[\overline{u_z'\left(\frac{p'}{\rho}+k'\right)}\right] \tag{54}$$

Rad fluktuacionih viskoznih sila se sređuje na sledeći način:

$$\begin{aligned}
&\frac{1}{2}\nu\left(\overline{\frac{\partial^2 u_r'^2}{\partial r^2}}+\frac{1}{r}\overline{\frac{\partial u_r'^2}{\partial r}}+\frac{1}{r^2}\overline{\frac{\partial^2 u_r'^2}{\partial \varphi^2}}+\overline{\frac{\partial^2 u_r'^2}{\partial z^2}}-\frac{2}{r^2}\overline{u_r'^2-u_\varphi'^2}-\frac{4}{r^2}\overline{\frac{\partial\, u_\varphi' u_r'}{\partial \varphi}}\right)+\\
&+\frac{1}{2}\nu\left(\overline{\frac{\partial^2 u_\varphi'^2}{\partial r^2}}+\frac{1}{r}\overline{\frac{\partial u_\varphi'^2}{\partial r}}+\frac{1}{r^2}\overline{\frac{\partial^2 u_\varphi'^2}{\partial \varphi^2}}+\overline{\frac{\partial^2 u_\varphi'^2}{\partial z^2}}+\frac{2}{r^2}\overline{u_r'^2-u_\varphi'^2}+\frac{4}{r^2}\overline{\frac{\partial\, u_r' u_\varphi'}{\partial \varphi}}\right)+\\
&+\frac{1}{2}\nu\left(\overline{\frac{\partial^2 u_z'^2}{\partial r^2}}+\frac{1}{r}\overline{\frac{\partial u_z'^2}{\partial r}}+\frac{1}{r^2}\overline{\frac{\partial^2 u_z'^2}{\partial \varphi^2}}+\overline{\frac{\partial^2 u_z'^2}{\partial z^2}}\right)=\nu\overline{\frac{\partial^2}{\partial r^2}\left(\frac{u_r'^2+u_\varphi'^2+u_z'^2}{2}\right)}+\\
&+\nu\overline{\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{u_r'^2+u_\varphi'^2+u_z'^2}{2}\right)}+\nu\overline{\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\left(\frac{u_r'^2+u_\varphi'^2+u_z'^2}{2}\right)}+\nu\overline{\frac{\partial^2}{\partial z^2}\left(\frac{u_r'^2+u_\varphi'^2+u_z'^2}{2}\right)}=\\
&=\nu\left[\frac{\partial k}{\partial r^2}+\frac{1}{r}\frac{\partial k}{\partial r}+\frac{1}{r^2}\frac{\partial k}{\partial \varphi^2}+\frac{\partial k}{\partial z^2}\right]
\end{aligned}\tag{55}$$

Disipacija kinetičke energije turbulencije glasi:

$$\begin{aligned}
&\frac{1}{2}\left\{-2\nu\left[\overline{\left(\frac{\partial u_r'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_r'}{\partial \varphi}\right)^2}+\overline{\left(\frac{\partial u_r'}{\partial z}\right)^2}-2\overline{\frac{u_\varphi'}{r^2}\frac{\partial u_r'}{\partial \varphi}}+\overline{\frac{u_\varphi'^2}{r^2}}\right]\right\}+\\
&+\frac{1}{2}\left\{-2\nu\left[\overline{\left(\frac{\partial u_\varphi'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_\varphi'}{\partial \varphi}\right)^2}+\overline{\left(\frac{\partial u_\varphi'}{\partial z}\right)^2}+2\overline{\frac{u_r'}{r^2}\frac{\partial u_\varphi'}{\partial \varphi}}+\overline{\frac{u_r'^2}{r^2}}\right]\right\}+\\
&+\frac{1}{2}\left\{-2\nu\left[\overline{\left(\frac{\partial u_z'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_z'}{\partial \varphi}\right)^2}+\overline{\left(\frac{\partial u_z'}{\partial z}\right)^2}\right]\right\}=\\
&=-\nu\left[\overline{\left(\frac{\partial u_r'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_r'}{\partial \varphi}-\frac{u_\varphi'}{r}\right)^2}+\overline{\left(\frac{\partial u_r'}{\partial z}\right)^2}\right]-\nu\left[\overline{\left(\frac{\partial u_\varphi'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_\varphi'}{\partial \varphi}+\frac{u_r'}{r}\right)^2}+\overline{\left(\frac{\partial u_\varphi'}{\partial z}\right)^2}\right]-\\
&-\nu\left[\overline{\left(\frac{\partial u_z'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_z'}{\partial \varphi}\right)^2}+\overline{\left(\frac{\partial u_z'}{\partial z}\right)^2}\right]
\end{aligned}\tag{56}$$

Konačan oblik transportne kinetičke energije turbulencije glasi:

$$\begin{aligned}
&\underbrace{\frac{\partial k}{\partial t}}_{a}+\underbrace{U_r\frac{\partial k}{\partial r}+\frac{U_\varphi}{r}\frac{\partial k}{\partial \varphi}+U_z\frac{\partial k}{\partial z}}_{b}=\underbrace{\overline{u_r'F_r'}+\overline{u_\varphi'F_\varphi'}+\overline{u_z'F_z'}}_{c}=\\
&\underbrace{-\left[\overline{u_r'u_r'}\frac{\partial U_r}{\partial r}+\overline{u_\varphi'u_\varphi'}\left(\frac{1}{r}\frac{\partial U_\varphi}{\partial \varphi}+\frac{U_r}{r}\right)+\overline{u_z'u_z'}\frac{\partial U_z}{\partial z}\right]}_{d}-\\
&\underbrace{-\left[\overline{u_r'u_\varphi'}\left(\frac{1}{r}\frac{\partial U_r}{\partial \varphi}+\frac{U_\varphi}{r}+\frac{\partial U_\varphi}{\partial r}\right)+\overline{u_\varphi'u_z'}\left(\frac{\partial U_\varphi}{\partial z}+\frac{1}{r}\frac{\partial U_z}{\partial \varphi}\right)+\overline{u_r'u_z'}\left(\frac{\partial U_r}{\partial z}+\frac{\partial U_z}{\partial r}\right)\right]}_{d}-\\
&\underbrace{-\frac{1}{r}\frac{\partial}{\partial r}\left[\overline{ru_r'\left(\frac{p'}{\rho}+k'\right)}\right]-\frac{1}{r}\frac{\partial}{\partial \varphi}\left[\overline{u_\varphi'\left(\frac{p'}{\rho}+k'\right)}\right]-\frac{\partial}{\partial z}\left[\overline{u_z'\left(\frac{p'}{\rho}+k'\right)}\right]}_{e}+\\
&\underbrace{+\nu\left(\frac{\partial k}{\partial r^2}+\frac{1}{r}\frac{\partial k}{\partial r}+\frac{1}{r^2}\frac{\partial k}{\partial \varphi^2}+\frac{\partial k}{\partial z^2}\right)}_{f}
\end{aligned}$$

$$\underbrace{-\nu\left[\overline{\left(\frac{\partial u_r'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_r'}{\partial \varphi}-\frac{u_\varphi'}{r}\right)^2}+\overline{\left(\frac{\partial u_r'}{\partial z}\right)^2}\right]-\nu\left[\overline{\left(\frac{\partial u_\varphi'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_\varphi'}{\partial \varphi}+\frac{u_r'}{r}\right)^2}+\overline{\left(\frac{\partial u_\varphi'}{\partial z}\right)^2}\right]}_{g}-$$
$$\underbrace{-\nu\left[\overline{\left(\frac{\partial u_z'}{\partial r}\right)^2}+\overline{\left(\frac{1}{r}\frac{\partial u_z'}{\partial \varphi}\right)^2}+\overline{\left(\frac{\partial u_z'}{\partial z}\right)^2}\right]}_{g} \tag{57}$$

gde članovi jednačine (57) imaju sledeće značenje: a – lokalna promena kinetičke energije turbulencije, b – konvektivna promena kinetičke energije turbulencije, c – rad fluktuacionih zapreminskih sila, d – produkcija kinetičke energije turbulencije (energija prenesena iz glavnog toka tangencijalnim turbulentnim naponima), e – turbulentna difuzija kinetičke energije turbulencije, f – rad fluktuacionih viskoznih sila, i g – brzina disipacija kinetičke energije turbulencije.

Transportna jednačina brzine disipacije kinetičke energije turbulencije

Tubulentna korelacija $\overline{\nu(\partial u_i^{'} / \partial x_k + \partial u_k^{'} / \partial x_i)\partial u_i^{'} / \partial x_k}$ je definisana kao brzina disipacije kinetičke energije ε. Međutim pri visokim Rejnoldsovim brojevima, proizvodi unakrsnih izvoda se mogu zanemariti, tako da se za ovaj parametar koristi aproksimativan izraz $\overline{\nu(\partial u_i^{'} / \partial x_k)^2}$. Dobijanje egzaktne transportne jednačine za ε se dobija kada se od jednačine za trenutne vrednosti turbulentne brzine oduzme njen vremenski osrednjeni deo, diferencira po x_l, pomnoži sa $2\nu(\partial u_i^{'} / \partial x_l)$ i osrednji po vremenu [6].

Izdvajanjem fluktuacionog dela N-S jednačina za sve tri koordinate i diferenciranjem po x_l, sledi relacija:

$$\frac{\partial}{\partial x_l}\left\{\rho\left[\frac{\partial u_r'}{\partial t}+U_r\frac{\partial u_r'}{\partial r}+u_r'\frac{\partial U_r}{\partial r}+u_r'\frac{\partial u_r'}{\partial r}+\frac{1}{r}\left(U_\varphi\frac{\partial u_r'}{\partial \varphi}+u_\varphi'\frac{\partial U_r}{\partial r}+u_\varphi'\frac{\partial u_r'}{\partial \varphi}\right)\right]+\right.$$
$$+\rho\left(U_z\frac{\partial u_r'}{\partial z}+u_z'\frac{\partial U_r}{\partial z}+u_z'\frac{\partial u_r'}{\partial z}-\frac{2U_\varphi u_\varphi'}{r}-\frac{u_\varphi'^2}{r}\right)+$$
$$+\rho\left[\frac{\partial u_\varphi'}{\partial t}+U_r\frac{\partial u_\varphi'}{\partial r}+u_r'\frac{\partial U_\varphi}{\partial r}+u_r'\frac{\partial u_\varphi'}{\partial r}+\frac{1}{r}\left(U_\varphi\frac{\partial u_\varphi'}{\partial \varphi}+u_\varphi'\frac{\partial U_\varphi}{\partial r}+u_\varphi'\frac{\partial u_\varphi'}{\partial \varphi}\right)\right]+$$
$$+\rho\left(U_z\frac{\partial u_\varphi'}{\partial z}+u_z'\frac{\partial U_\varphi}{\partial z}+u_z'\frac{\partial u_\varphi'}{\partial z}+\frac{U_r u_\varphi'}{r}+\frac{U_\varphi u_r'}{r}+\frac{u_r' u_\varphi'}{r}\right)+$$
$$+\rho\left[\frac{\partial u_z'}{\partial t}+U_r\frac{\partial u_z'}{\partial r}+u_r'\frac{\partial U_z}{\partial r}+u_r'\frac{\partial u_z'}{\partial r}+\frac{1}{r}\left(U_\varphi\frac{\partial u_z'}{\partial \varphi}+u_\varphi'\frac{\partial U_z}{\partial r}+u_\varphi'\frac{\partial u_z'}{\partial \varphi}\right)\right]+$$
$$\left.+\rho\left(U_z\frac{\partial u_z'}{\partial z}+u_z'\frac{\partial U_z}{\partial z}+u_z'\frac{\partial u_z'}{\partial z}\right)\right\}=$$
$$=\frac{\partial}{\partial x_l}\left[\rho F_r'-\frac{\partial p'}{\partial r}+\mu\left(\frac{\partial^2 u_r'}{\partial r^2}+\frac{1}{r}\frac{\partial u_r'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r'}{\partial \varphi^2}+\frac{\partial^2 u_r'}{\partial z^2}-\frac{u_r'}{r^2}-\frac{2}{r^2}\frac{\partial u_\varphi'}{\partial \varphi}\right)+\right.$$
$$+\rho F_\varphi'-\frac{1}{r}\frac{\partial p'}{\partial \varphi}+\mu\left(\frac{\partial^2 u_\varphi'}{\partial r^2}+\frac{1}{r}\frac{\partial u_\varphi'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\varphi'}{\partial \varphi^2}+\frac{\partial^2 u_\varphi'}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r'}{\partial \varphi}-\frac{u_\varphi'}{r^2}\right)+$$
$$\left.+\rho F_z'-\frac{\partial p'}{\partial z}+\mu\left(\frac{\partial^2 u_z'}{\partial r^2}+\frac{1}{r}\frac{\partial u_z'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z'}{\partial \varphi^2}+\frac{\partial^2 u_z'}{\partial z^2}\right)\right] \tag{58}$$

Nakon sprovedenog diferenciranja članova u zagradi prethodne jednačine, sledi:

$$
\begin{aligned}
&\rho\left[\underbrace{\frac{\partial}{\partial t}\left(\frac{\partial u_r'}{\partial x_l}\right)}_{\text{I}}+\underbrace{\frac{\partial}{\partial x_l}\left(U_r\frac{\partial u_r'}{\partial r}\right)}_{\text{III+V}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_r'\frac{\partial U_r}{\partial r}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_r'\frac{\partial u_r'}{\partial r}\right)}_{\text{VI+VII}}+\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}U_\varphi\frac{\partial u_r'}{\partial \varphi}\right)}_{\text{III+V}}\right]+\\
&+\rho\left[\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}u_\varphi'\frac{\partial U_r}{\partial r}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}u_\varphi'\frac{\partial u_r'}{\partial \varphi}\right)}_{\text{VI+VII}}\right]+\rho\left[\underbrace{\frac{\partial}{\partial x_l}\left(U_z\frac{\partial u_r'}{\partial z}\right)}_{\text{III+V}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_z'\frac{\partial U_r}{\partial z}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_z'\frac{\partial u_r'}{\partial z}\right)}_{\text{VI+VII}}\right]+\\
&\rho\left[-\underbrace{\frac{\partial}{\partial x_l}\left(\frac{2U_\varphi u_\varphi'}{r}\right)}_{\text{II+IV}}-\underbrace{\frac{\partial}{\partial x_l}\left(\frac{u_\varphi'^2}{r}\right)}_{\text{VI+VII}}\right]+\rho\left[\underbrace{\frac{\partial}{\partial t}\left(\frac{\partial u_\varphi'}{\partial x_l}\right)}_{\text{I}}+\underbrace{\frac{\partial}{\partial x_l}\left(U_r\frac{\partial u_\varphi'}{\partial r}\right)}_{\text{III+V}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_\varphi'\frac{\partial U_r}{\partial r}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_r'\frac{\partial u_\varphi'}{\partial r}\right)}_{\text{VI+VII}}\right]+\\
&+\rho\left[\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}U_\varphi\frac{\partial u_\varphi'}{\partial \varphi}\right)}_{\text{III+V}}+\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}u_\varphi'\frac{\partial U_\varphi}{\partial r}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}u_\varphi'\frac{\partial u_\varphi'}{\partial \varphi}\right)}_{\text{VI+VII}}\right]+\\
&+\rho\left[\underbrace{\frac{\partial}{\partial x_l}\left(U_z\frac{\partial u_\varphi'}{\partial z}\right)}_{\text{III+V}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_z'\frac{\partial U_\varphi}{\partial z}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_z'\frac{\partial u_\varphi'}{\partial z}\right)}_{\text{VI+VII}}+\underbrace{\frac{\partial}{\partial x_l}\left(\frac{U_r u_\varphi'}{r}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(\frac{U_\varphi u_r'}{r}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(\frac{u_r' u_\varphi'}{r}\right)}_{\text{VI+VII}}\right]+\\
&+\rho\left[\underbrace{\frac{\partial}{\partial t}\left(\frac{\partial u_z'}{\partial x_l}\right)}_{\text{I}}+\underbrace{\frac{\partial}{\partial x_l}\left(U_r\frac{\partial u_z'}{\partial r}\right)}_{\text{III+V}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_\varphi'\frac{\partial U_z}{\partial r}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_r'\frac{\partial u_z'}{\partial r}\right)}_{\text{VI+VII}}\right]+\\
&+\rho\left[\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}U_\varphi\frac{\partial u_z'}{\partial \varphi}\right)}_{\text{III+V}}+\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}u_\varphi'\frac{\partial U_z}{\partial r}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}u_\varphi'\frac{\partial u_z'}{\partial \varphi}\right)}_{\text{VI+VII}}\right]+\\
&+\rho\left[\underbrace{\frac{\partial}{\partial x_l}\left(U_z\frac{\partial u_z'}{\partial z}\right)}_{\text{III+V}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_z'\frac{\partial U_z}{\partial z}\right)}_{\text{II+IV}}+\underbrace{\frac{\partial}{\partial x_l}\left(u_z'\frac{\partial u_z'}{\partial z}\right)}_{\text{VI+VII}}\right]=\\
&=\underbrace{\rho\frac{\partial F_r'}{\partial x_l}}_{\text{VIII}}-\underbrace{\frac{\partial^2 p'}{\partial r\partial x_l}}_{\text{IX}}+\underbrace{\mu\frac{\partial}{\partial x_l}\left(\frac{\partial^2 u_r'}{\partial r^2}+\frac{1}{r}\frac{\partial u_r'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r'}{\partial \varphi^2}+\frac{\partial^2 u_r'}{\partial z^2}-\frac{u_r'}{r^2}-\frac{2}{r^2}\frac{\partial u_\varphi'}{\partial \varphi}\right)}_{\text{X}}+\\
&+\underbrace{\rho\frac{\partial F_\varphi'}{\partial x_l}}_{\text{VIII}}-\underbrace{\frac{\partial}{\partial x_l}\left(\frac{1}{r}\frac{\partial p'}{\partial \varphi}\right)}_{\text{IX}}+\underbrace{\mu\frac{\partial}{\partial x_l}\left(\frac{\partial^2 u_\varphi'}{\partial r^2}+\frac{1}{r}\frac{\partial u_\varphi'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\varphi'}{\partial \varphi^2}+\frac{\partial^2 u_\varphi'}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r'}{\partial \varphi}-\frac{u_\varphi'}{r^2}\right)}_{\text{X}}+\\
&+\underbrace{\rho\frac{\partial F_z'}{\partial x_l}}_{\text{VIII}}-\underbrace{\frac{\partial^2 p'}{\partial z\partial x_l}}_{\text{IX}}+\underbrace{\mu\frac{\partial}{\partial x_l}\left(\frac{\partial^2 u_z'}{\partial r^2}+\frac{1}{r}\frac{\partial u_z'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z'}{\partial \varphi^2}+\frac{\partial^2 u_z'}{\partial z^2}\right)}_{\text{X}}
\end{aligned}
\tag{59}
$$

Sledeći korak je množenje jednačine (59) sa $2\nu\partial u_i'/\partial x_l$, gde je $l=r,\varphi,z$ i deljenje sa ρ tako da se dobija jednačina sa sledećim članovima:

$$
\begin{aligned}
&2\nu\frac{\partial u_r'}{\partial x_l}\left[\frac{\partial}{\partial t}\left(\frac{\partial u_r'}{\partial x_l}\right)\right]+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left[\frac{\partial}{\partial t}\left(\frac{\partial u_\varphi'}{\partial x_l}\right)\right]+2\nu\frac{\partial u_z'}{\partial x_l}\left[\frac{\partial}{\partial t}\left(\frac{\partial u_z'}{\partial x_l}\right)\right]+ && \text{I}\\
&+2\nu\frac{\partial u_r'}{\partial x_l}\left[\frac{\partial U_r}{\partial x_l}\frac{\partial u_r'}{\partial r}+\frac{\partial}{\partial x_l}\left(\frac{U_\varphi}{r}\right)\frac{\partial u_r'}{\partial\varphi}+\frac{\partial U_z}{\partial x_l}\frac{\partial u_r'}{\partial z}\right]+ &&\\
&+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left(\frac{\partial U_r}{\partial x_l}\frac{\partial u_\varphi'}{\partial r}+\frac{\partial}{\partial x_l}\left(\frac{U_\varphi}{r}\right)\frac{\partial u_\varphi'}{\partial\varphi}+\frac{\partial U_z}{\partial x_l}\frac{\partial u_\varphi'}{\partial z}\right)+ &&\\
&+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left(\frac{\partial U_r}{\partial x_l}\frac{\partial u_z'}{\partial r}+\frac{\partial}{\partial x_l}\left(\frac{U_\varphi}{r}\right)\frac{\partial u_z'}{\partial\varphi}+\frac{\partial U_z}{\partial x_l}\frac{\partial u_z'}{\partial z}\right)+ && \text{II}\\
&+2\nu\frac{\partial u_r'}{\partial x_l}\left(U_r\frac{\partial^2 u_r'}{\partial x_l\partial r}+\frac{1}{r}U_\varphi\frac{\partial^2 u_r'}{\partial x_l\partial\varphi}+U_z\frac{\partial^2 u_r'}{\partial x_l\partial z}\right)+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left(U_r\frac{\partial^2 u_\varphi'}{\partial x_l\partial r}+\frac{1}{r}U_\varphi\frac{\partial^2 u_\varphi'}{\partial x_l\partial\varphi}+U_z\frac{\partial^2 u_\varphi'}{\partial x_l\partial z}\right)+ &&\\
&+2\nu\frac{\partial u_z'}{\partial x_l}\left(U_r\frac{\partial^2 u_z'}{\partial x_l\partial r}+\frac{1}{r}U_\varphi\frac{\partial^2 u_z'}{\partial x_l\partial\varphi}+U_z\frac{\partial^2 u_z'}{\partial x_l\partial z}\right)+ && \text{III}\\
&+2\nu\frac{\partial u_r'}{\partial x_l}\left[\frac{\partial u_r'}{\partial x_l}\frac{\partial U_r}{\partial r}+\frac{\partial}{\partial x_l}\left(\frac{u_\varphi'}{r}\right)\frac{\partial U_r}{\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial U_r}{\partial z}\right]+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left[\frac{\partial u_r'}{\partial x_l}\frac{\partial U_\varphi}{\partial r}+\frac{\partial}{\partial x_l}\left(\frac{u_\varphi'}{r}\right)\frac{\partial U_\varphi}{\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial U_\varphi}{\partial z}\right]+ && \text{IV}\\
&+2\nu\frac{\partial u_z'}{\partial x_l}\left[\frac{\partial u_r'}{\partial x_l}\frac{\partial U_z}{\partial r}+\frac{\partial}{\partial x_l}\left(\frac{u_\varphi'}{r}\right)\frac{\partial U_z}{\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial U_z}{\partial z}\right]+ &&\\
&+2\nu\frac{\partial u_r'}{\partial x_l}\left(u_r'\frac{\partial^2 U_r}{\partial x_l\partial r}+\frac{1}{r}u_\varphi'\frac{\partial^2 U_r}{\partial x_l\partial\varphi}+u_z'\frac{\partial^2 U_r}{\partial x_l\partial z}\right)+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left(u_r'\frac{\partial^2 U_\varphi}{\partial x_l\partial r}+\frac{1}{r}u_\varphi'\frac{\partial^2 U_\varphi}{\partial x_l\partial\varphi}+u_z'\frac{\partial^2 U_\varphi}{\partial x_l\partial z}\right)+ && \text{V}\\
&+2\nu\frac{\partial u_z'}{\partial x_l}\left(u_r'\frac{\partial^2 U_z}{\partial x_l\partial r}+\frac{1}{r}u_\varphi'\frac{\partial^2 U_z}{\partial x_l\partial\varphi}+u_z'\frac{\partial^2 U_z}{\partial x_l\partial z}\right)+ &&\\
&+2\nu\frac{\partial u_r'}{\partial x_l}\left[\frac{\partial u_r'}{\partial x_l}\frac{\partial u_r'}{\partial r}+\frac{\partial}{\partial x_l}\left(\frac{u_\varphi'}{r}\right)\frac{\partial u_r'}{\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial u_r'}{\partial z}\right]+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left[\frac{\partial u_r'}{\partial x_l}\frac{\partial u_\varphi'}{\partial r}+\frac{\partial}{\partial x_l}\left(\frac{u_\varphi'}{r}\right)\frac{\partial u_\varphi'}{\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial u_\varphi'}{\partial z}\right]+ && \text{VI}\\
&+2\nu\frac{\partial u_z'}{\partial x_l}\left[\frac{\partial u_r'}{\partial x_l}\frac{\partial u_z'}{\partial r}+\frac{\partial}{\partial x_l}\left(\frac{u_\varphi'}{r}\right)\frac{\partial u_z'}{\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial u_z'}{\partial z}\right]+ &&\\
&+2\nu\frac{\partial u_r'}{\partial x_l}\left(u_r'\frac{\partial^2 u_r'}{\partial x_l\partial r}+\frac{1}{r}u_\varphi'\frac{\partial^2 u_r'}{\partial x_l\partial\varphi}+u_z'\frac{\partial^2 u_r'}{\partial x_l\partial z}\right)+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left(u_r'\frac{\partial^2 u_\varphi'}{\partial x_l\partial r}+\frac{1}{r}u_\varphi'\frac{\partial^2 u_\varphi'}{\partial x_l\partial\varphi}+u_z'\frac{\partial^2 u_\varphi'}{\partial x_l\partial z}\right)+ && \text{VII}\\
&+2\nu\frac{\partial u_r'}{\partial x_l}\left(u_r'\frac{\partial^2 u_z'}{\partial x_l\partial r}+\frac{1}{r}u_\varphi'\frac{\partial^2 u_z'}{\partial x_l\partial\varphi}+u_z'\frac{\partial^2 u_z'}{\partial x_l\partial z}\right)= &&\\
&=2\nu\frac{\partial u_r'}{\partial x_l}\left(\frac{\partial F_r'}{\partial x_l}\right)+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left(\frac{\partial F_\varphi'}{\partial x_l}\right)+2\nu\frac{\partial u_z'}{\partial x_l}\left(\frac{\partial F_z'}{\partial x_l}\right)- && \text{VIII}\\
&-2\nu\frac{\partial u_r'}{\partial x_l}\left(\frac{1}{\rho}\frac{\partial^2 p'}{\partial r\partial x_l}\right)-2\nu\frac{\partial u_\varphi'}{\partial x_l}\left(\frac{1}{\rho}\frac{\partial^2 p'}{\partial\varphi\partial x_l}\right)-2\nu\frac{\partial u_z'}{\partial x_l}\left(\frac{1}{\rho}\frac{\partial^2 p'}{\partial z\partial x_l}\right)+ && \text{IX}\\
&2\nu\frac{\partial u_r'}{\partial x_l}\left[\nu\frac{\partial}{\partial x_l}\left(\frac{\partial^2 u_r'}{\partial r^2}+\frac{1}{r}\frac{\partial u_r'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r'}{\partial\varphi^2}+\frac{\partial^2 u_r'}{\partial z^2}-\frac{u_r'}{r^2}-\frac{2}{r^2}\frac{\partial u_\varphi'}{\partial\varphi}\right)\right]+ && \text{X}\\
&2\nu\frac{\partial u_\varphi'}{\partial x_l}\left[\nu\frac{\partial}{\partial x_l}\left(\frac{\partial^2 u_\varphi'}{\partial r^2}+\frac{1}{r}\frac{\partial u_\varphi'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\varphi'}{\partial\varphi^2}+\frac{\partial^2 u_\varphi'}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r'}{\partial\varphi}-\frac{u_\varphi'}{r^2}\right)\right] &&\\
&2\nu\frac{\partial u_z'}{\partial x_l}\left[\nu\frac{\partial}{\partial x_l}\left(\frac{\partial^2 u_z'}{\partial r^2}+\frac{1}{r}\frac{\partial u_z'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z'}{\partial\varphi^2}+\frac{\partial^2 u_z'}{\partial z^2}\right)\right] && (60)
\end{aligned}
$$

Posle preuređivanja prethodnih članova jednačina (60) se može napisati u sledećem obliku:

$$
\begin{aligned}
&\underset{\mathrm{I}}{\frac{\partial\varepsilon'}{\partial t}}+\underbrace{U_r\frac{\partial\varepsilon'}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\varepsilon'}{\partial\varphi}+U_z\frac{\partial\varepsilon'}{\partial z}}_{\mathrm{III}}=\\
&=\underbrace{-\left[\frac{1}{r}\frac{\partial}{\partial r}\; ru_r'\varepsilon'\;+\frac{1}{r}\frac{\partial}{\partial\varphi}\; u_\varphi'\varepsilon'\;+\frac{\partial}{\partial z}\; u_z'\varepsilon'\;\right]}_{\mathrm{VII}}-\\
&\underbrace{-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{2\nu}{\rho}\frac{\partial p'}{\partial x_l}\frac{\partial u_r'}{\partial x_l}\right)+\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\frac{2\nu}{\rho}\frac{\partial p'}{\partial x_l}\frac{\partial u_\varphi'}{\partial x_l}\right)+\frac{\partial}{\partial z}\left(\frac{2\nu}{\rho}\frac{\partial p'}{\partial x_l}\frac{\partial u_z'}{\partial x_l}\right)\right]}_{\mathrm{IX}}+\\
&\underbrace{+\frac{1}{r}\frac{\partial}{\partial r}\left(r\nu\frac{\partial\varepsilon'}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\nu\frac{1}{r}\frac{\partial\varepsilon'}{\partial\varphi}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial\varepsilon'}{\partial z}\right)}_{\text{prvi deo člana }\mathbf{X}}\underbrace{-2\nu\left[\left(\frac{\partial u_l'}{\partial x_i}\frac{\partial u_l'}{\partial x_k}+\frac{\partial u_i'}{\partial x_l}\frac{\partial u_k'}{\partial x_l}\right)\frac{\partial U_i}{\partial x_k}\right]}_{\mathrm{II+IV}}-\\
&\underbrace{-2\nu u_r'\left(\frac{\partial u_r'}{\partial x_l}\frac{\partial^2U_r}{\partial x_l\partial r}+\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial^2U_\varphi}{\partial x_l\partial r}+\frac{\partial u_z'}{\partial x_l}\frac{\partial^2U_z}{\partial x_l\partial r}\right)-2\nu\frac{u_\varphi'}{r}\left(\frac{\partial u_r'}{\partial x_l}\frac{\partial^2U_r}{\partial x_l\partial\varphi}+\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial^2U_\varphi}{\partial x_l\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial^2U_z}{\partial x_l\partial\varphi}\right)}_{\mathrm{V}}-\\
&\underbrace{-2\nu u_z'\left(\frac{\partial u_r'}{\partial x_l}\frac{\partial^2U_r}{\partial x_l\partial z}+\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial^2U_\varphi}{\partial x_l\partial z}+\frac{\partial u_z'}{\partial x_l}\frac{\partial^2U_z}{\partial x_l\partial z}\right)}_{\mathrm{V}}-\\
&\underbrace{-2\nu\frac{\partial u_r'}{\partial x_l}\left(\frac{\partial u_r'}{\partial x_l}\frac{\partial u_r'}{\partial r}+\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial u_r'}{\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial u_r'}{\partial z}\right)-2\nu\frac{\partial u_\varphi'}{\partial x_l}\left(\frac{\partial u_r'}{\partial x_l}\frac{\partial u_\varphi'}{\partial r}+\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial u_\varphi'}{\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial u_\varphi'}{\partial z}\right)}_{\mathrm{VI}}-\\
&\underbrace{-2\nu\frac{\partial u_z'}{\partial x_l}\left(\frac{\partial u_r'}{\partial x_l}\frac{\partial u_z'}{\partial r}+\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial u_z'}{\partial\varphi}+\frac{\partial u_z'}{\partial x_l}\frac{\partial u_z'}{\partial z}\right)}_{\mathrm{VI}}-\\
&\underbrace{-2\left(\nu\frac{\partial^2u_r'}{\partial r\partial x_l}\right)^2-2\left(\nu\frac{\partial^2u_\varphi'}{\partial r\partial x_l}\right)^2-2\left(\nu\frac{\partial^2u_z'}{\partial r\partial x_l}\right)^2-2\left(\nu\frac{\partial^2u_r'}{\partial\varphi\partial x_l}\right)^2-2\left(\nu\frac{\partial^2u_\varphi'}{\partial\varphi\partial x_l}\right)^2-2\left(\nu\frac{\partial^2u_z'}{\partial\varphi\partial x_l}\right)^2}_{\text{drugi deo člana }\mathbf{X}}-\\
&\underbrace{-2\left(\nu\frac{\partial^2u_r'}{\partial z\partial x_l}\right)^2-2\left(\nu\frac{\partial^2u_\varphi'}{\partial z\partial x_l}\right)^2-2\left(\nu\frac{\partial^2u_z'}{\partial z\partial x_l}\right)^2}_{\text{drugi deo člana }\mathbf{X}}\underbrace{+2\nu\frac{\partial u_r'}{\partial x_l}\left(\frac{\partial F_r'}{\partial x_l}\right)+2\nu\frac{\partial u_\varphi'}{\partial x_l}\left(\frac{\partial F_\varphi'}{\partial x_l}\right)+2\nu\frac{\partial u_z'}{\partial x_l}\left(\frac{\partial F_z'}{\partial x_l}\right)}_{\mathrm{VIII}}
\end{aligned}
\tag{61}
$$

Ovako preuređenu jednačinu potrebno je usrednjiti po pravilima Rejnoldsove statistike.

Konačno, uvođenjem definicije za brzinu disipacije kinetičke energije turbulencije $\varepsilon=\nu\overline{(\partial u_i'/\partial x_l)^2}$, dobija se egzaktna transportna jednačina za brzinu disipacije turbulentne kinetičke energije:

$$
\begin{aligned}
&\underset{a}{\frac{\partial\varepsilon}{\partial t}}+\underbrace{U_r\frac{\partial\varepsilon}{\partial r}+\frac{U_\varphi}{r}\frac{\partial\varepsilon}{\partial\varphi}+U_z\frac{\partial\varepsilon}{\partial z}}_{b}=\\
&=-\frac{1}{r}\frac{\partial}{\partial r}\left(\underbrace{\overline{ru_r'\varepsilon'}}_{c_1}+\underbrace{r\frac{2\nu}{\rho}\overline{\frac{\partial p'}{\partial x_l}\frac{\partial u_r'}{\partial x_l}}}_{c_2}-\underbrace{r\nu\frac{\partial\varepsilon}{\partial r}}_{c_3}\right)-\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\underbrace{\overline{u_\varphi'\varepsilon'}}_{c_1}+\underbrace{\frac{2\nu}{\rho}\overline{\frac{\partial p'}{\partial x_l}\frac{\partial u_\varphi'}{\partial x_l}}}_{c_2}-\underbrace{\frac{1}{r}\nu\frac{\partial\varepsilon}{\partial\varphi}}_{c_3}\right)-\\
&-\frac{\partial}{\partial z}\left(\underbrace{\overline{u_z'\varepsilon'}}_{c_1}+\underbrace{\frac{2\nu}{\rho}\overline{\frac{\partial p'}{\partial x_l}\frac{\partial u_z'}{\partial x_l}}}_{c_2}-\underbrace{\nu\frac{\partial\varepsilon}{\partial z}}_{c_3}\right)\underbrace{-2\nu\left[\left(\overline{\frac{\partial u_l'}{\partial x_i}\frac{\partial u_l'}{\partial x_k}}+\overline{\frac{\partial u_i'}{\partial x_l}\frac{\partial u_k'}{\partial x_l}}\right)\frac{\partial U_i}{\partial x_k}\right]}_{d}-\\
&\underbrace{-2\nu\left(\overline{u_r'\frac{\partial u_r'}{\partial x_l}}\frac{\partial^2U_r}{\partial x_l\partial r}+\overline{u_r'\frac{\partial u_\varphi'}{\partial x_l}}\frac{\partial^2U_\varphi}{\partial x_l\partial r}+\overline{u_r'\frac{\partial u_z'}{\partial x_l}}\frac{\partial^2U_z}{\partial x_l\partial r}\right)}_{e}-
\end{aligned}
$$

$$
\underbrace{-2\nu\left(\frac{1}{r}\overline{u_\varphi'\frac{\partial u_r'}{\partial x_l}}\frac{\partial^2 U_r}{\partial x_l\partial\varphi}+\frac{1}{r}\overline{u_\varphi'\frac{\partial u_\varphi'}{\partial x_l}}\frac{\partial^2 U_\varphi}{\partial x_l\partial\varphi}+\frac{1}{r}\overline{u_\varphi'\frac{\partial u_z'}{\partial x_l}}\frac{\partial^2 U_z}{\partial x_l\partial\varphi}\right)}_{e}-
$$
$$
\underbrace{-2\nu\left(\overline{u_z'\frac{\partial u_r'}{\partial x_l}}\frac{\partial^2 U_r}{\partial x_l\partial z}+\overline{u_z'\frac{\partial u_\varphi'}{\partial x_l}}\frac{\partial^2 U_\varphi}{\partial x_l\partial z}+\overline{u_z'\frac{\partial u_z'}{\partial x_l}}\frac{\partial^2 U_z}{\partial x_l\partial z}\right)}_{e}-
$$
$$
\underbrace{-2\nu\left(\overline{\frac{\partial u_r'}{\partial x_l}\frac{\partial u_r'}{\partial x_l}\frac{\partial u_r'}{\partial r}}+\overline{\frac{\partial u_r'}{\partial x_l}\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial u_r'}{\partial\varphi}}+\overline{\frac{\partial u_r'}{\partial x_l}\frac{\partial u_z'}{\partial x_l}\frac{\partial u_r'}{\partial z}}\right)}_{f}-
$$
$$
\underbrace{-2\nu\left(\overline{\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial u_r'}{\partial x_l}\frac{\partial u_\varphi'}{\partial r}}+\overline{\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial u_\varphi'}{\partial\varphi}}+\overline{\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial u_z'}{\partial x_l}\frac{\partial u_\varphi'}{\partial z}}\right)}_{f}-
$$
$$
\underbrace{-2\nu\left(\overline{\frac{\partial u_z'}{\partial x_l}\frac{\partial u_r'}{\partial x_l}\frac{\partial u_z'}{\partial r}}+\overline{\frac{\partial u_z'}{\partial x_l}\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial u_z'}{\partial\varphi}}+\overline{\frac{\partial u_z'}{\partial x_l}\frac{\partial u_z'}{\partial x_l}\frac{\partial u_z'}{\partial z}}\right)}_{f}-
$$
$$
\underbrace{-2\overline{\left(\nu\frac{\partial^2 u_r'}{\partial r\partial x_l}\right)^2}-2\overline{\left(\nu\frac{\partial^2 u_\varphi'}{\partial r\partial x_l}\right)^2}-2\overline{\left(\nu\frac{\partial^2 u_z'}{\partial r\partial x_l}\right)^2}-2\overline{\left(\nu\frac{\partial^2 u_r'}{\partial\varphi\partial x_l}\right)^2}-2\overline{\left(\nu\frac{\partial^2 u_\varphi'}{\partial\varphi\partial x_l}\right)^2}-2\overline{\left(\nu\frac{\partial^2 u_z'}{\partial\varphi\partial x_l}\right)^2}}_{g}-
$$
$$
\underbrace{-2\overline{\left(\nu\frac{\partial^2 u_r'}{\partial z\partial x_l}\right)^2}-2\overline{\left(\nu\frac{\partial^2 u_\varphi'}{\partial z\partial x_l}\right)^2}-2\overline{\left(\nu\frac{\partial^2 u_z'}{\partial z\partial x_l}\right)^2}}_{g}\underbrace{+2\nu\overline{\frac{\partial u_r'}{\partial x_l}\frac{\partial f_r'}{\partial x_l}}+2\nu\overline{\frac{\partial u_\varphi'}{\partial x_l}\frac{\partial f_\varphi'}{\partial x_l}}+2\nu\overline{\frac{\partial u_z'}{\partial x_l}\frac{\partial f_z'}{\partial x_l}}}_{h} \qquad (62)
$$

Fizičko tumačenje članova u dobijenoj egzaktnoj jednačini za brzinu disipacije kinetičke energije turbulencije (62) je sledeće:

- *a* – lokalna promena promenljive ε,
- *b* – konvektivna promena promenljive ε,
- *c* – difuzioni transport promenljive ε usled dejstva:
  - c_1 – fluktuirajuće brzine,
  - c_2 – fluktuirajućeg pritiska, i
  - c_3 – viskoznih efekata.
- *d* – produkcija promenljive ε usled deformacije usrednjenog toka,
- *e* – produkcija promenljive ε usled dejstva usrednjene vrtložnosti,
- *f* – produkcija promenljive ε usled samoizduženja vrtložnih vlakana,
- *g* – viskozna destrukcija promenljive ε, i
- *h* – produkcija promenljive ε usled dejstva zapreminskih sila.

Zaključak

Na osnovu uvida u dostupnu našu i stranu literaturu može se zaključiti da je vrlo retko detaljno prikazano i objašnjeno izvođenje turbulentnih transportnih jednačina u cilindričnim koordinatama, a posebno je taj nedostatak izražen za izvođenje jednačina turbulentne kinetičke energije i brzinu disipacije kinetičke energije turbulencije, tako da je ovim radom pokušano popuniti prazninu u literaturi u ovoj oblasti. Takođe, pored izvođenja turbulentnih transportnih jednačina, cilj je bio i da se ukaže na suštinsko razumevanje pojedinih članova jednačina.

Ovaj rad može biti od koristi studentima diplomskih i doktorskih studija, kao i inženjerima i istraživačima u praksi pri rešavanju inženjerskih problema i modeliranju turbulentnih strujanja koja su tako česta u termotehnici i energetici.

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Rad je nastao kao rezultat istraživanja na projektu „Povećanje energetske i ekološke efikasnosti procesa u ložištu na ugljeni prah i optimizacija izlazne površine energetskog parnog kotla primenom sopstvenih softverskih alata", ev. broj TR-33018, Ministarstva prosvete, nauke i tehnološkog razvoja Republike Srbije.

Oznake

A – površina, [m^2]
c_p – specifični toplotni kapacitet, [Jkg^{-1}K^{-1}]
F_i – zapreminske sile, [N]
G_e – generisanje toplote u jedinici zapremine, [Wm^{-1}K^{-1}]
h – entalpija, [Jkg^{-1}]
H_i – Lameovi koeficijenti, [Wm^{-1}K^{-1}]
k – turbulentna kinetička energija, [m^{2}s^{-1}]
k_p – toplotna provodljivost, [Wm^{-1}K^{-1}]
h – entalpija, [Jkg^{-1}]
$\vec{n}$ – jedinični vektor normalan na površinu,
q_i – krivolinijske ortogonalne koordinate, [m]
$\vec{q}$ – toplotni fluks, [Wm^{-2}]
p, p' – trenutna i srednja vrednost statičkog pritiska, [Pa]
r – koordinata u cilindričnom koordinatnom sistemu, [m]
$\vec{r}$ – radijus-vektor, [m]
s_i – dužine stranica paralelopipeda, [m]
T – temperatura, [K]
$\underline{\underline{T}}$ – tenzor napona, [Nm^{-2}]
t – vreme, [s]
$\vec{u}$ – vektor brzine, [ms^{-1}]
u_r, u_φ, u_z – trenutne vrednosti brzine, [ms^{-1}]
$\overline{u_r}, \overline{u_\varphi}, \overline{u_z}$ – vremenski osrednjene komponente brzine, [ms^{-1}]
U_r, U_φ, U_z – vremenski osrednjene komponente brzine, [ms^{-1}]
u_r', u_φ', u_z' – fluktuacione brzine, [ms^{-1}]
$\overline{u_i' u_j'}$ – komponente Rejnoldsovih napona, [m^{2}s^{-2}]
V – zapremina, [m^3]
z – koordinata u cilindričnom koordinatnom sistemu, [m]

Grčki simboli

ε – osrednjena vrednost disipacije turbulentne kinetičke energije, [m^{2}s^{-3}]
ε' – trenutna vrednost disipacije turbulentne kinetičke energije, [m^{2}s^{-3}]
μ – dinamička viskoznost, [kgm^{-1}s^{-1}]
ν – kinematska viskoznost, [m^{2}s^{-1}]
ρ – gustina, [kgm^{-3}]
Φ – deformacioni rad koji dovodi do disipacije, [W]
φ – ugao, koordinata u cilindričnom koordinatnom sistemu, [rad]

Indeksi

i, j, k, l	– indeksi koordinatnih osa
t	– turbulentni
'	– fluktuirajuća komponenta

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Abstract

Derivation of Transport Equations for Three-Dimensional Non-Isothermal Turbulent Flow in Cylindrical Coordinates

by

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Paper initially presents the elementary expressions in curvilinear coordinates, and afterwards, the following equations have been derived in cylindrical coordinates, respectively: equation of enthalpy (temperature), Navier-Stokes and Reynolds equation for turbulent flow, transport equations of Reynolds (turbulent) stresses, transport equation of kinetic energy of turbulence, and the transport equation of dissipation rate of turbulent kinetic energy. This paper has been attempted to fill the gap in our and foreign literature with respect to derivation of turbulent transport equations in cylindrical coordinates. This work may be useful to students of graduate and doctoral studies as well as engineers and researchers in practice in solving engineering problems and modeling of turbulent flows, commonly found in thermal engineering and energy.

Key words: *derivation of transport equation, turbulent flow, temperature, turbulence modeling, cylindrical coordinates*

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