

ORDERED B-METRIC SPACES AND GERAGHTY TYPE CONTRACTIVE MAPPINGS

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<https://dx.doi.org/10.5937/vojtehg65-13266>

FIELD: Mathematics, Subject Classification: 47H10, 54H25, 46Nxx
ARTICLE TYPE: Original Scientific Paper
ARTICLE LANGUAGE: English

Abstract:

The paper shows a new approach to proving the recent fixed point results in ordered b-metric as well as ordered metric spaces, established by several authors, with much shorter and nicer proofs. An example is given to illustrate our results.

Key words: fixed point, b-metric, comparable, well order, Geraghty mapping, b-Cauchy, b-complete.

Introduction and preliminaries

One of important generalizations of metric spaces are so-called b -metric spaces (type metric spaces by some authors). This concept was introduced by Bakhtin in 1989 and Czerwik in 1993.

Consistent with (Bakhtin, 1989, pp.26-37) and (Czerwik, 1993, pp.5-11), the following definition and results will be needed in the sequel.

Definition 1.1. (Bakhtin, 1989), (Czerwik, 1993) Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow [0, +\infty)$ is a b -metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

$$(b_1) \quad d(x, y) = 0 \text{ if and only if } x = y,$$

$$(b_2) \quad d(x, y) = d(y, x),$$

$$(b_3) \quad d(x, z) \leq s(d(x, y) + d(y, z)).$$

The pair (X, d) is called a b -etric space.

It should be noted that the class of b -metric spaces is effectively larger than that of metric spaces, since a b -metric is a metric when $s = 1$. The following example shows that, in general, a b -metric does not necessarily need to be a metric, see also (Aghajani, et al, 2014), (Abbas, et al, 2016, pp.1413-1429), (Ansari, et al, 2016), (Ding, et al, 2016, pp.151-164), (Djukić, et al, 2011), (Huang, et al, 2015a, pp.808-815), (Huaping, et al, 2015), (Huang, et al, 2015b, pp.800-807), (Hussain, et al, 2012), (Hussain, et al, 2013), (Jleli, et al, 2012, pp.175-192), (Jovanović, et al, 2010), (Kadelburg, et al, 2015, pp.57-67), (Khamsi, Hussain, 2010, pp.3123-3129), (Parvaneh, et al, 2013), (Roshan, et al, 2015), (Roshan, et al, 2014, pp.229-245), (Zabihi, Razani, 2014).

Example 1.1. Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, $p > 1$ is a real number. Then ρ is a b -metric with $s = 2^{p-1}$, but ρ is not a metric on X .

Otherwise, for more concepts such as b -convergence, b -completeness, b -Cauchy sequence and b -closed set in b -metric spaces, we refer the reader to (Aghajani, et al, 2014, pp.941-960), (Abbas, et al, 2016, pp.1413-1429), (Ansari, et al, 2016), (Djukić, et al, 2011), (Huang, et al, 2015a, pp.808-815), (Huaping, et al, 2015), (Huang, et al, 2015b, pp.800-807), (Hussain, et al, 2012), (Hussain, et al, 2013), (Jleli, et al, 2012, pp.175-192), (Jovanović, et al, 2010), (Kadelburg, et al, 2015, pp.57-67), (Khamsi, Hussain, 2010, pp.3123-3129), (Parvaneh, et al,

2013), (Roshan, et al, 2015), (Roshan, et al, 2014, pp.229-245), (Zabihi, Razani, 2014) and the references mentioned therein. Also, for the concepts such as partial order, comparable, well ordered, nondecreasing, increasing, dominated, dominating and other, we refer the reader to (Aghajani, et al, 2014, pp.941-960), (Abbas, et al, 2016, pp.1413-1429), (Ansari, et al, 2016).

The following three lemmas are very significant in the theory of a fixed point in the framework of metric and b -metric spaces. Also, we use these in the proof of our main results.

Lemma 1.2. (Aghajani, et al, 2014, pp.941-960, Lemma 2.1) Let (X, d) be a b -metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b -convergent to x, y respectively, then we have

$$\frac{1}{s^2}d(x, y) \leq \underline{\lim}_{n \rightarrow \infty} d(x_n, y_n) \leq \overline{\lim}_{n \rightarrow \infty} d(x_n, y_n) \leq s^2d(x, y). \quad (1.1)$$

In particular, if $x = y$, then we have $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$ we have

$$\frac{1}{s}d(x, z) \leq \underline{\lim}_{n \rightarrow \infty} d(x_n, z) \leq \overline{\lim}_{n \rightarrow \infty} d(x_n, z) \leq sd(x, z). \quad (1.2)$$

Lemma 1.3. (Jovanović, et al, 2010, Lemma 3.1) Let $\{y_n\}$ be a sequence in a b -metric space (X, d) with $s \geq 1$, such that

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \quad (1.3)$$

for some $\lambda \in [0, \frac{1}{s})$, and each $n = 1, 2, \dots$. Then $\{y_n\}$ is a b -Cauchy sequence in a b -metric space (X, d) .

Lemma 1.4. (Radenović, et al, 2012, pp.625-645, Lemma 2.1), (Jleli, et al, 2012, pp.175-192, Lemma 2.1) Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that $d(y_n, y_{n+1})$ is nonincreasing and that

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \quad (1.4)$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε^+ when $k \rightarrow \infty$:

$$d(y_{2m_k}, y_{2n_k}), d(y_{2m_k}, y_{2n_k+1}), d(y_{2m_k-1}, y_{2n_k}), d(y_{2m_k-1}, y_{2n_k+1}).$$

Main results

Let Ψ be the family of all nondecreasing functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ such that $\lim_{n \rightarrow \infty} \psi^n(t) = 0$ for all $t > 0$. If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

Our first result is the following:

Theorem 2.1. *Let (X, \preceq) be a partially ordered set and there exists a b -metric d on X such that (X, d) is a b -complete b -metric space. Suppose $s > 1$ and $f : X \rightarrow X$ is an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \circ fx_0$. Assume that*

$$s \cdot \frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)} \cdot d(fx, fy) \leq \psi(M(x, y)) + L \cdot N(x, y) \quad (2.1)$$

for all comparable elements $x, y \in X$, where $L \geq 0$,

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx) \cdot d(y, fy)}{1 + d(fx, fy)} \right\}$$

and

$$N(x, y) = \min \{ d(x, fx), d(x, fy), d(y, fx), d(y, fy) \}.$$

If

(1) f is continuous, or

(2) whenever $\{x_n\}$ is a nondecreasing sequence in X such that

$x_n \rightarrow u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Suppose that $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, \dots$, where $x_{n+1} = fx_n = f^n x_0$. In this case, we have $x_n \prec x_{n+1}$ for all $n = 0, 1, 2, \dots$.

Therefore, putting $x = x_n, y = x_{n+1}$ in (2.1) we shall prove that

$$d(x_{n+1}, x_{n+2}) \prec \frac{1}{s} d(x_n, x_{n+1}) \quad (2.2)$$

for all $n = 0, 1, 2, \dots$. Indeed, then (2.1) becomes

$$\begin{aligned}
 & s \frac{1 + sd(x_n, x_{n+1})}{1 + \frac{1}{2}d(x_n, x_{n+1})} d(x_{n+1}, x_{n+2}) \\
 & \leq \psi \left(\max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_{n+1}, x_{n+2})} \right\} \right) \\
 & \quad + L \min \{d(x_n, x_{n+1}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1}), d(x_{n+1}, x_{n+2})\}.
 \end{aligned}$$

Since, $1 < \frac{1 + sd(x_n, x_{n+1})}{1 + \frac{1}{2}d(x_n, x_{n+1})}, \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_{n+1}, x_{n+2})} < d(x_n, x_{n+1})$ and

$d(x_{n+1}, x_{n+1}) = 0$, we have $s \cdot d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1})$. Hence, (2.2) follows.

Further, using (2.2), we have

$$d(f^2x_n, f^2x_{n+1}) < \frac{1}{s}d(fx_n, fx_{n+1}) < \frac{1}{s^2}d(x_n, x_{n+1}).$$

As $\frac{1}{s^2} \in [0, \frac{1}{s})$, therefore by using Lemma 1.4, the sequence $\{f^2x_n\}_{n=0}^{+\infty} = \{x_2, x_3, \dots\}$ is a b -Cauchy sequence. This further implies that the sequence $\{fx_n\}_{n=0}^{+\infty} = \{x_1, x_2, \dots\}$ is a b -Cauchy sequence. Since (X, d) is b -Complete, $\{x_n\}$ b -converges to a point $u \in X$.

(1) First, we suppose that f is continuous. Therefore, we have

$$u = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} fx_n = f(\lim_{n \rightarrow \infty} x_n) = fu,$$

that is, u is a fixed point of f .

(2) Further, consider (2) of theorem holds. Using the assumption of (X, d, \preceq) , we have $x_n \preceq u$. Now, we show that $fu = u$. Firstly, we have

$$\frac{1}{s}d(u, fu) \leq d(u, x_{n+1}) + d(fx_n, fu).$$

Now, using the assumption $x_n \preceq u$ and inequality (2.1), we have

$$\frac{1}{s}d(u, fu) \leq d(u, x_{n+1}) + \frac{1 + \frac{1}{2}d(x_n, x_{n+1})}{s(1 + sd(x_n, u))} \cdot M(x_n, u)$$

$$1 + \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{s(1 + sd(x_n, u))} \cdot L \cdot N(x_n, u).$$

Since $M(x_n, u) \rightarrow 0$ and $N(x_n, u) \rightarrow 0$ as $n \rightarrow \infty$, the result follows, i.e., $fu = u$.

From Theorem 2.1, we have the following result which is an improvement from the corresponding results (Theorems 2.7 and 2.8) of (Ansari, et al, 2016).

Corollary 2.1. Let (X, \preceq) be a partially ordered set and there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Suppose $s > 1$ and $f : X \rightarrow X$ is an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Assume that

$$\frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)} \cdot d(fx, fy) \leq \beta(d(x, y)) \cdot M(x, y) + L \cdot N(x, y) \quad (2.3)$$

for all comparable elements $x, y \in X$,

where $L \geq 0$, $\beta : [0, +\infty) \rightarrow [0, \frac{1}{s})$ with $\beta(t_n) \rightarrow \frac{1}{s}$ implies $t_n \rightarrow 0$,

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx) \cdot d(y, fy)}{1 + d(fx, fy)} \right\}$$

and

$$N(x, y) = \min \{ d(x, fx), d(x, fy), d(y, fx), d(y, fy) \}.$$

If

(1) f is continuous, or

(2) whenever $\{x_n\}$ is a nondecreasing sequence in X such that

$x_n \rightarrow u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Since $\beta(d(x, y)) < \frac{1}{s}$, the condition (2.3) implies

$$s \cdot \frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)} \cdot d(fx, fy) \leq M(x, y) + L_1 \cdot N(x, y), \tag{2.4}$$

where $L_1 = s \cdot L$. On the similar lines of Theorem 2.1, we have the result.

On the similar lines of Theorem 2.1, we have the following result.

Theorem 2.2. *Let (X, \preceq) be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space (with parameter $s > 1$). Let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$s \cdot d(fx, fy) \leq \beta(d(x, y))M(x, y) + L \cdot N(x, y), \tag{2.5}$$

for all comparable elements $x, y \in X$,

where $L \geq 0$, $\beta : [0, +\infty) \rightarrow [0, \frac{1}{s})$ with $\beta(t_n) \rightarrow \frac{1}{s}$ implies $t_n \rightarrow 0$,

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx) \cdot d(y, fy)}{1 + d(fx, fy)} \right\}$$

and

$$N(x, y) = \min \{ d(x, fx), d(x, fy), d(y, fx), d(y, fy) \}.$$

If

(1) f is continuous, or

(2) whenever $\{x_n\}$ is a nondecreasing sequence in X such that

$x_n \rightarrow u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. The condition (2.5) implies

$$d(fx, fy) \leq \frac{1}{s^2} \cdot M(x, y) + L_1 \cdot N(x, y), \tag{2.6}$$

for all comparable elements $x, y \in X$, where $L_1 = \frac{L}{s} \geq 0$. The rest of the

proof is similar to Theorem 2.1.

Remark 2.1. *Since the proofs of the main results in (Ansari, et al, 2016), (Zabihi, Razani, 2014) are strongly dependent of Lemma 1.2 of Aghajani et al. (Aghajani, et al, 2014, pp.941-960), it is too complex to deal*

with them. Our approach in Theorems 2.1-2.2, as well as in Corollary 2.1 covers all the results of (Aghajani, et al, 2014, pp.941-960) without utilizing the lemma mentioned above. It is clear that our proofs are much shorter and nicer.

Also, it is not hard to see that the main results in (Abbas, et al, 2016, pp.1413-1429) have much shorter proofs by the application of our approach, that is, without using Lemma 1.2 of (Aghajani, et al, 2014, pp.941-960).

In the sequel, we consider all three results in the case where $s = 1$, that is, (X, d) is a standard metric space. Here we have to use Lemma 1.4 to obtain our results.

Theorem 2.3. Let (X, \preceq) be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose $f : X \rightarrow X$ is an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Assume that

$$\frac{1+d(x,y)}{1+\frac{1}{2}d(x,fx)} \cdot d(fx,fy) \leq \psi(M(x,y)) + L \cdot N(x,y) \quad (2.7)$$

for all comparable elements $x, y \in X$, where $L \geq 0$,

$$M(x,y) = \max \left\{ d(x,y), \frac{d(x,fx) \cdot d(y,fy)}{1+d(fx,fy)} \right\}$$

and

$$N(x,y) = \min \{ d(x,fx), d(x,fy), d(y,fx), d(y,fy) \}.$$

If

(1) f is continuous, or

(2) whenever $r\{x_n\}$ is a nondecreasing sequence in X such that

$x_n \rightarrow u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. First we suppose that $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, \dots$. Then, by taking $x = x_n, y = x_{n+1}$ in (2.7), we get

$$\frac{1+d(x_n, x_{n+1})}{1+\frac{1}{2}d(x_n, x_{n+1})} \cdot d(x_{n+1}, x_{n+2}) \leq \psi(M(x_n, x_{n+1})) + L \cdot N(x_n, x_{n+1}),$$

where

$$M(x_n, x_{n+1}) = \max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) \cdot d(x_{n+1}, x_{n+2})}{1+d(x_{n+1}, x_{n+2})} \right\} = d(x_n, x_{n+1}),$$

because $\frac{d(x_{n+1}, x_{n+2})}{1+d(x_{n+1}, x_{n+2})} < 1$, and

$$N(x_n, x_{n+1}) = \min \{d(x_n, x_{n+1}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1}), d(x_{n+1}, x_{n+2})\} = 0.$$

Since

$$\frac{1+d(x_n, x_{n+1})}{1+\frac{1}{2}d(x_n, x_{n+1})} > 1, \psi(M(x_n, x_{n+1})) < d(x_n, x_{n+1}) \text{ and } N(x_n, x_{n+1}) \quad (2.8)$$

becomes $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$,

i.e., $d(x_n, x_{n+1})$ is a decreasing sequence. Therefore, there exists $r \geq 0$

such that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r$. Assume that $r > 0$, from (2.8), we have

$$\frac{1+r}{1+\frac{1}{2}r} \cdot r \leq r \Leftrightarrow \frac{1}{2}r \leq 0,$$

which is a contradiction. Hence $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Now, we suppose that the sequence $\{x_n\}$ is not a Cauchy sequence in a metric space (X, d) . By putting $x = x_{m(k)}, y = x_{n(k)}$ in (2.7), we obtain

$$\frac{1+d(x_{m(k)}, x_{n(k)})}{1+\frac{1}{2}d(x_{m(k)}, x_{m(k)+1})} \cdot d(x_{m(k)+1}, x_{n(k)+1}) \leq \psi(M(x_{m(k)}, x_{n(k)})) + L \cdot N(x_{m(k)}, x_{n(k)}), \quad (2.9)$$

where

$$M(x_{m(k)}, x_{n(k)}) = \max \left\{ d(x_{m(k)}, x_{m(k)+1}), \frac{d(x_{m(k)}, x_{m(k)+1}) \cdot d(x_{n(k)}, x_{n(k)+1})}{1+d(x_{m(k)+1}, x_{n(k)+1})} \right\},$$

and

$$N(x_{m(k)}, x_{n(k)})$$

$$= \min\{d(x_{m(k)}, x_{m(k)+1}), d(x_{m(k)}, x_{n(k)+1}), d(x_{n(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1})\}$$

Now, letting to the limit in (2.9), as $k \rightarrow \infty$, and using Lemma 1.4, we get

$$\frac{1+\varepsilon}{1+\frac{1}{2}\varepsilon} \cdot \varepsilon \leq \psi(\varepsilon) + L \cdot 0 < \varepsilon \Leftrightarrow \frac{1}{2}\varepsilon \leq 0,$$

which is a contradiction. Hence the sequence $\{x_n\}$ is a Cauchy sequence. The rest of the proof is the same as in Theorem 2.1.

Corollary 2.2. Let (X, \preceq) be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Suppose $f : X \rightarrow X$ is an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Assume that

$$\frac{1+d(x,y)}{1+\frac{1}{2}d(x,fx)} \cdot d(fx,fy) \leq \beta(d(x,y)) \cdot M(x,y) + L \cdot N(x,y) \quad (2.10)$$

for all comparable elements $x, y \in X$,

where $L \geq 0, \beta : [0, +\infty) \rightarrow [0, 1)$ with $\beta(t_n) \rightarrow 1^-$ implies $t_n \rightarrow 0$,

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx) \cdot d(y,fy)}{1+d(fx,fy)}\right\}$$

and

$$N(x,y) = \min\{d(x,fx), d(x,fy), d(y,fx), d(y,fy)\}.$$

If

(1) f is continuous, or

(2) whenever $\{x_n\}$ is a nondecreasing sequence in X such that

$x_n \rightarrow u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Since $\beta(d(x,y)) < 1$, the condition (2.10) implies that

$$\frac{1+d(x,y)}{1+\frac{1}{2}d(x,fx)} \cdot d(fx,fy) \leq M(x,y) + L \cdot N(x,y). \quad (2.11)$$

On the similar lines of Theorem 2.3, we have the result.

Remark 2.2. It is not hard to see that both functions ψ and β in all results are superfluous. But in our next result, the function β is not superfluous.

Theorem 2.4. Let (X, \preceq) be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that

$$d(fx, fy) \leq \beta(d(x, y))M(x, y) + L \cdot N(x, y), \tag{2.12}$$

for all comparable elements $x, y \in X$, where

$L \geq 0, \beta : [0, +\infty) \rightarrow [0, 1)$ with $\beta(t_n) \rightarrow 1^-$ implies $t_n \rightarrow 0$,

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx) \cdot d(y, fy)}{1 + d(fx, fy)} \right\}$$

and

$$N(x, y) = \min \{ d(x, fx), d(x, fy), d(y, fx), d(y, fy) \}.$$

If

- (1) f is continuous, or
- (2) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \rightarrow u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

The following example support our theoretical result given with Corollary 2.1.

Example 2.3. Let $X = \{0, 1, 2\}$ and define the partial order \preceq on X by $\preceq := \{(0, 0), (1, 1), (2, 2), (0, 1)\}$. Consider the function $f : X \rightarrow X$ given as $f0 = f1 = 1, f3 = 0$ which is nondecreasing with respect to \preceq . Let $x_0 = 0$. Hence $fx_0 = f0 = 1$, so $x_0 \preceq fx_0$. Now, define the b-metric on X by $d(x, y) = (x - y)^2$ for all $x, y \in X$. Then (X, d) is a b-complete b-metric space with $s = 2$. It is easy to verify that this example satisfies all the

conditions of Corollary 2.1 for each $\beta : [0, \infty) \rightarrow [0, \frac{1}{2})$ with $t_n \rightarrow 0+$ whenever $\beta(t_n) \rightarrow \frac{1}{2}-$.

Finally, we formulate the following result (Geraghty fixed point theorem in the framework of a b -complete b -metric space):

Theorem 2.5. Let (X, d) be a b -complete b -metric space and let $s > 1$. Suppose that a mapping $f : X \rightarrow X$ satisfies the condition

$$d(fx, fy) \leq \beta(d(x, y))d(x, y),$$

for all $x, y \in X$, where $\beta : [0, \infty) \rightarrow [0, 1)$ with $t_n \rightarrow 0+$ whenever $\beta(t_n) \rightarrow 1-$ for each sequence $t_n \in (0, \infty)$.

Question. Prove or disprove Theorem 2.5.

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УПОРЯДОЧЕННЫЕ Б-МЕТРИЧЕСКИЕ ПРОСТРАНСТВА И СЖИМАЮЩИЕ ОТОБРАЖЕНИЯ ТИПА GERAGHTY

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ОБЛАСТЬ: математика

ВИД СТАТЬИ: оригинальная научная статья

ЯЗЫК СТАТЬИ: английский

Резюме:

В настоящей статье, благодаря новому методу, представлены новейшие результаты исследований неподвижной точки, проведенные разными авторами. Представление данных результатов подкреплены примерами.

Ключевые слова: неподвижная точка, б-метрика, сравнительный, упорядоченный, GERAGHTY-отображение, б-Коши, б-комплет.

УРЕЂЕНИ Б-МЕТРИЧКИ ПРОСТОРИ И КОНТРААКТИВНА ПРЕСЛИКАВАЊА ГЕРАГХТИЈЕВОГ ТИПА

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ОБЛАСТ: математика

ВРСТА ЧЛАНКА: оригинални научни чланак

ЈЕЗИК ЧЛАНКА: енглески

Сажетак:

Коришћењем новог приступа, у раду су представљени недавни резултате фиксне тачке, коју је установило више аутора, на много краћи и лепши начин. Наведен је и пример који то илуструје.

Кључне речи: фиксна тачка, б-метрика, упоредив, добро уређен, Герагхтијево пресликавање, б-Кошијев, б-комплет.

Paper received on / Дата получения работы / Датум пријема чланка: 24.01.2017.
Manuscript corrections submitted on / Дата получения исправленной версии работы /
Датум достављања исправки рукописа: 11. 03. 2017.
Paper accepted for publishing on / Дата окончательного согласования работы / Датум
коначног прихватања чланка за објављивање: 13. 03. 2017.

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