EXTENSIONS OF THE BANACH CONTRACTION PRINCIPLE IN MULTIPLICATIVE METRIC SPACES

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Abstract:

In this paper, we have proven several generalizations of the Banach contraction principle for multiplicative metric spaces. We have also derived the Cantor intersection theorem in the setup of multiplicative metric spaces. Non-trivial supporting examples are also given.

Key words: Multiplicative metric, Multiplicative open ball, Multiplicative Cauchy sequence, Multiplicative contraction.

Introduction

The study of fixed points of mappings satisfying certain contractive conditions has many fruitful applications in various branches of mathematics; hence, it has extensively been investigated by many authors (Rad, et al, nd), (Radenović, et al, nd), (Mustafa, et al, 2016, pp.110-116), (Radenović, et al, 2016, pp.38-40). The Banach contraction principle has been the most versatile and effective tool in the fixed-point theory (Banach, 1922, pp.133-181). Generalization of the Banach contraction principle has been one of the most investigated branches of research. Matthews (1994, pp.183-197) introduced the concept of partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Hitzler (2001) generalized the Banach contraction principle in the context of a dislocated metric space.

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Zeyada (2005, pp.111-114) improved the work of Hitzler in a dislocated quasi metric space. Shatanawia & Nashine (2012, pp.37-43) studied the Banach contraction principle for nonlinear contraction ina partial metric space. Suzuki (2008, pp.1861-1869) characterized metric completeness by the generalized Banach contraction principle. Boyd and Wong (1969, pp.458- 464) showed that the constant used in the Banach contraction principle can be replaced by an upper semi-continuous function. Hadžić and Pap (2001) extended the contraction principle to probabilistic metric. Jainet al. (2012, pp.252-258) generalized the Banach contraction principle for cone metric spaces. There have been a number of generalizations of a metric space. Some examples of such generalizations are given above. One such generalization is a multiplicative metric space, where Özavsar and Cevikel (2012) introduced the notion of multiplicative contraction mappings and derived some fixed-point results for such mappings on a complete multiplicative metric space.

Hxiaoju, et al. (2014) established some common fixed points for weak commutative mappings on a multiplicative metric space.

In the current paper, we establish an extension of the famous Banach contraction principle in multiplicative metric spaces. The Banach theorem is extended in two ways:

1. The contraction constant depends on the multiplicative distance between the points under consideration.
2. The behavior of $d(x; T x)$ is considered instead of the comparison of $d(T x, T y)$ and $d(x, y)$.

The derived results carry the fixed-point results of Dugundji and Granas (1982) in a metric space to a multiplicative metric space. Furthermore, to complete the proof of the extension of the Banach theorem, we also derived the Cantors intersection theorem in multiplicative metric spaces.

Definition 1.1. (Bashirov et al, 2008) A multiplicative metric on a nonempty set X is a mapping

$d: X \times X \to R$ satisfying the following condition:

(1) $d(x, y) \geq 1$ for all $x, y \in X$;

(2) $d(x, y) = 1$ if and only if $x = y$;

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(4) $d(x, z) \leq d(x, y). d(y, z)$ for all $x, y, z \in X$.

The pair (X,d) is called a multiplicative metric space.

Example 1.1. Let R_+^n denote the set of n-tuples of positive real numbers.

And let $d^*: R_+^n \times R_+^n \to R$ be defined as

$$d^*(x,y) = \left|\frac{x_1}{y_1}\right|^* \cdot \left|\frac{x_2}{y_2}\right|^* \cdot \left|\frac{x_3}{y_3}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*$$

Where $x=(x_1, x_2, \ldots, x_n)$, $y=(y_1, y_2, \ldots, y_n) \in R_+^n$ and $|.|^* : R_+ \to R_+$ is defined as

$$|a|^* = \begin{cases} a & \text{if } a \geq 1 \\ \frac{1}{a} & \text{if } a < 1 \end{cases}$$

Then, clearly, $d^*(x, y)$ is a multiplicative metric (Bashirov et al, 2008).

Example 1.2. Let (X,d) be a metric space, then the mapping d_a defined on X as follows is a multiplicative metric,

$$d_a(x,y) = a^{d(x,y)} \text{ where } a > 1.$$

The following definitions are given by Özavsar and Cevikel (2012).

Definition 1.2. Let (X,d) be a multiplicative metric space. If $a \in X$ and $r>1$, then a subset

$$B_r(a) = B(a;\ r) = \{x \in X : d(a;\ x) < r\}$$

of X is called a multiplicative open ball centered at a with the radius r. Analogously, one can define a multiplicative closed ball as

$$\overline{B}_r(a) = \overline{B}(a;\ r) = \{x \in X : d(a;\ x) \leq r\}$$

Definition 1.3. Let A be any subset of a multiplicative metric space (X,d). A point $x \in X$ is called a limit point of A if and only if $(A \cap B_\varepsilon(x)) - \{x\} \neq \varnothing$ for every $\varepsilon > 1$.

Definition 1.4. Let (X,d) and (Y,ρ) be given multiplicative metric spaces and $a \in X$. A function $f:(X,d) \to (Y,\rho)$ is said to be multiplicative continuous at a, if for given $\varepsilon > 1$, there exists a $\delta > 1$ such that

$$d(x,\ a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon \text{ or equivalently } f(B(a;\delta)) \subset B(f(a);\ \varepsilon).$$

Where $B(a;\delta)$ and $B(f(a);\ \varepsilon)$

are open balls in (X,d) and (Y,ρ) respectively. The function f is said to be continuous on X if it is continuous at each point of X.

Definition 1.5. A sequence $\{x_n\}$ in a multiplicative metric space (X,d) is said to be multiplicative convergent to a point $x \in X$ if for a given $\varepsilon > 1$ there exists a positive integer n_0 such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$ or equivalently, if for every multiplicative open ball $B_\varepsilon(x)$ there exists a positive integer n_0 such that $n \geq n_0 \Rightarrow x_n \in B_\varepsilon(x)$ then the sequence $\{x_n\}$ is said to bemultiplicative-convergent to a point $x \in X$ denoted by $xn \to x(n \to \infty)$.

Definition 1.6. A sequence $\{x_n\}$ in a multiplicative metric space *(X,d)* is said to be multiplicative Cauchy sequence if for every $\varepsilon>1$ there exists a positive integer n_0 such that $d(x_n, x_m) < \varepsilon$ for all n, $m \geq n_0$.

Definition 1.7. A multiplicative metric space *(X,d)* is said to be complete if every multiplicative Cauchy sequence in *X* converges in *X* in the multiplicative sense.

Definition 1.8. Let *(X,d)* be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real number α where $0<\alpha<1$ such that

$$d(f(x_1), f(x_2)) \leq d(x_1, x_2)^\alpha \text{ for all } x_1, x_2 \in X.$$

Theorem 1.1. In a multiplicative metric space, every multiplicative convergent sequence is a multiplicative Cauchy sequence.

Lemma 1.1. A sequence $\{x_n\}$ in a multiplicative metric space *(X,d)* is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1 \ (n, m \to \infty)$

Theorem 1.2. (*Banach Contraction Principle*): Let *(X,d)* be a multiplicative metric space and let *f: X→X* be a multiplicative contraction. If *(X,d)* is complete, then *f* has a unique fixed point.

Main Results

In this section, we are attempting to extend the famous Banach contraction principle into multiplicative metric spaces.

Theorem 2.1. Let *(M,d)* be a complete multiplicative metric space and let *T: M→M*. Also assume that for each $\alpha>1$ there is a $\gamma(\alpha)>1$ such that if $d(x,Tx)<\gamma(\alpha)$ then $T(B\alpha(x)) \subset B\alpha(x)$.

If $d(T^n y, T^{n+1} y) \to 1$ for some $y \in M$, then the sequence $\{T^n y\}$ converges to a fixed point of T .

Proof. We first show that $\{T^n y\}$ is a multiplicative Cauchy sequence. Let, for the sake of brevity, define $T^n y = y_n$. Given $\alpha>1$, choose a natural number n_0 so that $d(y_n, y_{n+1}) < \gamma\sqrt{\alpha}$ for all $n \geq n_0$.

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Since $d(y_n,y_{n+1})=d(y_n,Ty_n)<\gamma\sqrt{\alpha}$ we get $T(B_{\sqrt{\alpha}}(y_n))\subset B_{\sqrt{\alpha}}(y_n)$. This gives $y_{n+1}=Ty_n\in B_{\sqrt{\alpha}}(y_n)$ and $T^j y_n = Ty_{n+j}\in B_{\sqrt{\alpha}}(y_n)$ by induction for all $j\geq 0$.

Then $d(y_k,y_j)\leq d(y_k,y_{n_0}).d(y_{n_0},y_j)<\sqrt{\alpha}.\sqrt{\alpha}=\alpha$ for all $j,k\geq n_0$.

It means $\{y_n\}$ = $\{T^n y\}$ is a multiplicative Cauchy sequence and, due to the completeness of *M*, converges to some point $z\in M$. Now we claim that $z=Tz$. Suppose by way of contradiction that $z\neq Tz$ then $d(z, Tz)=\beta>1$.

Choose $z_n\in B_{\sqrt[3]{\beta}}(z)$ such that $d(z_n,z_{n+1})<\gamma\sqrt[3]{\beta}$. Then, by the hypothesis of the theorem $T(B_{\sqrt[3]{\beta}}(z_n))\subset B_{\sqrt[3]{\beta}}(z_n)$. Therefore, $Tz\in B_{\sqrt[3]{\beta}}(z_n)$.

But since $d(Tz,z)\leq d(Tz,z_n).\ d(z_n,z)\Rightarrow d(Tz,z_n)\geq\dfrac{d(Tz,z)}{d(z_n,z)}\geq\dfrac{\beta}{\sqrt[3]{\beta}}=\beta^{\frac{2}{3}}$.

As $\beta^{\frac{2}{3}}>\sqrt{\beta}$ for β >1.

Therefore, $Tz\notin B_{\sqrt[3]{\beta}}(z_n)$ gives a contradiction. Hence, $Tz=z$. This completes the proof.

Theorem 2.2. Let *(M,d)* be a complete multiplicative metric space, and let $T: M\rightarrow M$ be a mapping satisfying

$$d(Tx,Ty)\leq\delta[d(x,y)]. \tag{2.1}$$

Where $\delta:[1,\infty)\rightarrow[1,\infty)$ is any non-decreasing (not necessarily continuous) mapping such that $\delta^n(t)\rightarrow 1$ for each fixed $t>1$.

Then the sequence $\{T^n x\}$ converges to a fixed point of T in M.

Proof. We claim that $\delta(t)<t$ for each $t>1$; because if $t\leq\delta(t)$ for some $t>1$, then by monotonicity $\delta(t)\leq\delta[\delta(t)]$, which by induction implies that $t\leq\delta^n(t)$ for all $n>0$, implying that $t\leq 1$, which is a contradiction. Now, by equation (2.1), we have $d(T^n x,T^{n+1}x)\leq\delta^n d(x,Tx)$. Hence $d(T^n x,T^{n+1}x)\rightarrow 1$ for each $x\in M$.

Let α be given, and $\gamma(\alpha)=\dfrac{\alpha}{\delta(\alpha)}>1$. If $d(x,Tx)<\gamma(\alpha)$ for any $u\in B(x,\alpha)$, using multiplicative triangular inequality we have

$$d(Tu,x) \le d(Tu,Tx).d(Tx,x) \le \delta[d(u,x)].\gamma \le \delta(\alpha).\frac{\alpha}{\delta(\alpha)} = \alpha.$$

It means $Tu \in B(x,\alpha)$. The rest of the proof is followed by Theorem 2.1.

Theorem 2.3. *Let (M,d) be a complete multiplicative metric space and T: M →M be a map satisfying*

$$d(Tx,Ty) \le d(x,y)^{\delta(x,y)}. \tag{2.2}$$

Where $\delta : M \times M \to [1,\infty)$ has the property that for any closed interval *[a, b] ⊂ [1, ∞),* $\sup\{\delta(x,y) \mid a \le d(x,y) \le b\} = \mu(a,b) < 1$, here $\mu(a,b)$ denotes a maximum value of $\delta(x,y)$ for all $x,y \in [a,b]$. Then T has a unique fixed point $\upsilon \in M$ and $T^n x \to \upsilon$ for each $x \in M$.

Proof. Using condition (2.2), for any positive integer *n* and $x \in M$, we have

$$d(T^{n+1}x,T^{n+2}x) \le d(T^n x,T^{n+1}x)^{\delta(T^n x,T^{n+1}x)}$$

$$\le d(T^n x,T^{n+1}x)^{\mu(T^n x,T^{n+1}x)} < d(T^n x,T^{n+1}x)$$

That is $\{d(T^n x,T^{n+1}x)\}_n^\infty$ is a decreasing sequence and therefore converges to some $\epsilon \ge 1$. We claim that $\epsilon = 1$.Suppose on the contrary that $\epsilon > 1$. Obviously, there will be some positive integer n_0 such that $d(T^n x,T^{n+1}x) \in [\varepsilon,\varepsilon+1]$ for all $n \ge n_0$. We can choose an integer $m \ge n_0$ and let $\kappa \in \mu(\varepsilon,\varepsilon+1)$, we get by inductio

$$\varepsilon \le d(T^{m+p}x,T^{m+1+p}x) \le d(T^m x,T^{m+1}x)^{\kappa^p} \le (\varepsilon+1)^{\kappa^p} \quad \text{for all } p \ge 0.$$

As $\kappa < 1$, so letting $p \to \infty$, we have $\varepsilon \le 1$, which is a contradiction.

Now let $\alpha > 1$, $\mu = \mu(\sqrt{\alpha},\alpha)$ and $\gamma = \min\{\sqrt{\alpha}, \alpha^{1-\mu}\}$. Let $d(x,Tx) < \gamma$ and $z \in B(x,\alpha)$. Using multiplicative triangular inequality, we have $d(Tz,x) \le d(Tz,Tx).d(Tx,x)$. We distinguish the following two cases:

Case-1: $d(x,z) < \sqrt{\alpha}$, then

$$d(Tz,x) \le d(Tz,Tx).d(Tx,x) < d(z,x).d(Tx,x) < \sqrt{\alpha}.\sqrt{\alpha} = \alpha$$

Case-2: $\sqrt{\alpha} \le d(x,z) < \alpha$, then

$$d(Tz,x) \le d(Tz,Tx).d(Tx,x) \le d(z,x)^{\delta(z,x)}.d(Tx,x) \le \alpha^{\mu}.\alpha^{1-\mu} = \alpha.$$

In both cases, $T(B_{\alpha}(x)) \subset B_{\alpha}(x)$. Consequently, the existence of the fixed point of T follows from Theorem 2.1.

For the uniqueness of the fixed point T, consider $Tz = z \neq w = Tw$ where $x, w \in M$. Using (2.2), we have, $d(z,w) = d(Tz,Tw) \le d(z,w)^{\delta(z,w)}$. Which for $\delta(z,w) < 1$, gives a contradiction. Hence $z = w$.

Definition 2.1. The sequence of non-empty sets $\{S_n\}$ in a multiplicative metric space M is said to be a nested sequence of sets, if

1) $S_n \supseteq S_{n+1}$, $n = 1,2,...$
2) The diameter $\delta(S_n)$ of S_n tends to 1 as $n \to \infty$.

Theorem 2.4. (Cantor's Intersection Theorem)A multiplicative metric space *(M,d)* is complete if and only if every nested sequence of closed sets has a non-empty intersection.

Proof. Suppose the multiplicative metric space M is complete and let $S_1 \supseteq S_2 \supseteq S_3 \supseteq ... \supseteq S_n \supseteq ...$ be a nested sequence of closed sets. Select a point x_n in S_n, $n = 1,2,...$ We show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 1$ be given. As $\delta(S_n) \to 1$ for $n \to 1$. There will be a positive integer n_0 such that $\delta(S_n) = \sup\limits_{a,b \in S_n} d(a,b) < \varepsilon$ for $n \ge n_0$.

Now x_m is in $S_n \supseteq S_m$ for $m \ge n$. Therefore, $x_m, x_n \in S_{n_0}$ for all $m,n > n_0$ and $d(x_m, x_n) < \varepsilon$. Hence, $\{x_n\}$ is a Cauchy sequence in M: Due to the completeness of M, $x_n \to x \in M$. Next, we are going to show that $x \in \bigcap\limits_{n=1}^{\infty} S_n$. For any integer $n \ge n_0$, the elements $x_n, x_{n+1}, ...$ are all in S_n. As x is the limit point of the set of these points of S_n, so x is the limit point of S_n as well. Also, as S_n is closed, therefore $x \in S_n$ for $n \ge n_0$. Hence,

$x \in \bigcap_{n=1}^{\infty} S_n \neq \phi.$ Conversely, suppose every nested sequence of closed sets has a non-empty intersection. We shall show that M is complete. Let $\{x_n\}$ be a Cauchy sequence in M. Then for every $\varepsilon > 1$ there will be a positive integer n_0 such that $d(x_m, x_n) < \varepsilon \ \forall \ m, n \geq n_0$.

Take $\varepsilon = 2^{\frac{1}{2}}$ and let n_1 be a positive integer such that $d(x_n, x_{n_1}) < 2^{\frac{1}{2}} \quad n_1 > n_0$.

Let $S_1 = \overline{B}(x_{n_1}, 2)$, take $\varepsilon = 2^{\frac{1}{4}}$ and let n_2 be a positive integer such that $d(x_{n_1}, x_{n_2}) < 2^{\frac{1}{4}} \quad n_2 > n_1$.

Let $S_2 = \overline{B}(x_{n_2}, 2^{\frac{1}{2}})$. Again take $\varepsilon = 2^{\frac{1}{8}}$ and let n_3 be a positive integer such that $d(x_{n_2}, x_{n_3}) < 2^{\frac{1}{8}} \quad n_3 > n_2$.

Let $S_3 = \overline{B}(x_{n_3}, 2^{\frac{1}{4}})$. Clearly S_1, S_2 and S_3 are closed and $S_1 \supseteq S_2 \supseteq S_3$. Continuing in the same way, choose $n_1 < n_2 < n_3 < \ldots < n_k < \ldots$ and closed sets $S_1 \supseteq S_2 \supseteq S_3 \supseteq \ldots \supseteq S_k \supseteq \ldots$ with $S_k = \overline{B}(x_{n_k}, 2^{\frac{1}{2^{k-1}}})$. As $\delta(S_k) \to 1$ when $k \to \infty$,therefore these sets form a nested sequence of closed sets. By our assumption $\bigcap_{k=1}^{\infty} S_k \neq \phi.$ Let $x \in \bigcap_{k=1}^{\infty} S_k$, then for some

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integer k_0, $x \in S_k$ for all $k \geq k_0$. That is $d(x_{n_k}, x) < 2^{\frac{1}{2^{k-1}}}$, $k \geq k_0$. It means $x_{n_k} \to x$. But $\{x_{n_k}\}$ is a subsequence of a Cauchy sequence $\{x_n\}$, therefore, $x_n \to x \in M$. This completes the proof.

The Banach theorem can also be extended in another way, where the behavior of $d(x,Tx)$ is considered instead of comparing $d(Tx,Ty)$ and $d(x,y)$. Many of such generalizations relay on the following general principle involving minimizing sequences for suitable real valued functions:

Theorem 2.5. Let *(M,d)* be a complete multiplicative metric space and $\psi : M \to [1, \infty)$ be an arbitrary (not necessarily continuous) function. Assume that $\inf\{\psi(x).\psi(y) \mid d(x,y) \geq \beta\} = \lambda(\beta) > 1$ for all $\beta > 1$ (2.3).

Then each sequence $\{x_n\}$ in M such that $\psi(x_n) \to 1$ converges to one and the same point $z \in M$.

Proof. Let $S_n = \{x \mid \psi(x) \leq \psi(x_n)\}$. Any finite family of these nonempty sets has a nonempty intersection. We shall show that $\delta(S_n) \to 1$. As $\psi(x_n) \to 1$; so, for any given $\varepsilon > 1$, there will be a positive integer n_0 such that $\psi(x_n) < \sqrt{\lambda(\varepsilon)}$ for all $n \geq n_0$.

For any $x, y \in S_n$ with $n \geq n_0$ we have $\psi(x).\psi(y) < \lambda(\varepsilon)$. Condition (2.3) gives $d(x,y) < \varepsilon$, so $\delta(S_n) \leq \varepsilon$. But $\varepsilon > 1$ is arbitrary, so $\delta(S_n) \to 1$. Moreover, as $\delta(\overline{S}_n) = \delta(S_n) \to 1$, so, using Cantor's Intersection Theorem 2.4, we conclude that there is unique $z \in \bigcap_{n=1}^{\infty} S_n$. Since $x_n \in \overline{S}_n$ for each *n*, therefore $x_n \to z$. Now let $\{y_n\}$ be another sequence with $\psi(y_n) \to 1$, *therefore* $\psi(x_n).\psi(y_n) \to 1$, arguing as before and using relation (2.3), it follows that $d(x_n, y_n) \to 1$ and therefore $y_n \to z$.

The following theorem is an obvious consequence of the above result.

Theorem 2.6. Let *(M,d)* be a complete multiplicative metric space and $F_1 : M \to [1, \infty)$ be a continuous mapping. Assume the function $\psi(x) = d(x, F_1(x))$ satisfying condition (2.3) and $\inf_{x \in M} d(x, F_1(x)) = 1$. Then F_1 has a unique fixed point.

Proof. Notice that the Banach fixed-point theorem in a multiplicative metric space follows from theorem 2.6. If $d(F_1(x), F_1(y)) \le d(x, y)^{\alpha}$ where $\alpha \in (0,1)$. Then condition (2.3) is valid for $\psi(x) = d(x, F_1(x))$, because

$$d(x, y)^{1-\alpha} = \frac{d(x, y)}{d(x, y)^{-\alpha}} \le \frac{d(x, y)}{d(F_1(x), F_1(y))} \le d(x, F_1(x)).d(y, F_1(y)).$$

Using $\inf\limits_{x\in M} d(x, F_1(x)) = 1$, we get that $d(F_1^n(x), F_1^{n+1}{}_1(x)) \to 1$ for each $x \in M$. This completes the proof.

The following corollary is readily derivable from the Banach Contraction principle

Corollary 2.1. Let *(M,d)* be a complete multiplicative metric space and $B = B(x_0, r) = \{x \mid d(x, x_0) < r\}$ where $r > 1$.

Let $T: B \to M$ be a mapping such that $d(Tx, Ty) \le d(x, y)^{\lambda}$ for all $x, y \in B$ where $\lambda \in (0,1)$. If $d(Tx_0, x_0) \le r^{1-\lambda}$, then T has a unique fixed point.

Proof. Choose $\varepsilon < r$ such that $d(Tx_0, x_0) \le \varepsilon^{1-\lambda} < r^{1-\lambda}$. Next, we show that *T* maps the closed ball $C = \{x \mid d(x, x_0) \le \varepsilon\}$ into itself. If $x \in C$, then using the contractive condition of *T* and multiplicative triangular inequality we have $d(Tx, x_0) \le d(Tx, Tx_0).d(Tx_0, x_0) \le d(x, x_0)^{\lambda}.\varepsilon^{1-\lambda} \le \varepsilon^{\lambda}.\varepsilon^{1-\lambda} = \varepsilon$.

As *C* is closed, so the Banach Contraction Principle completes the proof.

We conclude with the following example which supports Theorem 2.3.

Example 2.1. Let $M = [0.01, 1]$. Consider the multiplicative metric $d: M \times M \to [1, \infty)$ defined by $d(x, y) = e^{|\frac{1}{x} - \frac{1}{y}|}$. Then *(M,d)* is a complete multiplicative metric space. The mapping $T: M \to M$ defined by $T(x) = \frac{3}{5 + x}$, satisfies the following multiplicative contractive condition $d(Tx, Ty) \le d(x, y)^{\delta(x,y)}$, where $\delta: M \times M \to [0, \infty)$ defined $\delta(x, y) = \frac{x.y}{2}$, has the property that for any closed interval

$$[a, b] \subset [1, \infty), \ \sup\{\delta(x, y) \mid a \le d(x, y) \le b\} = \mu(a, b) < 1.$$

Obviously, T has a unique fixed point $0.5413812651 \in M$.

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РАСШИРЕНИЕ БАНАХОВЫХ ПРИНЦИПОВ СЖАТИЯ В МУЛЬТИПЛИКАТИВНОМ МЕТРИЧЕСКОМ ПРОСТРАНСТВЕ

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ВИД СТАТЬИ: оригинальная научная статья
ЯЗЫК СТАТЬИ: английский

Резюме:

В данной статье мы доказали несколько обобщений Банаховых принципов сжатия в мультипликативном метрическом пространстве. Мы также развили применяемуюКанторовутеорему подмножеств при образовании мультипликативных метрических пространств,подтвердив ее нетривиальными примерами.

Ключевые слова: мультипликативная метрика, мультипликативный открытый шар, последовательность Коши, мультипликативное сжатие.

ПРОШИРИВАЊЕ БАНАХОВОГ ПРИНЦИПА КОНТРАКЦИЈЕ НА МУЛТИПЛИКАТИВНЕ МЕТРИЧКЕ ПРОСТОРЕ

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Сажетак:

У овом раду је доказано неколико генерализација Банаховог принципа контракције за мултипликативне метричке просторе. Такође, развијена је Канторова теорема интерсекције при образовању мултипликативних метричких простора, подржана нетривијалним примерима.

Кључне речи: мултипликативна метрика, мултипликативна отворена кугла, мултипликативни Кошијев низ, мултипликативна контракција.

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