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ОРИГИНАЛЬНЫЕ НАУЧНЫЕ СТАТЬИ
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SOME CRITICAL REMARKS ON THE PAPER "A NOTE ON THE METRIZABILITY OF TVS-CONE METRIC SPACES"

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Abstract:

This short and concise note provides a detailed exposition of the approach and results established by (Lin et al, 2015, pp.271-279). We show that the obtained results are not particularly surprising and new. Namely, using an old result due to K. Deimling it is indicated that tvs-cone metric spaces over

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solid cones are actually cone metric spaces over normal solid cones. Hence, there are only cone metric spaces over normal solid cones or over normal non-solid cones. One question still unanswered is whether an ordered topological vector space with a non-normal non-solid cone exists.

Key words: tvs-cone metric space, metrizable, solid, normal, non-normal.

Introduction, preliminaries and results

We will denote by E a topological vector space with the zero vector θ. A subset C of E is called a cone in E if and only if:

(a) C is nonempty and closed in E,
(b) $x, y \in C$ and $a, b \in R^{+}$ imply $ax + by \in C$,
(c) $x, -x \in C$ implies $x = \theta$.

For more details we refer the reader to (Ansari et al, 2016), (Deimling, 1985), (Filipović et al, 2011), (Janković et al, 2011), (Köthe, 1969), (Shaefer, 1971) and (Wong & Ng, 1973).

For a given cone C, a partial ordering $\leq$ with respect to C is introduced in the following way: $x \leq y$ if and only if $y - x \in C$. We write $x < y$ to indicate that $x \leq y$, but $x \neq y$. If $y - x \in \operatorname{int} C$, we write $x << y$. It is clear that < and << are not relations of partial order.

The pair (E, C) is called an ordered topological vector space. Throughout the paper, C stands for a solid cone, i.e. $\operatorname{int} C \neq \phi$. In most papers it turns out that this assumption is essential (Amini-Harandi & Fakhar, 2010), (Cakalli et al, 2012), (Du, 2010), (Proinov, 2013), (Khani & Pourmahdian, 2011), (Lin et al, 2015), (Simić, 2011), (Vandergraft, 1967) and (Zabrejko, 1997).

The following result from (Wong & Ng, 1973) is important in the framework of ordered topological vector spaces.

Proposition 1.1. Let (E, C) be an ordered topological vector space. Then e is an interior point of C, i.e., $e \in \operatorname{int} C$, if and only if

$$[-e, e] = \{x \in E : -e \leq x \leq e\} = (e - C) \cap (C - e)$$

is the neighborhood of θ.

Under the assumption $E \neq \{\theta\}$, we have the following corollary.

Corollary 1.2. Let (E, C) be an ordered topological vector space with $\operatorname{int} C \neq \phi$. Then

(a) $\theta \notin \operatorname{int} C,$

(b) $\lambda \operatorname{int} C = \operatorname{int} C$ if $\lambda > 0$ and $C + \operatorname{int} C = \operatorname{int} C,$

(c) (E, C) is Hausdorff ordered topological vector space.

Let us note that the property (c) is an immediate consequence of Theorem 2.1 (2) from (Jachymski & Klima, 2016).

For each $c \in \operatorname{int} C$ there exists a norm $\|\cdot\|_c$ on the vector space E where $\|\cdot\|_c$ is the Minkowski functional on $E = E_c = \bigcup_{n \in N} n[-c, c]$, i.e.,

$$\|x\|_c = \inf\{\lambda > 0 : x \in \lambda[-c, c]\}.$$

It is obvious that all norms $\|\cdot\|_c$, $c \in \operatorname{int} C,$ are equivalent. Further, we get that $(E, \|\cdot\|_c, C)$ is an ordered normed space with the normal-solid cone C. For more details see (Deimling, 1985, pp.230, Proposition 19.9). Therefore, $\operatorname{int} C$ under the topology on E is a subset of $\operatorname{int} C$ under the norm topology, i.e., $(\operatorname{int} C)_E \subseteq (\operatorname{int} C)_{\|\cdot\|_c}$. However, the converse is also valid. Hence, $(\operatorname{int} C)_E = (\operatorname{int} C)_{\|\cdot\|_c}$.

From what has already been proved, it is clear that the notion of tvs-cone metric space in the sense of Du, see (Du, 2010), is the same as that of (Lin et al, 2015, Definition 2.3).

According to (Deimling, 1985) and (Kadelburg et al, 2016), we are thus led to the following strengthening conclusions.

Proposition 1.3. Each tvs-cone metric space over a solid cone in the sense of the both definitions (Du, 2010) and (Lin et al, 2015) is in fact a cone metric space over a normal solid cone. Therefore, the well known results, (Amini-Harandi & Fakhar, 2010), (Du, 2010), (Kadelburg et al, 2011), (Khani & Pourmahdian, 2011) give a positive answer to Question 1.1 that arises in (Lin et al, 2015).

Proposition 1.4. The claims from (Lin, 2015) are the immediate consequences of the corresponding results in the framework of cone metric spaces over normal solid cones, see (Huang & Zhang, 2007), (Janković et al, 2011), (Kadelburg et al, 2011), (Simić, 2011).

Proposition 1.5. If C is a solid cone in some ordered topological vector space E, then it is a normal solid cone under a suitable defined norm. For all details see (Deimling, 1985, Proposition 19.9).

Remark 1.6. It is worth pointing out that in (Du, 2010), (Fierro, 2016), (Kadelburg et al, 2016), (Lin et al, 2015) and (Simić, 2011) the assumption:

(E,C) is Hausdorff, is superfluous. Applying Corollary 1.2 (c) we conclude that the condition $\operatorname{int} C \neq \phi$ implies that the ordered topological vector space (E,C) is Hausdorff.

In the theory of abstract metric spaces, quite a few proofs are based on c-sequences introduced in (Alnafei et al, 2011). Namely, for a sequence $\{x_n\}$ in the solid cone C of a real ordered topological vector space E, we say that it is a c-sequence if for every interior point c of the cone C there exists $n_0 \in \mathbb{N}$ such that $x_n \ll c$ whenever $n \geq n_0$. If C is normal in E, then $\{x_n\}$ is a c-sequence if and only if $x_n \to \theta$ in E. However, if the cone C is a non-normal solid cone, then the previous is not true, i.e., a c-sequence does not always converge to θ as $n \to \infty$. We refer the readers to (Radenović et al, 2017), (Xu & Radenović, 2014), (Xu et al, 2016), (Đorđević et al, 2011) and (Kadelburg et al, 2011) for more details on c-sequences.

Theorem 1.7. Let $(E,\|\cdot\|)$ be an ordered (real) Banach space with the underlying solid cone C and $\{x_n\}$ be a sequence in C. Then $\{x_n\}$ is a c-sequence if and only if $\|x_n\|_e \to \theta$ as $n \to \infty$ where

$$\|x\|_e = \inf\{\lambda > 0 : x \in \lambda[-e,e], e \in \operatorname{int} C\},$$

i.e., $\|\cdot\|_e$ is the Minkowski functional.

Proof. First, $\|\cdot\|_e$ is a norm in E. If C is a normal solid cone, then this norm is equivalent to the given one, and $x_n \xrightarrow{\|\cdot\|} \theta$ if and only if $x_n \xrightarrow{\|\cdot\|_e} \theta$.

If C is not normal, then C is a normal and solid cone under the norm $\|\cdot\|_e$ for each $e \in \operatorname{int} C$. Since $\|\cdot\|$ and $\|\cdot\|_e$ give the same interior points, then by the first case $\{x_n\}$ is a c-sequence if and only if $\|x_n\|_e \to \theta$. Consequently, in each normal cone, the conditions $x_n \to \theta$ and $\{x_n\}$ is a c-sequence are equivalent.

Conclusion

Note that we have actually proved that there exist cone metric spaces over normal solid cones or over normal non-solid cones. Therefore, the results obtained in (Lin et al, 2015) are not new but rather the consequences of known facts (Huang & Zhang, 2007), (Janković et al, 2011), (Kadelburg et al, 2011) and (Simić, 2011). We can conclude that the existence of ordered topological vector spaces with a non-normal non-solid cone is still an open question.

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НЕКОТОРЫЕ КРИТИЧЕСКИЕ ЗАМЕЧАНИЯ О РАБОТЕ «ЗАМЕТКИ О МЕТРИЗУЕМОСТИ ТВП-КОНИЧЕСКИХ ПРОСТРАНСТВ»

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Резюме:

В представленных, в данной статье, заметках приведен подробный обзор методов и полученных результатов исследования группы авторов, во главе с Шой Линь (Lin et al, 2015, pp.271-279). Мы в свою очередь доказываем, что их результаты не являются инновационными. В частности, при применении известных результатов К. Деймлинга установлено, что ТВП-коничеческие метрические пространства с конусами с непустой внутренностью фактически являются метрическими пространствами с нормальной конической и непустой внутренностью. Следовательно, существуют только конические метрические пространства с нормальными конусами, внутренность которых непуста или конусами, которые нормальны, но с пустой внутренностью. До сих пор не установлено упорядоченное топологическое векторное простанство с конусом, который не является нормальным, а внутренность которого не пуста.

Ключевые слова: твп-коническое метрическое пространство, метризуемость, конус с непустой внутренностью, нормальный, ненормальный.

НЕКЕ КРИТИЧКЕ НАПОМЕНЕ О РАДУ „БЕЛЕШКА О МЕТРИЗАБИЛНОСТИ ТВП-КОНУСНИХ МЕТРИЧКИХ ПРОСТОРА"

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ЈЕЗИК ЧЛАНКА: енглески

Сажетак:

Ова кратка и прегледна белешка даје детаљан извештај о приступу и резултатима до којих су дошли Шоу Лин и група аутора (Lin et al, 2015, pp.271-279). У чланку је показано да њихови резултати нису нарочито изненађујући и нови. У ствари, коришћењем једног познатог К. Демлинговог резултата назначено је да су твп-конусни метрички простори са конусима који имају непразну унутрашњост заправо конусни метрички простори са нормалним конусима и непразним унутрашњостима. Стога, постоје само конусни метрички простори са нормалним конусима чија унутрашњост није празна или са конусима који су нормални, али са празним унутрашњостима. Још увек се не зна да ли постоји уређен тополошки векторски простор са конусом који није нормалан и чија унутрашњост није празна.

Кључне речи: твп-конусни метрички простор, метризабилан, конус са непразном унутрашњошћу, нормалан, није нормалан.

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