ПРЕГЛЕДНИ ЧЛАНЦИ

ОБЗОРНЫЕ СТАТЬИ

REVIEW PAPERS

ON SOME KNOWN FIXED POINT RESULTS IN THE COMPLEX DOMAIN: SURVEY

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Abstract:

In this survey paper, we consider some known results from the fixed point theory with complex domain. The year 1926 is very significant for this subject. This is the beginning of the research and application of the fixed point theory in complex analysis. The Denjoy-Wolf theorem, together with the Banach contraction principle, is one of the main tools in the mathematical analysis.

Keywords: fixed point, Jordan curve, analytic function, complex Banach space.

Introduction and preliminaries

Let f be an analytic map of the unit disk $D = \{z \in C : |z| < 1\}$ into

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itself. Now, we want to consider the fixed point of f and especially a lemma implies that f has at most one fixed point in D and some maps have no fixed point. If $f:\overline{D}\to\overline{D}$ is a continuous map, then f must have a fixed point in $\overline{D}$. From now on, we make the assumption that f is an analytic function on the open disk D.

Definition 1.1 *Suppose that f is an analytic map of the unit disk D into itself. We say that $a\in D$ is a fixed point of f if $f(a)=a$. Also, $a\in\partial D$ is a fixed point of f if $\lim_{r\to 1^-} f(ra)=a$.*

The Julia-Caratheodory theorem implies that if $a\in\partial D$ is a fixed point of f, then $\lim_{r\to 1^-} f(ra)$ exists (we call it $f'(a)$) and $0<f'(a)\le\infty$.

Theorem 1.2 (*The Denjoy-Wolf theorem from 1926.*) *If f is an analytic map of D into itself, but not the identity map, there is a unique fixed point a of f in $\overline{D}$ such that $|f'(a)|\le 1$.*

The point a in the above theorem is called the Denjoy-Wolff point of f.

The Schwarz-Pick Lemma implies that f has at most one fixed point in D and if f has a fixed point in D, it must be the Denjoy-Wolf point.

Example 1.3 1) *The mapping $f(z)=\dfrac{z+\frac{1}{2}}{1+\frac{z}{2}}$ is an automorphism of D with the fixed points 1 and -1, but the Denjoy-Wolff point is $a=1$ because $f'(1)=\frac{1}{3}$ $\left(f'(-1)=3\right)$.*

2) The mapping $f:D\to D$ given by $f(z)=\dfrac{z}{2-z^2}$ has three fixed points: 0, 1, and -1. The Denjoy-Wolff point is $a=0$ since $f'(0)=\frac{1}{2}$ and $f'(\pm 1)=3$.

3) The mapping $f(z)=\dfrac{2z^3+1}{2+z^3}$ is an inner function fixing 1 and -1, with the Denjoy-Wolff point $a=1$ because $f'(1)=1$ (note that $f'(-1)=9$).

4) The inner function $f(z)=\exp\left(\dfrac{z+1}{z-1}\right)$ has a fixed point in D which is the Denjoy-Wolff point $(a\approx .21365)$, and infinitely many fixed points on ∂D.

Definition 1.4 *Let $f:\overline{D}\to\overline{D}$ be an analytic map. The set*

$$F=\left\{z\in\overline{D}:\lim_{r\to 1^-} f(rz)=z\right\}$$

is called the fixed point set of f.

Theorem 1.5 *If f is an analytic function that maps the unit disk into itself, then there exists $E\subset F$ which has linear measure zero.*

Example 1.6 *Let K be a compact set of measure zero in ∂D. There is a function f analytic in D and continuous on $\overline{D}$ such that $f(D)\subset D$ and the fixed point set of f is $\{0\}\cup K$.*

Theorem 1.7 *Let f be a univalent analytic function that maps the unit disk into itself. Then the set F has capacity zero.*

Example 1.8 For a given compact set K in ∂D of capacity zero and a point $a\in\partial D\setminus K$, there exists an analytic and univalent function in D (say f) such that $f(D)\subset D$ and $F=\{a\}\cup K$.

In the next theorem, it is required that the Denjoy-Wolff point $a=b_0$ be normalized, i.e., $b_0=0$ or $b_0=1$.

Theorem 1.9 Let f be an analytic function with $f(D)\subset D$ and suppose that $b_0,b_1,\ldots,b_n$ are fixed points of f.

• If $b_0 = 0$, then

$$\sum_{j=1}^{n} \frac{1}{f'(b_j)-1} \le \operatorname{Re} \frac{1+f'(0)}{1-f'(0)}.$$

• If $b_0 = 1$ and $f'(1) = 1$, then

$$\sum_{j=1}^{n} \frac{1}{f'(b_j)-1} \le \frac{f'(1)}{1-f'(1)}.$$

• If $b_0 = 1$ and $0 < f'(1) < 1$, then

$$\sum_{j=1}^{n} \frac{|1-b_j|^2}{f'}(b_j) - 1 \le 2\operatorname{Re}\left(\frac{1}{f(0)} - 1\right).$$

Moreover, the equality holds if and only if f is the Blaschke product of order $n+1$ in case $1)$ or of order n in cases $2)$ and $3)$.

If f has infinitely many fixed points, then the appropriate inequality holds for any choice of finitely many fixed points. In particular, only countably many fixed points of f can have a finite angular derivative.

If $b_0, b_1, b_2, \ldots$ are countably many fixed points for which $f'(b_j) < \infty$, then the corresponding infinite sum converges and the appropriate inequality holds.

With the assumption that an analytic function f is univalent and that the Denjoy-Wolff point is $b_0 = 0$ or $b_0 = 1$, we have the next assertion.

Theorem 1.10 *Let f be a univalent analytic function with $f(D) \subset D$ and suppose $b_0, b_1, \ldots, b_n$ are fixed points of f.*

• If $b_0 = 0$, then

$$\sum_{j=1}^{n} \left(\log f'(b_j)\right)^{-1} \le 2 \operatorname{Re} B^{-1}$$

where $B=\lim_{r\to 1^-}\log\left(\frac{f(rb_1)}{f'(0)rb_1}\right)$ and $\lim_{z\to 0}\log\left(\frac{f(z)}{z}\right)=\log f'(0)$.

• If $b_0=0$ and $0<f'(1)<1$, then $\sum_{j=1}^{n}\left(\log f'(b_j)\right)^{-1}\le -\left(\log f'(1)\right)^{-1}$.

Moreover, the equality holds if and only if f is embeddable in a semigroup and $f(D)=D$ with n analytic arcs removed in case $1)$ or $n-1$ in case $2)$.

• If $b_0=1$ and $f'(1)=1$, then $\sum_{j=1}^{n}c_j^2\left(\log f'(b_j)\right)^{-1}\le 2\log\frac{1-|f(0)|^2}{|f'(0)|}$,

where

$$c_j=\lim_{r\to 1^-}\mathrm{Im}\left(\log\left(\frac{1}{b_j}\cdot\frac{f(rb_j)-f(0)}{1-f(0)f(rb_j)}\cdot\frac{1-f(0)\overline{f(r)}}{f(r)-f(0)}\right)\right).$$

Remark 1.11 In (Anderson & Vasil'ev, 2008, pp.101–110), the authors proved for $b_0=0$, case $1)$, the inequality

$$\prod_{j=1}^{n}f'(b_j)^{2\alpha_j^2}\ge\frac{1}{|f'(0)|},$$

where $\alpha_j\ge 0$ and $\sum_{j=1}^{n}\alpha_j=1$. The equality holds only for the unique solution of a given complex differential equation with a given initial condition.

Considering $b_0=0$, case $3)$ and the assumption that f is embeddable in a continuous semigroup, in (Contreras et al, 2006, pp.125-142) was proved that

$$\sum_{j=1}^{n}\frac{1-\mathrm{Re}\,b_j}{\log f'(b_j)}\le\mathrm{Re}\frac{1}{G(0)}=\mathrm{Re}\,\sigma'(0),$$

where G is the infinitesimal generator of the semi-group and σ is the map from the linear fractional model.

In case $3)$, an equality condition is not included and maybe this inequality is not the best possible. Also, a sharp inequality to describe the general case has not yet been obtained.

Main results

It is known that a continuous mapping of a simply connected, closed, bounded set of the Euclidean plane into itself has at least one fixed point.

Let F be an analytic function in some domain S of the complex plane. Standard results in real numerical analysis show that the equation $z = F(z)$ has at least one solution, called a fixed point of F. If S is bounded and simply connected, F is continuous on the closure $\overline{S}$ of S, and $F(\overline{S}) \subset \overline{S}$. If the mapping F is a contraction, then there is a unique fixed point, and the iteration sequence defined by $z_{n+1} = F(z_n)$, $n = 0,1,2,\ldots$ converges to the fixed point for every choice of $z_0 \in \overline{S}$. If S is convex, then a necessary and sufficient condition for the mapping to be a contraction is that the derivative F' of F satisfies $|F'(z)| \le k$, $z \in S$, where $k < 1$.

The purpose of the next theorem is to show that the conclusion is not affected if we replace the assumption that F is a contraction by the condition that F is an analytic function.

Theorem 2.1 *(Henrici, 1969)*

We first prove a reduced form of the theorem. Let S denote the interior of a Jordan curve Γ, let F be analytic in S and continuous on $S \cup \Gamma$, and let $F(S \cup \Gamma) \subset S$. Then F has exactly one fixed point, and the iteration sequence defined by $z_{n+1} = F(z_n)$, $n = 0,1,2,\ldots$ converges to the fixed point for arbitrary $z_0 \in S \cup \Gamma$.

Note that the function $F(z) = \frac{1}{2} z^{100}$ in $|z| \le 1$ satisfies the hypothesis and $|F'(z)|$ is arbitrary large.

Proof. First let S be the unit disk. By the hypothesis, we have

$$r := \max_{|z|\le 1} |F(z)| < 1. \tag{1}$$

To prove the existence of a zero of $z = F(z)$, we apply the Rouche theorem (Ahlfors, 1953) with z in the role of the 'big' function and $F(z)$ in the role of the 'small' function. Then, using (1), we can conclude that $z - F(z)$ and z have the same number of zeros inside $|z| = 1$, namely exactly one.

With s denote the unique fixed point and let

$$t(z) = \frac{z - s}{1 - z\bar{s}}.$$

This is a linear transformation which maps $|z| \le 1$ onto itself and sends s into 0. Hence, the function $G = t \circ F \circ t^{-1}$ has the fixed point 0 and it is continuous mapping of $|z| \le 1$ onto a closed subset of $|z| < 1$. Hence,

$$k := \sup_{|z|\le 1} |G(z)| < 1.$$

We can certainly assume that $k > 0$, since otherwise, G and F are constant and the proof is straightforward. The function $k^{-1}G$ vanishes at 0 and is bounded by 1, hence by the Schwarz lemma (Ahlfors, 1953) we have $k^{-1}|G(z)| \le |z|$ and consequently,

$$|G(z)| \le k|z| \tag{2}$$

for all z such that $|z| \le 1$.

Let $w_n = t(z_n)$. Since

$$w_{n+1} = t(z_{n+1}) = f(F(z_n)) = t(F(t^{-1}(w_n))) = G(w_n),$$

we conclude from (2) that $|w_{n+1}| \le k|w_n|$, hence that $|w_n| \le k^n |w_0|$, and finally that $w_n \to 0$.

From the above, it follows that the iteration sequence converges to the fixed point, i.e., $z_n = t^{-1}(w_n) \to t^{-1}(0) = s$.

We now turn to the case where S is an arbitrary Jordan domain. By the Osgood-Caratheodory theorem (Caratheodory, 1960), there exists a function q that maps S conformally onto $|z| < 1$ and $S \cup \Gamma$ continuously and one-to-one onto $|z| \le 1$. It is easily seen that the function $H = g \circ F \circ g^{-1}$ satisfies the hypotheses of the theorem for the unit disk. Furthermore, if the points z_n are defined by $z_{n+1} = F(z_n)$, $n = 0,1,2,\dots$ and $w_n = g(z_n)$, then $w_{n+1} = H(w_n)$. Consequently, the validity of the theorem for the unit disk implies the validity for the general case.

We add some problems related to the previous theorem.

- It can be shown that $k \le \dfrac{2r}{1+r^2}$.
- If S is the unit disk, than $|z_n - s| \le (1+r)k^n$, $n = 0,1,2,\dots$
- Let $F'(s) = F''(s) = \cdots = F^{(m-1)}(s) = 0$, $F^{(m)}(s) \ne 0$, for some integer $m > 1$. If S is the unit disk, show that

$$|z_n - s| \le (1+r)k^1 + k^2 + \cdots + m^{n-1}, \quad n \in \mathrm{N}.$$

- A research problem is whether similar results can be established for systems of analytic equations.

A set S is said to lie strictly inside a subset D of a Banach space if there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \mathrm{D}$ whenever $x \in S$. The following theorem may be viewed as a holomorphic version of the Banach's contraction mapping theorem.

Theorem 2.2 (Earle & Hamilton, 1970, pp.61–65)

Let D be a nonempty domain in a complex Banach space X and $h : \mathrm{D} \to \mathrm{D}$ a bounded holomorphic function. If $h(\mathrm{D})$ lies strictly inside D, then h has a unique fixed point in D.

Proof. Let us construct a metric ρ, called the CFR-pseudometric, with the contraction h. Define $\alpha(x,v) = \sup\{|g'(x)v| : g : \mathrm{D} \to \Delta \text{holomorphic}\}$ for $x \in \mathrm{D}$ and $v \in X$ and set

$$L(\gamma) = \int_0^1 \alpha(\gamma(t), \gamma'(t))t$$

for γ in the set Γ of all curves in D with piecewise continuous derivative. Clearly α specifies a seminorm at each point of D. We view $L(\gamma)$ as the length of the curve γ measured with respect to α. Define

$$\rho(x,y)=\inf\{L(\gamma):\gamma\in\Gamma,\gamma(0)=x,\gamma(1)=y\}$$

for $x,y\in$ D. It is easy to verify that ρ is a pseudometric on D.

Let $x\in$ D and $v\in X$. By the chain rule, we have

$$(g\circ h)'(x)v=g'(h(x))h'(x)v$$

for any holomorphic function $g:\mathrm{D}\to\Delta$. Hence,

$$\alpha(h(x),h'(x)v)\le\alpha(x,v). \tag{3}$$

By integrating this and applying the chain rule, we obtain $L(h\circ\gamma)\le L(\gamma)$ for all $\gamma\in\Gamma$ and thus the Schwarz-Pick inequality $\rho(h(x),h(y))\le\rho(x,y)$ holds for all $x,y\in$ D.

Now, by hypothesis, there exists an $\varepsilon>0$ such that $B_\varepsilon(x)\subseteq$ D whenever $x\in$ D. We may assume that D is bounded by replacing D by the subset

$$\cup\{B_\varepsilon(h(x)):x\in\mathrm{D}\}.$$

Fix t with $0<t<\dfrac{\varepsilon}{\delta}$, where δ denotes the diameter of $h(\mathrm{D})$. Given $x\in$ D, define

$$h_1(y)=h(y)+t[h(y)-h(x)]$$

and note that h_1: D $\to$ D is holomorphic. Given $x\in$ D and $v\in X$, it follows from $h_1'(x)v=(1+t)h'(x)v$ and (3) with h replaced by h_1 that

$$\alpha(h(x),h'(x)v)\le\frac{1}{1+t}\alpha(x,v).$$

Integrating this as before, we obtain

$$\rho(h(x),h(y))\le\frac{1}{1+t}\rho(x,y)$$

for all $x,y\in$ D.

Now pick a point $x_0\in$ D and let $\{x_n\}$ be the sequence of iterates given by $x_n=h^n(x_0)$. Then $\{x_n\}$ is a ρ-Cauchy sequence by the proof of the contraction mapping theorem.

Since X is complete in the norm metric, it suffices to show that there exists a constant $m > 0$ such that

$$\rho(x, y) \geq m\|x - y\| \tag{4}$$

for all $x, y \in$ D. Since D is bounded, we may take $m = \frac{1}{d}$, where d is the diameter of D. Given $x \in$ D and $v \in X$, define $\rho(y) = ml(y - x)$, where $l \in X^*$ with $\|l\| = 1$. Then $g:$ D $\to \Delta$ is holomorphic and $Dg(x)v = ml(v)$. Hence $\alpha(x, v) \geq m\|v\|$ by the Hahn-Banach theorem. Integrating as before, we obtain (4).

The previous result, so called the Earle-Hamilton theorem, is still applied in cases where the holomorphic function does not necessarily map its domain strictly inside itself. The following fixed point theorem is a consequence of two applications of the Earle-Hamilton theorem.

Theorem 2.3 (Khatskevich et al, 1995, pp.305–316, S. Reic et al, 1996, pp.1–44)

Let D *be a nonempty bounded convex domain in a Banach space and* $h:$ D $\to$ D *a holomorphic function having a uniformly continuous extension to* $\overline{\text{D}}$. *If there exists an* $\varepsilon > 0$ *such that* $\|h(x) - x\| \geq \varepsilon$ *whenever* $x \in \partial$D, *then* h *has a unique fixed point in* D.

The hypothesis that $\|h(x) - x\| \geq \varepsilon$ for all $x \in \partial$D is satisfied when D contains the origin and $\sup_{x \in \partial \text{D}} \frac{\|h(x)\|}{\|x\|} < 1$. Considerably stronger results have been obtained for the case where D is the open unit ball of a Hilbert space.

Theorem 2.4 *(*Goebel et al, 1980, pp.1011–1021)

Let D be the open unit ball of a Hilbert space and $h: B \to B$ a holomorphic function. If there is a point x_0 in B such that the sequence $\{h^n(x_0)\}$ of iterates lies strictly inside B, then h has a fixed point in B. If $x \in \partial$D, then h has a unique fixed point in D.

It is more complicated to obtain fixed points for nonexpansive mappings which are not contractive. One important result is that a nonexpansive self-mapping of a closed bounded convex set in a uniformly convex Banach space has a fixed point. In (Goebel et al, 1980, pp.1011–1021), it was shown that the CRF-metric ρ in the open unit ball B of a Hilbert space is uniformly convex and the fixed point theorem for holomorphic self-mappings of B was obtained.

Theorem 2.5 Let B be the open unit ball of a Hilbert space and $h: B \to B$ an arbitrary function satisfying the Schwarz-Pick inequality:

$$\rho(h(x), h(y)) \leq \rho(x, y)$$

for all $x, y \in B$. If h has a continuous extension to $\overline{B}$, then h has a fixed point in $\overline{B}$.

Corollary 2.6 If $h: B \to B$ is a holomorphic function that has a continuous extension to $\overline{B}$, then h has a fixed point in $\overline{B}$.

For a treatment of the Cartesian products of the Hilbert balls, we refer the reader to (Kuczumow et al, 2001, pp.437–515).

Szhwarz lemma and its application in the fixed point theory

In this section, we will restrict our attention to the paper (Xu et al, 2016, 2016:84) where the sharp estimates of a boundary fixed point is obtained using the Schwarz lemma. This lemma provides a very powerful tool for studying several research fields in complex analysis. For example, almost all results in the geometric function theory rely heavily on the Schwarz lemma (Ahlfors, 1953), (Anderson & Vasil'ev, 2008, pp.101–110), (Beardon, 1990, pp.41–150), (Beardon, 1997, pp.1257–1266), (Budzynska et al, 2012, pp. 504–512), (Budzynska et al, 2013a, 621–648), (Budzynska et al, 2013b, pp.747–756), (Burckel, 1981, pp.396–407), (Caratheodory, 1960), (Contreras et al, 2006, pp.125-142), (Cowen, 2010), (Cowen, 1981, pp.69–95), (Cowen & Pommerenke, 1982, pp.271–289), (Denjoy, 1926, pp. 255–257), (Earle & Hamilton, 1970, pp.61–65), (Goebel et al, 1980, pp.1011–1021), (Goebel, 1982, pp.1327–1334), (Harris, 2003, pp.261–274), (Hayden & Suffridge, 1971, pp.419–422), (Hayden & Suffridge, 1976, pp.95–105), (Henrici, 1969), (Julia, 1918, pp.47–295), (Khatskevich et al, 1995, pp.305–316), (Kuczumow, 1984, pp.417-419), (Kuczumow et al, 2001, pp.437–515), (Lemmens et al, 2016), (Mateljević, 1998, pp.1-4), (Reich & Shoikhet, 1996, pp.1-44), (Rudin, 1978, pp.25–28), (Suffridge, 1974, pp.309-314), (Wolff, 1926), (Xu et al, 2016, 2016:84). On the other hand, the Schwarz lemma at the boundary is also useful in complex analysis, and various interesting results have been obtained (Ahlfors, 1953), (Anderson & Vasil'ev, 2008, pp.101–110), (Beardon, 1990, pp.41–150), (Beardon, 1997, pp.1257–1266), (Budzynska et al, 2012, pp. 504–512), (Budzynska et al, 2013a, pp.621–648), (Budzynska et al, 2013b, pp.747–756), (Burckel, 1981, pp.396–407), (Caratheodory, 1960), (Contreras et al, 2006, pp.125-142), (Cowen, 2010), (Cowen, 1981, pp.69–95), (Cowen & Pommerenke, 1982, pp.271–289), (Denjoy, 1926, pp.255–257).

We will summarize without proofs the relevant material on (Xu et al, 2016, 2016:84). First we set up the notation and the terminology.

Let D denote the unit disk in C. With the notion $H(D,D)$, we have the class of holomorphic self-mappings of D. Here, N stands for the set of all positive integers. The boundary point $\xi \in \partial D$ is called a fixed point of $f \in H(D,D)$ if

$$f(\xi) = \lim_{r \to 1^-} f(r\xi) = \xi.$$

The classification of the boundary fixed points is given at the begging of this survey. This classification can be done via the value of the angular derivative

$$f'(\xi) = \angle \lim_{z \to \xi} \frac{f(z) - \xi}{z - \xi},$$

which belongs to $(0,\infty)$ due to the Julia-Caratheodory theorem (see Julia, 1918, pp.47–295). This theorem also asserts that the finite angular derivative at the boundary fixed point ξ exists if and only if the holomorphic function $f'(z)$ has the finite angular limit $\angle \lim_{z \to \xi} f(z)$. For a boundary fixed point ξ of f, if $f'(\xi) \in (0,\infty)$, then ξ is called a regular fixed point. The regular fixed point is attractive if $f'(\xi) \in (0,1)$, neutral if $f'(\xi) = 1$, or repulsive if $f'(\xi) \in (1,\infty)$.

By the Julia-Caratheodory theorem (Julia, 1918, pp.47–295) and the Wolf lemma (Wolff, 1926), if $f \in H(D,D)$ with no interior fixed point, then there exists a unique regular boundary fixed point ξ such that

$f'(\xi) \in (0,1]$ and if $f \in H(D,D)$ with an interior fixed point, then $f'(\xi) > 1$ for any boundary fixed point $\xi \in \partial D$.

The following known results are very significant.

Theorem 3.1 Assume that $f \in H(D,D)$ has a regular boundary fixed point 1 and $f(0) = 0$. Then

$$f'(1) \geq \frac{2}{1+|f'(0)|}.$$

Moreover, the equality holds if and only if f is of the form

$$f(z) = -z\frac{a-z}{1-az}, \quad z \in D,$$

for some constant $a \in (-1,0]$.

The next theorem is the improvement of the previous ones. It was announced 60 years later and showed how to dispense with the assumption $f(0) = 0$.

Theorem 3.2 If $f \in H(D,D)$ with $\xi = 1$ as its regular boundary fixed point, then

$$f'(1) \geq \frac{2(1-|f(0)|)^2}{1-|f(0)|^2+|f'(0)|}.$$

Finally, the previous result has been improved and the better estimate has been obtained.

Theorem 3.3 If $f \in H(D,D)$ with $\xi = 1$ as its regular boundary fixed point, then

$$f'(1) \geq \frac{2}{\operatorname{Re}\left(\frac{1-f^2(0)+f'(0)}{(1-f(0))^2}\right)}.$$

For a fuller treatment and a deeper discussion of fixed point results in complex domain, we refer the reader to (Xu et al, 2016, 2016:84) and the references given there.

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О НЕКОТОРЫХ ИЗВЕСТНЫХ РЕЗУЛЬТАТАХ О НЕПОДВИЖНОЙ ТОЧКЕ В КОМПЛЕКСНОМ ДОМЕНЕ: ИССЛЕДОВАНИЕ

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ОБЛАСТЬ: математика
ВИД СТАТЬИ: обзорная статья
ЯЗЫК СТАТЬИ: английский

Резюме:

В данной обзорной статье представлены некоторые известные результаты теорий о неподвижной точке над комплексным доменом. Надо подчеркнуть, что 1962 год является ключевым для данной области. Ведь именно тогда начали проводиться исследования по применению теорий о неподвижной точке в рамках комплексного анализа. Теорема единственности для рядов Вольфа-Данжуа наряду с Банаховым принципом сжатия, становятся главным методом (результатом) математического анализа.

Ключевые слова: неподвижная точка, кривая Жордана, аналитические функции, полное Банахово пространство.

НЕКИ ПОЗНАТИ РЕЗУЛТАТИ ИЗ НЕПОКРЕТНЕ ТАЧКЕ У КОМПЛЕКСНОМ ДОМЕНУ: ИСТРАЖИВАЊЕ

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ОБЛАСТ: математика
ВРСТА ЧЛАНКА: прегледни чланак
ЈЕЗИК ЧЛАНКА: енглески

Сажетак:

У овом прегледном раду разматрани су неки познати резултати из теорије непокретне тачке над комплексним доменом. Истраживање и примена теорије непокретне тачке у комплексној анализи започети су 1926, године. Теорема Denjoy-Wolf, заједно са Банаховим принципом контракције, једно је од главних оруђа (резултата) математичке анализе.

Кључне речи: непокретна тачка, Жорданове криве, аналитичке функције, комплексан Банахов простор.

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