

SOME FIXED POINT THEOREMS IN b_2 -METRIC SPACES

Poom Kumam^a, Zoran D. Mitrović^b, Mirjana V. Pavlović^c

^a King Mongkut's University of Technology Thonburi, Faculty of Science, KMUTT-Fixed Point Theory and Applications Research Group, Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Bangkok, Kingdom of Thailand, e-mail: poom.kumam@mail.kmutt.ac.th,

ORCID iD: <http://orcid.org/0000-0002-5463-4581>

^b University of Banja Luka, Faculty of Electrical Engineering, Banja Luka, Republic of Srpska, Bosnia and Herzegovina, e-mail: zoran.mitrovic@etf.unibl.org,

ORCID iD: <http://orcid.org/0000-0001-9993-9082>

^c University of Kragujevac, Faculty of Science, Department of Mathematics and Informatics, Kragujevac, Republic of Serbia, e-mail: mpavlovic@kg.ac.rs,

ORCID iD: <http://orcid.org/0000-0001-6257-8666>

DOI: 10.5937/vojtehg67-21221; <https://doi.org/10.5937/vojtehg67-21221>

FIELD: Mathematics (Mathematics Subject Classification: primary 47H10, secondary 54H25)

ARTICLE TYPE: Original Scientific Paper

ARTICLE LANGUAGE: English

Abstract:

In this paper, we first prove a result that gives a sufficient condition for the convergence of the sequences in the b_2 -metric space. Next, we give some fixed point theorems in the b_2 -metric space. Some of our results are the corresponding generalizations of the known results in the b_2 -metric space, which is confirmed by some examples.

Keywords: fixed points, common fixed points, 2-metric space, b_2 -metric space.

ACKNOWLEDGMENT:

The second author is grateful for the financial support from the Ministry for Scientific and Technological Development, Higher Education and Information Society of Republika Srpska (Savremena istraživanja u teoriji fiksne tačke: metrički i topološki pristup, 1255007). The third author is grateful for the financial support from the Ministry of Education and Science and Technological Development of the Republic of Serbia (Metode numeričke i nelinearne analize sa primenama, 174002).

Introduction and Preliminaries

The applications of fixed point theorems are very important in diverse disciplines of mathematics, engineering and economics. The origin of the fixed theory is dated to the last quarter of the nineteenth century. The work of S. Banach in 1922 known as the Banach contraction principle is the starting point of the metric fixed point theory.

More on fixed point results and contractive conditions, the reader can find in (Ćirić, 2003), (Agarwal et al, 2015), (Kirk & Shahzad, 2014).

Theorem 1.1 (Banach contraction principle) Let (X, d) be a complete metric space. Let T be a contractive mapping on X , that is, one for which there is a $\lambda \in [0, 1)$ satisfying

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$. Then, there exists a unique fixed point $x \in X$ of T .

This theorem is a forceful tool in the nonlinear analysis. It has many applications and has been extended by a great number of authors. Although the famous Banach contraction principle was proved in the metric space, after 1990 many new modifications of the definition of the metric space appeared.

From now on, \mathbb{R} and \mathbb{N} will denote the set of real numbers and natural numbers, respectively. Let us recall the definitions of the b-metric spaces, the rectangular b-metric spaces, the 2-metric spaces, and the b_2 -metric spaces.

In the papers of Bakhtin (Bakhtin, 1989, pp.26-37) and Czerwik (Czerwik, 1993, pp.5-11), the notion of the b-metric space was introduced and some fixed point theorems for single-valued and multi-valued mappings in the b-metric spaces were proved.

Definition 1.2 Let X be a nonempty set and $s \geq 1$ a given real number. The function $d : X \times X \rightarrow [0, \infty)$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

- $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A triplet (X, d, s) is called a b-metric space with the coefficient s .

In the paper (George et al, 2015, pp.1005-1013), the authors introduced the concept of a rectangular b-metric space, which is not necessarily Hausdorff and which generalizes the concept of the metric space, the rectangular metric space (RMS) and the b-metric space.

Definition 1.3 (George et al, 2015, pp.1005-1013) Let X be a nonempty set and $s \geq 1$ a given real number. The function $d : X \times X \rightarrow [0, \infty)$ is said to be a rectangular b-metric if the following conditions are satisfied:

(RbM1) $d(x, y) = 0$ if and only if $x = y$;

(RbM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(RbM3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$.

A triplet (X, d, s) is called a rectangular b-metric space with the coefficient s (in short RbMS).

Also in (George et al, 2015, pp.1005-1013), the concept of convergence in such spaces is similar to that of standard metric spaces.

In the paper (Gähler, 1963, pp.115-118), Gähler introduced the concept of the 2-metric space.

Definition 1.4 (Gähler, 1963, pp.115-118) Let X be a nonempty set and the mapping $d : X \times X \times X \rightarrow \mathbb{R}$ satisfies:

(1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(2) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$

(3) The symmetry:

$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$
for all $x, y, z \in X$.

(4) The rectangle inequality:

$d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)$ for all $x, y, z, t \in X$.

Then d is called a 2-metric on X and (X, d) is called a 2-metric space.

Many generalizations of the concept of metric spaces are established, and several papers are published on the topic of the b-metric spaces (see (Aleksić et al, 2018), (Aydi, 2016, pp.2417-2433), (Czerwik, 1993, pp.5-11), (Czerwik, 1998, pp.263-276), (Dung & Le Hang, 2016, pp.267-284), (Miculescu & Mihail, 2017, pp.2153-2163) and others), of the rectangular b-metric spaces, see (George et al, 2015, pp.1005-1013), (Mitrović & Radenović, 2017, pp.3087-3095), (Mitrović & Radenović, 2017, pp.401-407) and others) and of the 2-metric spaces, see (Ahmed, 2009, pp.2914-2920), (Aliouche & Simpson, 2012, pp.668-690), (Deshpande & Chouhan, 2011,

pp.37-55), (Dung & Le Hang, 2013), (Fadail et al, 2015, pp.533-548), (Freese et al, 1992, pp.391-400), (Gähler, 1963, pp.115-118), (Iseki, 1975, 133-136), (Iseki, 1976, pp.127-135), (Lahiri et al, 2011, pp.337-352), (Lal & Singh, 1978, pp.137-143), (Naidu & Prasad, 1986, pp.974-993), (Popa et al, 2010, pp.105-120) and others).

In the paper (Mustafa et al, 2014), the notion of the b_2 -metric space was introduced and some fixed point theorems in the b_2 -metric spaces proved.

Definition 1.5 (Mustafa et al, 2014) Let X be a nonempty set, $s \geq 1$ be a real number and let $d : X \times X \times X \rightarrow \mathbb{R}$ satisfies:

(1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(2) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$.

(3) The symmetry:

$$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$$

for all $x, y, z \in X$.

(4) The rectangle inequality:

$$d(x, y, z) \leq s[d(x, y, t) + d(y, z, t) + d(z, x, t)]$$

for all $x, y, z, t \in X$.

Then d is called a b_2 -metric on X and (X, d, s) is called a b_2 -metric space.

Remark 1.6 Note that, $d(x, y, z) \geq 0$ for all $x, y, z \in X$. Applying the rectangle inequality, we get

$$d(x, y, y) \leq s[d(z, y, y) + d(x, z, y) + d(x, y, z)].$$

By (2) and the symmetry of d , we obtain $d(x, y, z) \geq 0$.

Note that a 2-metric space is included in the class of the b_2 -metric spaces with the coefficient $s = 1$.

Example 1 (Mustafa et al, 2014)

1. Let $X = [0, +\infty)$ and $d(x, y, z) = (xy + yz + zx)^p$ if $x \neq y \neq z \neq x$, and otherwise $d(x, y, z) = 0$, where $p \geq 1$ is a real number.

From convexity of the function $f(x) = x^p$ for $x \geq 0$, then by Jensen inequality we have

$$(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p).$$

So, (X, d, s) is a b_2 -metric space with $s \leq 3^{p-1}$.

2. Let a mapping $d : \mathbb{R}^3 \rightarrow [0, +\infty)$ be defined by

$$d(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}.$$

Then d is a b_2 -metric on \mathbb{R} , i.e., the following inequality holds:

$$d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t),$$

for all $x, y, z, t \in \mathbb{R}$. From the convexity of the function $f(x) = x^p$ on $[0, +\infty)$ for $p \geq 1$, we obtain that

$$d^p(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}^p$$

is a b_2 -metric on \mathbb{R} with $s \leq 3^{p-1}$.

Definition 1.7 (Mustafa et al, 2014) Let $\{x_n\}$ be a sequence in a b_2 -metric space (X, d, s) .

1. $\{x_n\}$ is said to be b_2 -convergent to $x \in X$, written as $\lim_{n \rightarrow \infty} x_n = x$, if for all $a \in X$, $\lim_{n \rightarrow \infty} d(x_n, x, a) = 0$.

2. $\{x_n\}$ is said to be a b_2 -Cauchy sequence in X if for all $a \in X$, $\lim_{n, m \rightarrow \infty} d(x_n, x_m, a) = 0$.

3. (X, d) is said to be b_2 -complete if every b_2 -Cauchy sequence is a b_2 -convergent sequence.

Definition 1.8 (Mustafa et al, 2014) Let (X, d, s) be a b_2 -metric spaces and let $f : X \rightarrow X$ be a mapping. Then f is said to be b_2 -continuous at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d(z, x, a) < \delta$ for all $a \in X$ imply that $d(fz, fx, a) < \varepsilon$. The mapping f is b_2 -continuous on X if it is b_2 -continuous at all $z \in X$.

Remark 1.9 Let (X, d) be b_2 -metric spaces. Then a mapping $f : X \rightarrow X$ is b_2 -continuous at a point $x \in X$ if and only if it is b_2 -sequentially continuous at x , that is, whenever $\{x_n\}$ is b_2 -convergent to x , $\{fx_n\}$ is b_2 -convergent to $f(x)$.

This paper is to derive theorems of Banach, Reich and Jungck in the b_2 -metric spaces. Also, we obtain some results in partially ordered b_2 -metric spaces.

One sequence convergence test in the b_2 -metric space

The inequality given in the next Lemma 2.1 is key to proving our main results.

Lemma 2.1 Let (X, d, s) be a b_2 -metric space and $\{x_n\}$ a sequence in X . Suppose that $\lambda \in (0,1)$ and let c be a real nonnegative number such that

$$d(x_m, x_n, a) \leq \lambda d(x_{m-1}, x_{n-1}, a) + c(\lambda^m + \lambda^n) \quad (2.1)$$

for all $m, n \in \mathbb{N}$ and $a \in X$. Then $\{x_n\}$ is a b_2 -Cauchy sequence in X .

Proof. Let $m, n \in \mathbb{N}$, $a \in X$ and $p \in \mathbb{N}$ fixed such that $p > -2 \log_\lambda s$.

From (2.1) we have that

$$d(x_{m+k}, x_{n+k}, a) \leq \lambda^k d(x_m, x_n, a) + ck\lambda^k(\lambda^m + \lambda^n) \quad (2.2)$$

for all $m, n, k \in \mathbb{N}$ and $a \in X$. We have

$$\begin{aligned} d(x_m, x_n, a) &\leq s[d(x_{m+p}, x_n, a) + d(x_m, x_{m+p}, a) + d(x_m, x_n, x_{m+p})] \\ &\leq s[d(x_{m+p}, x_n, a) + \lambda^m d(x_0, x_p, a) \\ &\quad + cm\lambda^m(1 + \lambda^p) + \lambda^m d(x_0, x_n, x_p) + cm\lambda^m(1 + \lambda^p)] \\ &\leq s[d(x_{m+p}, x_n, a) + \lambda^m d(x_0, x_p, a) + \lambda^m d(x_0, x_n, x_p)] \\ &\quad + 4scm\lambda^m. \end{aligned}$$

Next, as it is

$$\begin{aligned} d(x_0, x_n, x_p) &\leq s[d(x_{n+p}, x_n, x_p) + d(x_0, x_{n+p}, x_p) + d(x_0, x_n, x_{n+p})] \\ &\leq s[\lambda^n d(x_p, x_0, x_p) + cn\lambda^n(\lambda^p + 1) + \lambda^p d(x_0, x_n, x_0) \\ &\quad + cp\lambda^p(1 + \lambda^n) + \lambda^n d(x_0, x_0, x_p) + cn\lambda^n(1 + \lambda^p)] \\ &\leq 2cs(2n\lambda^n + p), \end{aligned}$$

and

$$\begin{aligned} d(x_{m+p}, x_n, a) &\leq s[d(x_{n+p}, x_n, a) + d(x_{m+p}, x_{n+p}, a) + d(x_{m+p}, x_n, x_{n+p})] \\ &\leq s[\lambda^n d(x_p, x_0, a) + cn\lambda^n(1 + \lambda^p) + \lambda^p d(x_m, x_n, a) \\ &\quad + cp\lambda^p(\lambda^m + \lambda^n) + \lambda^p d(x_m, x_n, x_n) + cp\lambda^p(\lambda^m + \lambda^n)] \\ &\leq s[\lambda^n d(x_p, x_0, a) + \lambda^p d(x_m, x_n, a)] \\ &\quad + 2cs[p(\lambda^m + \lambda^n) + n\lambda^n], \end{aligned}$$

we obtain the following

$$d(x_m, x_n, a) \leq s^2 [\lambda^n d(x_p, x_0, a) + \lambda^p d(x_m, x_n, a) + s\lambda^m d(x_0, x_p, a)] + 2cs^2 [p(\lambda^m + \lambda^n) + n\lambda^n] + 2s^2c\lambda^m(2n\lambda^n + p).$$

So,

$$(1 - s^2\lambda^p)d(x_m, x_n, a) \leq s^2\lambda^n d(x_p, x_0, a) + s\lambda^m d(x_0, x_p, a) \tag{2.3}$$

$$+ 2cs^2 [2p\lambda^m + (p+n)\lambda^n + 2n\lambda^{m+n}], \tag{2.4}$$

how is it $p > -2\log_\lambda s$ we have $1 - s^2\lambda^p > 0$, therefore, we obtain that $\{x_n\}$ is a b_2 -Cauchy.

Lemma 2.2 Let (X, d, s) be a b_2 -metric space and $\{x_n\}$ a sequence in X . Suppose that $\lambda \in [0, 1)$ such that

$$d(x_m, x_n, a) \leq \lambda d(x_{m-1}, x_{n-1}, a) \text{ for all } m, n \in \mathbb{N}, a \in X.$$

Then

$$d(x_m, x_n, a) \leq \frac{s^2(\lambda^m + \lambda^n)}{1 - s^2\lambda^p} d(x_0, x_p, a), \tag{2.5}$$

for all $m, n, p \in \mathbb{N}$, $p > -2\log_\lambda s$ and $a \in X$.

Proof. It follows directly from Lemma 2.1, if we put $c = 0$, see (2.4).

Remark 2.3 Note that both Lemma 2.1 and Lemma 2.2 improve the result of Lemma 1.6. in (Fadail et al, 2015, pp.533-548).

A Theorem of Jungck in the b_2 -metric space

The following Theorem is the version of the Theorem of Jungck (Jungck, 1976, pp.261-263) in b_2 -metric spaces.

Theorem 3.1 Let T and I be commuting mappings of a b_2 -complete b_2 -metric space (X, d, s) into itself satisfying the inequality

$$d(Tx, Ty, a) \leq \lambda d(Ix, Iy, a) \tag{3.1}$$

for all $x, y, a \in X$, where $0 < \lambda < 1$. If the range of I contains the range of T and if I is b_2 -continuous, then T and I have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Then Tx_0 and Ix_0 are well defined. Since $Tx_0 \in I(X)$, there is any $x_1 \in X$ such that $Ix_1 = Tx_0$. In general, if x_n is chosen, then we choose a point x_{n+1} in X such that $Ix_{n+1} = Tx_n$. We show that $\{Ix_n\}$ is a b_2 -Cauchy sequence. From (3.1) we have

$$d(Ix_m, Ix_n, a) = d(Tx_{m-1}, Tx_{n-1}, a) \leq \lambda d(Ix_{m-1}, Ix_{n-1}, a).$$

So,

$$d(Ix_m, Ix_n, a) \leq \lambda d(Ix_{m-1}, Ix_{n-1}, a), \text{ for all } m, n \in \mathbb{N}, a \in X. \quad (3.2)$$

From Lemma 2.2 we obtain

$$d(Ix_m, Ix_n, a) \leq \frac{s^2(\lambda^m + \lambda^n)}{1 - s^2 p} d(Ix_0, Ix_p, a), \quad (3.3)$$

for all $m, n \in \mathbb{N}$, $a \in X$ and for some $p \in \mathbb{N}$ such that it is $p > -2 \log_\lambda s$. Thus, we obtain that $\{Ix_n\}$ is a b_2 -Cauchy sequence in X . By the b_2 -completeness of X , there exists $u \in X$ such that

$$\lim_{n \rightarrow \infty} Ix_n = \lim_{n \rightarrow \infty} Tx_{n-1} = u.$$

Now, since I is b_2 -continuous, (3.1) implies that both I and T are b_2 -continuous. Since T and I commute, we obtain

$$Iu = I\left(\lim_{n \rightarrow \infty} Tx_n\right) = \lim_{n \rightarrow \infty} ITx_n = \lim_{n \rightarrow \infty} TIX_n = T\left(\lim_{n \rightarrow \infty} Ix_n\right) = Tu.$$

Let $v = Iu = Tu$. We get $Tv = Tlu = ITu = Iv$. If $Tu \neq Tv$, from (3.1) we obtain

$$d(Tu, Tv, a) \leq \lambda d(Iu, Iv, a) = \lambda d(Tu, Tv, a) < d(Tu, Tv, a),$$

a contradiction. So we have $Tu = Tv$, and finally we obtain $Tv = Iv = v$ i.e. v is a common fixed point for T and I . Condition (3.1) implies that v is the unique common fixed point.

Theorem 3.2 Let T and I be commuting mappings of a complete 2-metric space (X, d, s) into itself satisfying the inequality

$$d(Tx, Ty, a) \leq \lambda d(Ix, Iy, a) \tag{3.4}$$

for all $x, y, a \in X$, where $0 < \lambda < 1$. If the range of I contains the range of T and I is continuous, then T and I have a unique common fixed point.

From Theorem 3.1, we obtain the following variant of the Banach theorem in b_2 -metric spaces.

Theorem 3.3 Let (X, d, s) be a b_2 -complete b_2 -metric space and $T : X \rightarrow X$ a mapping satisfying:

$$d(Tx, Ty, a) \leq \alpha d(x, y, a) \tag{3.5}$$

for all $x, y, a \in X$, where $\alpha \in [0, 1)$. Then T has a unique fixed point.

Proof. Put $I(x) = x, x \in X$ in Theorem 3.1.

We obtain the following result as a consequence of Theorem 3.1 if we put $K = \frac{1}{\lambda}$ and T to be the identity map, i.e. $T(x) = x, x \in X$.

Theorem 3.4 Let I be a continuous onto mapping of a b_2 -complete b_2 -metric space (X, d, s) . If there exists $K > 1$ such that

$$d(Ix, Iy, a) \geq Kd(x, y, a)$$

for all $x, y, a \in X$, then I has a unique fixed point.

The Reich theorem in b_2 -metric spaces

The following theorem is the analogue of the Reich contraction principle (Reich, 1971, pp.121-124) in the b_2 -metric space.

Theorem 4.1 Let (X, d, s) be a b_2 -complete b_2 -metric space and $T : X \rightarrow X$ be a mapping satisfying:

$$d(Tx, Ty, a) \leq \alpha d(x, y, a) + \beta d(x, Tx, a) + \gamma d(y, Ty, a) \tag{4.1}$$

for all $x, y, a \in X$, where α, β, γ are nonnegative constants with $\alpha + \beta + \gamma < 1$ and $\min\{\beta, \gamma\} < \frac{1}{s}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. From condition (4.1) we have that

$$d(x_{n+1}, x_n, a) \leq \alpha d(x_n, x_{n-1}, a) + \beta d(x_n, x_{n+1}, a) + \gamma d(x_{n-1}, x_n, a).$$

Therefore,

$$d(x_{n+1}, x_n, a) \leq \frac{\alpha + \gamma}{1 - \beta} d(x_n, x_{n-1}, a). \quad (4.2)$$

Put $r = \frac{\alpha + \gamma}{1 - \beta}$. We have that $r \in [0, 1)$. It follows from (4.2) that

$$d(x_{n+1}, x_n, a) \leq r^n d(x_1, x_0, a) \text{ for all } n \geq 1. \quad (4.3)$$

From conditions (4.1) and (4.3), we obtain

$$\begin{aligned} d(x_m, x_n, a) &\leq \alpha d(x_{m-1}, x_{n-1}, a) + \beta d(x_{m-1}, x_m, a) + \gamma d(x_{n-1}, x_n, a) \\ &\leq \alpha d(x_{m-1}, x_{n-1}, a) + \beta r^{m-1} d(x_0, x_1, a) + \gamma r^{n-1} d(x_0, x_1, a) \\ &= \alpha d(x_{m-1}, x_{n-1}, a) + (\beta r^{m-1} + \gamma r^{n-1}) d(x_0, x_1, a) \end{aligned}$$

From this, together with Lemma 2.1 (we can put

$$\lambda = \alpha, c = \max\{\beta, \gamma\} d(x_0, x_1, a)$$

for all $m, n \in \mathbb{N}$, note that if $r = 0$ then the proof is trivial) we conclude that $\{x_n\}$ is Cauchy. By the b_2 -completeness of (X, d, s) there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = x^*. \quad (4.4)$$

Now we obtain that x^* is the unique fixed point of T . Namely, we have

$$\begin{aligned}
 d(x^*, Tx^*, a) &\leq s[d(x_{n+1}, Tx^*, a) + d(x^*, x_{n+1}, a) + d(x^*, Tx^*, x_{n+1})] \\
 &= s[d(Tx_n, Tx^*, a) + d(x^*, x_{n+1}, a) + d(x^*, Tx^*, Tx_n)] \\
 &\leq s[\alpha d(x_n, x^*, a) + \beta d(x_n, x_{n+1}, a) + \gamma d(x^*, Tx^*, a) \\
 &\quad + d(x^*, x_{n+1}, a) + \alpha d(x^*, x^*, x_n) + \beta d(x^*, x^*, Tx^*) \\
 &\quad + \gamma d(x^*, x_n, x_{n+1})]
 \end{aligned}$$

and

$$\begin{aligned}
 d(Tx^*, x^*, a) &\leq s[d(x_{n+1}, x^*, a) + d(Tx^*, x_{n+1}, a) + d(Tx^*, x^*, x_{n+1})] \\
 &= s[d(x_{n+1}, x^*, a) + d(Tx^*, Tx_n, a) + d(Tx^*, x^*, Tx_n)] \\
 &\leq s[d(x_{n+1}, x^*, a) + \alpha d(x^*, x_n, a) + \beta d(x^*, Tx^*, a) \\
 &\quad + \gamma d(x_n, x_{n+1}, a) + \alpha d(x^*, x^*, x_n) + \beta d(x^*, x^*, Tx^*) \\
 &\quad + \gamma d(x^*, x_n, x_{n+1})]
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} d(x^*, x_n, a) = 0$, $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}, a) = 0$,

$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}, x^*) = 0$ and $\min(\beta, \gamma) < \frac{1}{s}$, we have $d(x^*, Tx^*, a) = 0$ for

all $a \in X$ i. e., $Tx^* = x^*$ (Axiom (1) in Definition 1.5).

For uniqueness, let y^* be another fixed point of T . Then it follows from (4.1) that

$$\begin{aligned}
 d(x^*, y^*, a) &= d(Tx^*, Ty^*, a) \leq \alpha d(x^*, y^*, a) + \beta d(x^*, Tx^*, a) + \gamma d(y^*, Ty^*, a) \\
 &= \alpha d(x^*, y^*, a) < d(x^*, y^*, a)
 \end{aligned}$$

is a contradiction. Therefore, we must have $d(x^*, y^*, a) = 0$, i.e., $x^* = y^*$.

From Theorem 4.1, we obtain the following variant of the Kannan theorem (Kannan, 1968, pp.71-76) in b_2 -metric spaces.

Theorem 4.2 Let (X, d, s) be a b_2 -complete b_2 -metric space and $T : X \rightarrow X$ be a mapping satisfying:

$$d(Tx, Ty, a) \leq \beta d(x, Tx, a) + \gamma d(y, Ty, a) \tag{4.5}$$

for all $x, y, a \in X$, where β, γ nonnegative constants with $\beta + \gamma < 1$ and $\min\{\beta, \gamma\} < \frac{1}{s}$. Then T has a unique fixed point.

A result in partial order b_2 -metric space

Let F_s denote the class of all functions $\beta : [0, \infty) \rightarrow \left[0, \frac{1}{s}\right)$ satisfying the following condition:

$$\beta(t_n) \rightarrow \frac{1}{s} \text{ as } n \rightarrow \infty \text{ implies } t_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In the paper (Mustafa et al, 2014), Mustafa et al. obtain the following result in partially ordered b_2 -metric spaces.

Theorem 5.1 (Mustafa et al, 2014, Theorem 1) Let (X, \preceq) be a partially ordered set and suppose that there exists a b_2 -metric d on X such that (X, d, s) is a b_2 -complete b_2 -metric space. Let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that

$$sd(fx, fy, a) \leq \beta(d(x, y, a))M(x, y, a) \quad (5.1)$$

for all $a \in X$ and for all comparable elements $x, y \in X$, where

$$M(x, y, a) = \max\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}\right\}.$$

If f is b_2 -continuous, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

In the further, we consider that $M(x, y, a)$ is given as in Theorem 5.1.

In the paper (Fadail et al, 2015, pp.533-548), Fadail et al. generalize, complement and improve Theorem 5.1 in several directions.

Theorem 5.2 (Fadail et al, 2015, pp.533-548, Theorem 2.1) Let (X, \preceq) be a partially ordered set and suppose that there exists a b_2 -metric d on X such that (X, d, s) is a b_2 -complete b_2 -metric space with $s > 1$. Let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that

$$s^\varepsilon d(fx, fy, a) \leq \beta(d(x, y, a))M(x, y, a) \tag{5.2}$$

for all $a \in X$ and for all comparable elements $x, y \in X$, where $\varepsilon \in (0,1]$. If f is b_2 -contuniuous then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Using Lemma 2.1, we get the following result.

Theorem 5.3 Let (X, \preceq) be a partially ordered set and suppose that there exists a b_2 -metric d on X such that (X, d, s) is a b_2 -complete b_2 -metric space with $s > 1$. Let $f : X \rightarrow X$ be an increasing mapping with respect to \preceq such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that exists $\lambda \in (0,1]$ such that

$$d(fx, fy, a) \leq \lambda M(x, y, a), \tag{5.3}$$

for all $a \in X$ and for all comparable elements $x, y \in X$. If f is b_2 -continuous, then f has a fixed point. Moreover, the set of the fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Since $x_0 \preceq fx_0$ and f is an increasing function, we obtain that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq \dots \preceq f^n x_0 \preceq f^{n+1}x_0 \preceq \dots.$$

Since $x_n \preceq x_{n+1}$, where $x_{n+1} = fx_n, n \in \mathbb{N}$, from (5.3) we obtain

$$\begin{aligned} d(x_n, x_{n+1}, a) &\leq \lambda \max \left\{ d(x_{n-1}, x_n, a), \frac{d(x_{n-1}, x_n, a)d(x_n, x_{n+1}, a)}{1 + d(x_n, x_{n+1}, a)} \right\} \\ &\leq \lambda d(x_{n-1}, x_n, a). \end{aligned}$$

Using induction, we conclude that

$$d(x_n, x_{n+1}, a) \leq \lambda^n d(x_0, x_1, a), \tag{5.4}$$

for all $n \in \mathbb{N}$, $a \in X$. Further from conditions (5.3), we have

$$\begin{aligned} d(x_m, x_n, a) &\leq \lambda \max \left\{ d(x_{m-1}, x_{n-1}, a), \frac{d(x_{m-1}, x_m, a)d(x_{n-1}, x_n, a)}{1 + d(x_m, x_n, a)} \right\} \\ &\leq \lambda \max \{ d(x_{m-1}, x_{n-1}, a), d(x_{m-1}, x_m, a)d(x_{n-1}, x_n, a) \}. \end{aligned}$$

Now, from inequality (5.4), it follows

$$\begin{aligned}
d(x_m, x_n, a) &\leq \lambda \max\{d(x_{m-1}, x_{n-1}, a), \lambda^{m-1}d(x_0, x_1, a)\lambda^{n-1}d(x_0, x_1, a)\} \\
&\leq \lambda d(x_{m-1}, x_{n-1}, a) + \lambda^{m+n-1}d^2(x_0, x_1, a) \\
&\leq \lambda d(x_{m-1}, x_{n-1}, a) + (\lambda^m + \lambda^n)d^2(x_0, x_1, a) \\
&= \lambda d(x_{m-1}, x_{n-1}, a) + c(\lambda^m + \lambda^n),
\end{aligned}$$

where $c = d^2(x_0, x_1, a)$. Now, because of Lemma 2.1, we get that $\{x_n\}$ is a b_2 -Cauchy sequence in (X, d) . The rest of the proof is the same as in (Mustafa et al, 2014) (Steps IV and V).

Note that condition (5.1) implies (5.2) and condition (5.2) implies (5.3).

Example 2 Let $X = \{(a,0) : a \in [0, +\infty)\} \cup \{(0,2)\}$ and let $d(x, y, z)$ denote the square of the area of a triangle with the vertices $x, y, z \in X$, e.g.,

$$d((a,0), (b,0), (0,2)) = (a-b)^2.$$

Then d is a b_2 -metric with the parameter $s = 2$. Introduce an order in X by

$$(a,0) \preceq (b,0) \Leftrightarrow a \geq b,$$

with all other pairs of distinct points in X incomparable.

Consider the mapping $f : X \rightarrow X$ given by

$$f(a,0) = (\lambda a, 0) \text{ for } a \in [0, +\infty) \text{ and } f(0,2) = (0,2),$$

and the function $\beta \in F_2$ given as

$$\beta(t) = \frac{1+t}{2+4t} \text{ for } t \in [0, +\infty).$$

Then f is an increasing mapping with $(a,0) \preceq f(a,0)$ for each $a \geq 0$.

1. If $\lambda = \frac{1}{3}$ then the assumptions of Teoreme 5.1 are satisfied (Mustafa et al, 2014, Example 3).

2. If $\lambda = \frac{1}{2}$ then the assumptions of Theorem 5.2 are satisfied (Fadail et al, 2015, pp.533-548, Example 2.6), but we cannot apply Theorem 5.1.

3. If $\lambda \in \left(\frac{1}{2}, 1\right)$ then the assumptions of Theorem 5.3 are satisfied, but we cannot apply Theorem 5.2.

References

- Agarwal, R.P., Karapinar, E., O'Regan, D., & Roldán-López-de-Hierro, A.F. 2015. *Fixed Point Theory in Metric Type Spaces*. Cham: Springer Nature. Available at: <https://doi.org/10.1007/978-3-319-24082-4>.
- Ahmed, M.A. 2009. A common fixed point theorem for expansive mappings in 2-metric spaces and its application. *Chaos, Solitons and Fractals*, 42(5), pp.2914-2920. Available at: <https://doi.org/10.1016/j.chaos.2009.04.034>.
- Aleksić, S., Mitrović, Z.D., & Radenović, S. 2018. A fixed point theorem of Jungck in $b_v(s)$ -metric spaces. *Periodica Mathematica Hungarica*, 77(2), pp.224-231. Available at: <https://doi.org/10.1007/s10998-018-0236-1>.
- Aliouche, A., & Simpson, C. 2012. Fixed points and lines in 2-metric spaces. *Advances in Mathematics*, 229(1), pp.668-690. Available at: <https://doi.org/10.1016/j.aim.2011.10.002>.
- Aydi, H. 2016. Implicit contractive pair of mappings on quasi b -metric spaces and an application to integral equations. *J. Nonlinear Convex Anal*, 17(12), pp.2417-2433. Available at: https://scholar.google.com/scholar?cluster=11178112980716461515&hl=en&as_sdt=2005&scioldt=0,5. Accessed:30.03.2019.
- Bakhtin, I.A., 1989. The contraction mapping principle in quasimetric spaces. *Func. An., Gos. Ped. Inst. Unianowsk*, 30, pp.26-37.
- Czerwik, S. 1993. Contraction mappings in b -metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis*, 1, pp.5-11. Available at: <https://dml.cz/handle/10338.dmlcz/120469>. Accessed:30.03.2019.
- Czerwik, S. 1998. Nonlinear setvalued contraction mappings in b -metric spaces. *Atti Sem. Mat. Univ. Modena*, 46, pp.263-276.
- Ćirić, Lj. 2003. *Fixed Point Theory: Contraction Mapping Principle*. Belgrade: Faculty of Mechanical Engineering.
- Deshpande, B., & Chouhan, S. 2011. Common fixed point theorems for hybrid pairs of mappings with some weaker conditions in 2-metric spaces. *Fasc. Math*, 46, pp.37-55. Available at: [http://www.math.put.poznan.pl/artykuly/FM46\(2011\)-DeshpandeB-ChouhanS.pdf](http://www.math.put.poznan.pl/artykuly/FM46(2011)-DeshpandeB-ChouhanS.pdf). Accessed: 30.03.2019.
- Dung, N.V., & Le Hang, V.T. 2016. On relaxations of contraction constants and Caristi's theorem in b -metric spaces. *Journal of Fixed Point Theory and Applications*, 18(2), pp.267-284. Available at: <https://doi.org/10.1007/s11784-015-0273-9>.
- Dung, N.V., & Le Hang, V.T. 2013. Fixed point theorems for weak C -contractions in partially ordered 2-metric spaces. *Fixed Point Theory and Applications*, 2013:161. Available at: <https://doi.org/10.1186/1687-1812-2013-161>.
- Fadail, Z.M., Ahmad, A.G.B., Ozturk, V., & Radenović, S. 2015. Some remarks on fixed point results of b_2 -metric spaces. *Far East Journal of Mathematical Sciences (FJMS)*, 97(5), pp.533-548. Available at: https://doi.org/10.17654/fjmsjul2015_533_548.

Freese, R.W., Cho, J.Y., & Kim, S.S. 1992. Strictly 2-convex linear 2-normed spaces. *J. Korean Math. Soc.*, 29(2), pp.391-400. Available at: <http://pdf.medrang.co.kr/kms01/JKMS/29/JKMS-29-2-391-400.pdf>. Accessed: 30.03.2019.

Gähler, S. 1963. 2-metrische Räume und ihre topologische Struktur. *Mathematische Nachrichten*, 26(1-4), pp.115-148 (in German). Available at: <https://doi.org/10.1002/mana.19630260109>.

George, R., Radenović, S., Reshma, K.P., & Shukla, S. 2015. Rectangular b -metric space and contraction principles. *Journal of Nonlinear Sciences and Applications*, 8(6), pp.1005-1013. Available at: <https://doi.org/10.22436/jnsa.008.06.11>.

Iseki, K. 1975. Fixed point theorems in 2-metric spaces. *Math. Semin Notes*, 3, pp.133-136.

Iseki, K. 1976. Mathematics on 2-normed spaces. *Bull. Korean Math. Soc*, 13(2), pp.127-135.

Jungck, G. 1976. Commuting Mappings and Fixed Points. *The American Mathematical Monthly*, 83(4), pp.261-263. Available at: <https://doi.org/10.1080/00029890.1976.11994093>.

Kannan, R. 1968. Some results on fixed points. *Bull. Calcutta Math. Soc*, 60, pp.71-76.

Kirk, W., & Shahzad, N. 2014. *Fixed Point Theory in Distance Spaces*. Cham: Springer Nature. Available at: <https://doi.org/10.1007/978-3-319-10927-5>.

Lahiri, B.K., Das, P., & Dey, L.K. 2011. Cantor's Theorem in 2-Metric Spaces and its Applications to Fixed Point Problems. *Taiwanese Journal of Mathematics*, 15(1), pp.337-352. Available at: <https://doi.org/10.11650/twjmath/1500406178>.

Lal, S.N., & Singh, A.K. 1978. An analogue of Banach's contraction principle for 2-metric spaces. *Bulletin of the Australian Mathematical Society*, 18(1), pp.137-143. Available at: <https://doi.org/10.1017/s0004972700007887>.

Miculescu, R., & Mihail, A. 2017. New fixed point theorems for set-valued contractions in b -metric spaces. *Journal of Fixed Point Theory and Applications*, 19(3), pp.2153-2163. Available at: <https://doi.org/10.1007/s11784-016-0400-2>.

Mitrović, Z.D., & Radenović, S. 2017. The Banach and Reich contractions in $b_{\psi}(s)$ -metric spaces. *Journal of Fixed Point Theory and Applications*, 19(4), pp.3087-3095. Available at: <https://doi.org/10.1007/s11784-017-0469-2>.

Mitrović, Z.D., & Radenović, S. 2017. A common fixed point theorem of Jungck in rectangular b -metric spaces. *Acta Mathematica Hungarica*, 153(2), pp.401-407. Available at: <https://doi.org/10.1007/s10474-017-0750-2>.

Mustafa, Z., Parvaneh, V., Roshan, J.R., & Kadelburg, Z. 2014. b_2 -metric spaces and some fixed point theorems. *Fixed Point Theory and Applications*, 2014:144. Available at: <https://doi.org/10.1186/1687-1812-2014-144>.

Naidu, S.V.R., & Prasad, J.R. 1986. Fixed point theorems in 2-metric spaces. *Indian J. Pure Appl. Math*, 17(8), pp.974-993.

Popa, V., Imdad, M., & Ali, J. 2010. Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces. *Bull. Malays. Math. Sci. Soc.*, 33(1), pp.105-120. Available at: <http://emis.ams.org/journals/BMMSS/pdf/v33n1/v33n1p8.pdf>. Accessed:30.03.2019.

Reich, S. 1971. Some Remarks Concerning Contraction Mappings. *Canadian Mathematical Bulletin*, 14(1), pp.121-124. Available at: <https://doi.org/10.4153/cmb-1971-024-9>.

НЕКОТОРЫЕ ТЕОРЕМЫ О НЕПОДВИЖНЫХ ТОЧКАХ В b_2 -МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

Лум Кумам^а, Зоран Д. Митрович^б, Мирьяна В. Павлович^в

^а Технологический университет короля Монгкута в Тхонбури, Естественно-математический факультет, Исследовательская группа по теории неподвижных точек и их применению, Центр теоретических и вычислительных наук, Здание научной лаборатории, г. Бангкок, Королевство Таиланд

^б Университет в г. Баня-Лука, Электротехнический факультет, г. Баня-Лука, Республика Сербская, Босния и Герцеговина

^в Университет в г. Крагуевац, Естественно-математический факультет, г. Крагуевац, Республика Сербия

РУБРИКА ГРНТИ: 27.00.00 МАТЕМАТИКА;
27.39.27 Нелинейный функциональный анализ

ВИД СТАТЬИ: оригинальная научная статья
ЯЗЫК СТАТЬИ: английский

Резюме:

Во введении данной статьи мы доказали результат, который является достаточным условием для сходимости последовательностей в b_2 -метрическом пространстве. Далее представлены некоторые теоремы о неподвижной точке в b_2 -метрическом пространстве. Некоторые из наших результатов представляют собой соответствующие обобщения известных результатов в b_2 -метрическом пространстве, в качестве доказательства приведены примеры.

Ключевые слова: неподвижные точки, обобщенные неподвижные точки, 2-метрическое пространство, b_2 -метрическое пространство.

НЕКЕ ТЕОРЕМЕ О ФИКСНОЈ ТАЧКИ У b_2 -МЕТРИЧКИМ ПРОСТОРИМАПум Кумам^а, Зоран Д. Митровић^б, Мирјана В. Павловић^в^а Технолошки универзитет Краља Монгаста у Тонбурију, Природно-математички факултет, Истраживачка група теорије и примене фиксних тачака, Центар за теоријске и рачунске науке, Зграда научне лабораторије, Бангкок, Краљевина Тајланд^б Универзитет у Бањој Луци, Електротехнички факултет, Бања Лука, Република Српска, Босна и Херцеговина^в Универзитет у Крагујевцу, Природно-математички факултет, Крагујевац, Република Србија

ОБЛАСТ: математика (математичка тематска класификација: примарна 47Н10, секундарна 54Н25)

ВРСТА ЧЛАНКА: оригинални научни рад

ЈЕЗИК ЧЛАНКА: енглески

Сажетак:

У овом раду прво је доказан резултат који даје довољан услов за конвергенцију низова у b_2 -метричком простору. Такође, наведене су неке теореме о фиксној тачки у b_2 -метричком простору. Неки резултати представљају одговарајуће генерализације познатих резултата у b_2 -метричком простору, а примери су презентирани да то потврде.

Кључне речи: фиксне тачке, заједничке фиксне тачке, 2-метрички простор, b_2 -метрички простор.

Paper received on / Дата получения работы / Датум пријема чланка: 05.04.2019.

Manuscript corrections submitted on / Дата получения исправленной версии работы / Датум достављања исправки рукописа: 10.05.2019.

Paper accepted for publishing on / Дата окончательного согласования работы / Датум коначног прихватања чланка за објављивање: 12.05.2019.

© 2019 The Authors. Published by Vojnotehnički glasnik / Military Technical Courier (www.vtg.mod.gov.rs, втг.мо.упр.срб). This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution license (<http://creativecommons.org/licenses/by/3.0/rs/>).

© 2019 Авторы. Опубликовано в «Военно-технический вестник / Vojnotehnički glasnik / Military Technical Courier» (www.vtg.mod.gov.rs, втг.мо.упр.срб). Данная статья в открытом доступе и распространяется в соответствии с лицензией «Creative Commons» (<http://creativecommons.org/licenses/by/3.0/rs/>).

© 2019 Аутори. Објавио Војнотехнички гласник / Vojnotehnički glasnik / Technical Courier (www.vtg.mod.gov.rs, втг.мо.упр.срб). Ово је чланак отвореног приступа и дистрибуира се у складу са Creative Commons licencom (<http://creativecommons.org/licenses/by/3.0/rs/>).

