

TWO LAPLACIAN ENERGIES AND THE RELATIONS BETWEEN THEM

Ivan Gutman

University of Kragujevac, Faculty of Science,
Kragujevac, Republic of Serbia,
e-mail: gutman@kg.ac.rs,
ORCID iD: <https://orcid.org/0000-0001-9681-1550>

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Abstract:

Introduction/purpose: The Laplacian energy (LE) is the sum of absolute values of the terms $\mu_i - 2m/n$, where μ_i , $i=1,2,\dots,n$, are the eigenvalues of the Laplacian matrix of the graph G with n vertices and m edges. In 2006, another quantity Z was introduced, based on Laplacian eigenvalues, which was also named „Laplacian energy“. Z is the sum of squares of Laplacian eigenvalues. The aim of this work is to establish relations between LE and Z .

Results: Lower and upper bounds for LE are deduced, in terms of Z .

Conclusion: The paper contributes to the Laplacian spectral theory and the theory of graph energies. It is shown that, as a rough approximation, LE is proportional to the term $(Z-4m^2/n)^{1/2}$.

Keywords: Laplacian spectrum (of graph), Laplacian energy.

Introduction

Let M be a real symmetric square matrix of order n . Let $\zeta_1, \zeta_2, \dots, \zeta_n$, be the eigenvalues of M , and let $\zeta_1 + \zeta_2 + \dots + \zeta_n = \zeta$. Then the energy of M is defined as (Nikiforov, 2007), (Gutman & Furtula, 2019):

$$E(M) = \sum_{i=1}^n \left| \zeta_i - \frac{\zeta}{n} \right| \quad (1)$$

By using different matrices, one arrives at different „energies“. The first among them is the (ordinary) graph energy, based on the

eigenvalues of the (0,1)-adjacency matrix of a graph (Li et al, 2012), (Ramane 2020). It was introduced in 1978. Since then, more than 170 various „energies“ have been considered in the literature; for details see (Gutman & Furtula, 2019). In this paper, we are concerned with the *Laplacian energy*.

Let G be a simple graph possessing n vertices and m edges. Label its vertices by v_1, v_2, \dots, v_n . Let $\deg(v_i)$ be the degree (= number of first neighbors) of the vertex v_i . The Laplacian matrix of G , denoted by $L(G)$, is the square matrix of order n , whose (i,j) -element is

$$L(G)_{ij} = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ \deg(v_i) & \text{if } i = j \end{cases}$$

For details of the theory of Laplacian matrices and their spectra see (Grone et al, 1990), (Mohar, 1992), (Merris, 1994).

Let $\mu_1, \mu_2, \dots, \mu_n$ be the Laplacian eigenvalues of the graph G , i.e., the eigenvalues of $L(G)$. Then the Laplacian energy of G is

$$LE = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|. \quad (2)$$

The Laplacian energy was introduced in 2006 by the Chinese mathematician Bo Zhou and the present author (Gutman & Zhou, 2006). Since then, its theory was elaborated in due detail, see (Das & Mojallal, 2014), (Pirzada & Ganie, 2015), (Andriantiana, 2016), (Gutman & Furtula, 2019), (Gutman, 2020), and the references cited therein.

In the same year when the concept of the Laplacian energy was conceived (Gutman & Zhou, 2006), a paper was published in which an unrelated Laplacian-spectral quantity was defined, and also named „Laplacian energy“ (Lazić, 2006). The quantity put forward in (Lazić, 2006) is

$$Z = \sum_{i=1}^n \mu_i^2. \quad (3)$$

In what follows we refer to Z as to the fake Laplacian energy.

It is evident that Z , Eq. (3), violates the general conditions that an „energy“ needs to satisfy, see Eq. (1). The right-hand side of Eq. (3) is just the second spectral moment of the Laplacian eigenvalues. Naming it „energy“ was a misnomer. This was immediately recognized by all

mathematicians who did research of the true Laplacian energy, Eq. (2), and the paper (Lazić, 2006) was simply ignored.

The inventors of the Laplacian energy (Gutman & Zhou, 2006), as well as the scholars who later studied it, were solely interested in its mathematical properties. However, in recent years, the Laplacian energy has gained popularity for a variety of technical applications, mainly in the area of image analysis and pattern recognition (Luyuan et al, 2010), (Song et al, 2010), (Meng & Xiao, 2011), (Xiao et al, 2011), (Huigang et al, 2013), (Bai et al, 2014), (Deepa et al, 2016), (Pournami & Govindan, 2017), (Zou et al, 2018). In all the quoted papers, the Laplacian energy was computed according to Eq. (2).

Not all scholars who work on applications of the Laplacian energy are experts on its mathematical theory, and some of them seem to have learned about the Laplacian energy by means of Google search. Therefore, it happened that in some papers, instead of the true Laplacian energy, a group of authors used Z , Eq. (3) (Qi et al, 2012), (Qi et al, 2013), (Qi et al, 2015). It may be that there are more such erroneous works, spread in the non-mathematical literature.

The existence of papers in which the fake Laplacian energy is used, motivated us to examine the actual (mathematical) relation between LE and Z .

Relating the two Laplacian energies

In (Gutman 2020), it was pointed out that the relations

$$\sum_{i=1}^n \mu_i = 2m$$

and

$$\sum_{i=1}^n \mu_i^2 = 2m + \sum_{i=1}^n \deg(v_i)^2$$

are well known (Grone et al, 1990). There it was shown that

$$\sum_{i=1}^n \mu_i^{*2} = 2m + \sum_{i=1}^n \deg(v_i)^2 - \frac{4m^2}{n}$$

where

$$\mu_i^* = \mu_i - \frac{2m}{n}.$$

Recall that

$$\sum_{i=1}^n \mu_i^* = 0 \quad \text{and} \quad \sum_{i=1}^n |\mu_i^*| = LE. \quad (4)$$

Bearing in mind Eq. (3), we get

$$Z = 2m + \sum_{i=1}^n \deg(v_i)^2 \quad (5)$$

and

$$\sum_{i=1}^n \mu_i^{*2} = Z - \frac{4m^2}{n}. \quad (6)$$

Starting with

$$\sum_{i=1}^n \sum_{j=1}^n (|\mu_i^*| - |\mu_j^*|)^2 \geq 0$$

and using Eqs. (4) and (6), we get

$$n \sum_{i=1}^n \mu_i^{*2} + n \sum_{j=1}^n \mu_j^{*2} - 2 \sum_{i=1}^n \sum_{j=1}^n |\mu_i^*| \cdot |\mu_j^*| = 2n \left(Z - \frac{4m^2}{n} \right) - 2LE^2 \geq 0$$

from which,

$$LE \leq \sqrt{n \left(Z - \frac{4m^2}{n} \right)}. \quad (7)$$

Starting with

$$LE^2 = \left(\sum_{i=1}^n |\mu_i^*| \right)^2 = \sum_{i=1}^n \mu_i^{*2} + 2 \sum_{i<j} |\mu_i^*| \cdot |\mu_j^*| \geq \sum_{i=1}^n \mu_i^{*2} + 2 \left| \sum_{i<j} \mu_i^* \cdot \mu_j^* \right|$$

and taking into account that because of (4),

$$2 \left| \sum_{i<j} \mu_i^* \cdot \mu_j^* \right| = \left| \sum_{i=1}^n \sum_{j=1}^n \mu_i^* \cdot \mu_j^* - \sum_{i=1}^n \mu_i^{*2} \right| = \sum_{i=1}^n \mu_i^{*2}$$

we get

$$LE^2 \geq 2 \sum_{i=1}^n \mu_i^{*2}$$

i.e.

$$LE \geq \sqrt{2 \left(Z - \frac{4m^2}{n} \right)}. \quad (8)$$

Combining (7) and (8), we arrive at

$$\sqrt{2 \left(Z - \frac{4m^2}{n} \right)} \leq LE \leq \sqrt{n \left(Z - \frac{4m^2}{n} \right)}. \quad (9)$$

Discussion

From the bounds (9), we see that, as a rough approximation, there should exist a linear relation between the Laplacian energy (LE) and the term $\sqrt{Z - 4m^2/n}$, with Z standing for the fake Laplacian energy. As the first guess, we may have

$$LE \approx \frac{\sqrt{n} + \sqrt{2}}{2} \sqrt{Z - \frac{4m^2}{n}}. \quad (10)$$

The approximation (10), as well as any other approximation based on the bounds (7) and (8), is of poor quality. Namely, in contrast to the Laplacian energy, the right-hand side of (10) is structure-insensitive. This, of course, is the consequence of the structure-insensitivity of the fake Laplacian energy, Z .

For instance, if the graph G is regular of degree r , then $Z = nr(r+1)$. If the graph G has n_a vertices of degree a , and n_b vertices of degree b , so that $n_a + n_b = n$, then

$$Z = \frac{2m - nb}{a - b} a(a + 1) + \frac{2m - na}{b - a} b(b + 1)$$

and Z is independent of the parameters n_a and n_b . Thus, for the chemically important class of (molecular) graphs with vertices of degree two and three, $Z = 6(2m - n)$, independent of any other structural detail. Then the same holds also for the right-hand side of Eq. (10).

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ДВЕ ЭНЕРГИИ ЛАПЛАСА И ИХ СООТНОШЕНИЕ

Иван Гутман

Крагуевацкий университет, Естественно-математический факультет,
г. Крагуевац, Республика Сербия

РУБРИКА ГРНТИ: 27.29.19 Краевые задачи и задачи на собственные значения для обыкновенных дифференциальных уравнений и систем уравнений

ВИД СТАТЬИ: оригинальная научная статья

ЯЗЫК СТАТЬИ: английский

Резюме:

Введение/цель: Энергия Лапласа (LE) представляет собой сумму абсолютных значений $\mu_{-2m/n}$, где μ_i , $i=1,2,\dots,n$ являются собственными значениями G графы матрицы Лапласа с вершинами n и ребром m . В 2006 году была введена величина Z , основанная на характерных значениях Лапласа, которая получила название «Лапласова энергия». Z – это сумма квадратов собственных значений Лапласа. Целью данной работы является установление соотношений между LE и Z .

Результаты: Нижняя и верхняя границы для LE выводятся из функции Z .

Выводы: Статья способствует спектральной теории Лапласа и теории энергии графов. В грубой аппроксимации было показано, что LE пропорциональна $(Z-4m^2/n)^{1/2}$.

Ключевые слова: Лапласов спектр (граф), энергия Лапласа.

ДВЕ ЛАПЛАСОВЕ ЕНЕРГИЈЕ И ОДНОСИ МЕЂУ ЊИМА

Иван Гутман

Универзитет у Крагујевцу, Природно-математички факултет,
Крагујевац, Република Србија

ОБЛАСТ: математика

ВРСТА ЧЛАНКА: оригинални научни рад

ЈЕЗИК ЧЛАНКА: енглески

Сажетак:

Увод/циљ: Лапласова енергија (LE) јесте сума апсолутних вредности појмова $\mu-2t/n$, где су μ_i , $i=1,2,\dots,n$, сопствене вредности Лапласове матрице графа G са n врхова и t ивица. Године 2006. уведена је друга величина Z , заснована на Лапласовим својственим вредностима, која је такође названа „Лапласова енергија”. Z је сума квадрата Лапласових својствених вредности. Циљ овог рада је налажење односа између LE и Z .

Резултати: Доња и горња граница за LE одређене су као функције од Z .

Закључак: Рад доприноси Лапласовој спектралној теорији и теорији енергије графова. Показано је да је, као груба апроксимација, LE пропорционална са $(Z-4m^2/n)^{1/2}$.

Кључне речи: Лапласов спектар (графа), Лапласова енергија.

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