# SOLUTIONS AND ULAM-HYERS STABILITY OF DIFFERENTIAL INCLUSIONS INVOLVING SUZUKI TYPE MULTIVALUED MAPPINGS ON *b*-METRIC SPACES

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*Abstract:*

*Introduction/purpose: This paper presents coincidence and common fixed points of Suzuki type $(\alpha_* - \psi)-$ multivalued operators on b-metric spaces.*

*Methods: The limit shadowing property was discussed as well as the well-posedness and the Ulam-Hyers stability of the solution for the fixed point problem of such operators.*

*Results: The upper bound of the Hausdorff distance between the fixed point sets is obtained. Some examples are presented to support the obtained results.*

*Conclusion: The application of the obtained results establishes the existence of differential inclusion.*

*Keywords: b-metric space, multi-valued mapping, fixed point problems, Ulam-Hyers stability, initial value problem.*

## Introduction and preliminaries

Euclidean distance is an important measure of ”nearness” between two real or complex numbers. Fréchet (1905) introduced the concept of a metric to obtain the distance between two arbitrary objects. Since then, this notion has been generalized further in one to many directions, see (An et al, 2015a), among which one of the most important generalizations is the concept of a b-metric initiated by (Czerwik, 1993). For more details of b-metric spaces see (Aleksić et al, 2018), (Hussain et al, 2012), (Kirk & Shahzad, 2014) and the references therein.

**Definition 1.1** Let $X$ be a nonempty set. A mapping $d: X \times X \to [0, +\infty)$ is said to be a b-metric on $X$ if there exists some real constant $b \geqslant 1$ such that for any $x, y, z \in X$, the following condition holds:

**$a_1$):** $d(x, y) = 0$ if and only if $x = y$;

**$a_2$):** $d(x, y) = d(y, x)$;

**$a_3$):** $d(x, y) \leqslant bd(x, z) + bd(z, y)$

*The pair* $(X, d)$ *is termed a b-metric space with b-metric constant* $b$. Every metric is b-metric for $b = 1$ but the converse does not hold in general (Ćirić et al, 2012), (Czerwik, 1993), (Singh & Prasad, 2008).

In the sequel, the letters, $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}^+$ will denote the set of all nonnegative real numbers, the set of all real numbers, the set of all natural numbers and the set of all nonnegative integer numbers, respectively.

Let $(X, d)$ be a b-metric space and $P(X)$ a collection of all subsets of $X$. Denote $Cl(X), CB(X)$, and $K(X)$ by the collection of closed, closed and bounded and compact subsets of $X$, respectively.

Let $U, V \in P(X)$. The gap functional $D$, the excess generalized function $\rho$, the Pompeiu-Hausdorff generalized functional $H$, and the functional $\delta$ induced by a b-metric $d$ on $X$ are defined as:

$$(1)\quad D(U,V) = \begin{cases} \inf_{u\in U, v\in V} d(u,v), \text{if } U \neq V \neq \emptyset \neq U, \\ 0, \text{if } U = V = \emptyset, \\ \infty, \text{otherwise.} \end{cases}$$

$$(2)\quad \rho(U,V) = \begin{cases} \sup_{u\in U} D(u,V), \text{if } U \neq V \neq \emptyset \neq U, \\ 0, \text{if } U = \emptyset, \\ \infty, \text{if } V = \emptyset, U \neq \emptyset. \end{cases}$$

$$(3)\quad H(U,V) = \begin{cases} \max\{\rho(U,V), \rho(V,U)\}, \text{if } U \neq V \neq \emptyset \neq U, \\ 0, \text{if } U = V = \emptyset, \\ \infty, \text{otherwise.} \end{cases}$$

$$\delta(U,V) = \begin{cases} \sup_{u\in U, v\in V} d(u,v), \text{if } U \neq V \neq \emptyset \neq U, \\ 0, \text{if } U = V = \emptyset, \\ \infty, \text{otherwise.} \end{cases}$$

An et al (2015b) studied the topological properties of b-metric spaces and stated that a b-metric is not necessarily continuous in each variable. If a b-metric is continuous in one variable, then it is continuous in other variable. A ball $B(u_0, \epsilon) = \{v \in X: d(u_0, v) < \epsilon\}$ in a b-metric space $(X, d)$ is not necessarily an open set. A ball is an open set if $d$ is continuous in one variable.

Let $(X, d)$ be a b-metric space. We call $(f, T)$ a hybrid pair of mappings if $f: X \to X$ and $T: X \to CB(X)$.

A mapping $f$ is called a contraction if there is some real constant $r \in [0,1)$ such that for any $u, v \in X$, we have $d(fu, fv) \leqslant rd(u, v)$.

A point $u$ in $X$ is a fixed point of $f$, if $u = fu$, a fixed point of $T$, if $u \in Tu$, a coincidence point of $(f, T)$ if $fu \in T$, and a common fixed point of $(f, T)$ if $u = fu \in Tu$. Denote $F(f), F(T)$ by the fixed points of $f$ and $T$, respectively, and $C(f, T)$ and $F(f, T)$ by coincidence and common fixed point of $(f, T)$, respectively.

**Definition 1.2,** *compare with (Abbas et al, 2012). A pair $(f,T)$ is w-compatible if $f(Tu) \subseteq T(fu)$ for all $u \in C(f,T)$. The mapping $f$ is $T$-weakly commuting at some point $u \in X$ if $f^2(u) \in T(fu)$.*

Using an axiom of choice, Haghi et al (2011) proved the following lemma.

**Lemma 1.3** *(Haghi et al, 2011) Let $f: X \to X$ be a self-mapping of a nonempty set $X$, then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f$ is one-to-one on $E$.*

**Lemma 1.4,** *compare (Rus et al, 2003). Let $(X,d)$ be a b-metric space, $U,V \in P(X)$. If there exists a $\lambda > 0$ such that for each $u \in U$, there exists a $v \in V$ such that $d(u,v) \leqslant \lambda$, and for each $v \in V$, there exists a $u \in U$ such that $d(u,v) \leqslant \lambda$, then $H(U,V) \leqslant \lambda$.*

We need following lemmas given in (Czerwik, 1993), (Singh & Prasad, 2008).

**Lemma 1.5** *Let $(X,d)$ be a b-metric space, $u,v \in X, \{u_n\}$ a sequence in $X$ and $U,V \in CB(X)$. The following statements hold:*

**$b_1$-:** *$(CB(X),H)$ is a b-metric space and $(CB(X),H)$ is complete whenever $(X,d)$ is complete*;

**$b_2$-:** $D(u,V) \leqslant H(U,V)$ *for all* $u \in U$;

**$b_3$-:** $D(u,U) \leqslant bd(u,v) + bD(v,U)$;

**$b_4$-:** *for $h > 1$ and $u \in U$, there is a $v \in V$ such that $d(u,v) \leqslant hH(U,V)$*;

**$b_5$-:** *for every $h > 0$ and $u \in U$, there is a $v \in V$ such that $d(u,v) \leqslant H(U,V) + h$*;

**$b_6$-:** $D(u,U) = 0$ *if and only if* $u \in \overline{U} = U$;

**$b_7$-:** $d(u_0,u_n) \leqslant bd(u_0,u_1)+\ldots+b^{n-1}d(u_{n-2},u_{n-1}) + b^{n-1}d(u_{n-1},u_n)$;

**$b_8$-:** *$\{u_n\}$ is a Cauchy sequence if and only if for $\epsilon > 0$, there exists $n(\epsilon) \in N$ such that for each $n,m \geqslant n(\epsilon)$ we have $d(u_n,u_m) < \epsilon$*;

**$b_9$-:** *$\{u_n\}$ is a convergent sequence if and only if there exists $u \in X$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in N$ such that for all $n \geqslant n(\epsilon), d(u_n,u) < \epsilon$.*

A sequence $\{u_n\}$ is convergent to $u \in X$ if and only if $\lim_{n\to+\infty} d(u_n,u) = 0$.

A subset $Y \subset X$ is closed if and only if for each sequence $\{u_n\}$ in $Y$ that converges to an element $u$, $u \in Y$. A subset $Y \subset X$ is bounded if diam$(Y)$ is finite, where diam$(Y) = \sup\{d(u,v): u,v \in Y\}$. A b-metric space $(X,d)$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

Following are some known classes of mappings given in (Berinde, 1993), (Berinde, 1996), (Berinde, 1997), (Bota et al, 2015), (Rus, 2001).

Let $\psi, \varphi: \mathbb{R}^+ \to \mathbb{R}^+$, then

**c₁-:** $\Psi_1 = \left\{\psi: \psi \text{ is increasing}, \lim\limits_{n\to+\infty} \psi^n(t) = 0, \text{for any } t \geqslant 0\right\}$.

The elements in this class are called comparison functions. If $\psi \in \Psi_1$, then the n$^{th}$ iterate of $\psi$ is a comparison function, $\psi$ is continuous at $t = 0$, and $\psi(t) < t$, for any $t > 0$.

**c₂-:** $\Psi_2 = \{\psi: \sum_{n=1}^{\infty} \psi^n(t) < +\infty \text{ for all } t > 0 \text{ and} \psi \text{ is nondecreasing}\}$.

**c₃-:** $\Psi_3 = \left\{\begin{matrix} \psi: \psi \text{ is increasing, there exists a } n_0 \in \mathbb{N}, a \in (0,1), \\ \text{a sequence } u_n \geqslant 0 \text{ such that } \sum_{n=1}^{\infty} u_n < +\infty \text{and} \\ \psi^{n+1}(t) \leqslant a\psi^n(t) + u_n \text{ for all } n \geqslant n_0,\ t \geqslant 0 \end{matrix}\right\}$

is called the class of $(c)$ −comparison functions.

**c₄-:** $\Psi_4 = \left\{\begin{matrix} \psi: \psi \text{ is increasing, there exists a } n_0 \in \mathbb{N}, a \in (0,1), \\ b \geqslant 1, \text{a sequence } u_n \geqslant 0 \text{ such that } \sum_{n=1}^{\infty} u_n < +\infty \text{and} \\ b^{n+1}\psi^{n+1}(t) \leqslant ab^n\psi^n(t) + u_n \text{ for all } n \geqslant n_0,\ t \geqslant 0 \end{matrix}\right\}$

is known as the class of $(b)$ −comparison functions.

Note that $\Psi_2 \subseteq \Psi_1$. If $b = 1$, then $\Psi_3 = \Psi_4$.

**Lemma 1.6** *(Berinde, 1993) If $\psi \in \Psi_4$ with $b \geqslant 1$, then the series $\sum_{n=0}^{\infty} b^n\psi^n(t)$ converges for all $t \in \mathbb{R}^+$, and $r_b(t) = \sum_{n=0}^{\infty} b^n\psi^n(t)$ is increasing and continuous at $t = 0$.*

In the light of the above lemma, $\Psi_4 \subseteq \Psi_1$.

**Lemma 1.7** *(Păcurar, 2010) If $\psi \in \Psi_4$ with $b \geqslant 1$, and $\{a_n\} \subseteq \mathbb{R}^+$ is such that $\lim\limits_{n\to+\infty} a_n = 0$, then*

$$\lim_{k\to+\infty} \sum_{n=0}^{\infty} b^{k-n}\psi^{k-n}(a_n) = 0.$$

**Example 1.8** Let $\psi(t) = qt$, where $q \in [0,1)$ and $t \in \mathbb{R}^+$. Consider $\sum_{k=0}^{\infty} u_k(t)$, where $u_k(t) = b^k\psi^k(t)$ and $b \geqslant 1$. If $t = 0$, then $\sum_{k=0}^{\infty} u_k(t)$

converges trivially. If $t > 0$, then by the generalized ratio test (Berinde, 1993), $\sum_{k=0}^{\infty} u_k(t)$ is convergent for any $t > 0$. Hence for some $n_0 \in \mathbb{N}, b^{n+1}\psi^{n+1}(t) \leqslant ab^n\psi^n(t) + u_n$ for all $n \geqslant n_0$ and $t \geqslant 0$. Consequently, we have $\psi \in \Psi_4$.

The Banach contraction principle (BCP) (Banach, 1922) states that a contraction mapping on a complete metric space has a unique fixed point.

Let $\alpha: X \times X \to \mathbb{R}^+$. A mapping $f: X \to X$ is called an $\alpha$ −admissible if for all $u, v \in X, \alpha(u, v) \geqslant 1$ implies that $\alpha(fu, fv) \geqslant 1$. Samet et al (2012), Theorem 1, obtained the following generalization of BCP.

***Theorem 1.9*** *Let $(X, d)$ be a complete metric space and $f: X \to X$ an $\alpha$ −admissible mapping. Suppose that there is an element $u_0$ in $X$ with $\alpha(u_0, fu_0) \geqslant 1$. If for any $u, v \in X$, there exists $\psi \in \Psi_2$ such that $\alpha(u, v)d(fu, fv) \leqslant \psi(d(u, v))$, then $F(f)$ is nonempty provided that $f$ is continuous.*

Suzuki (2008) provided an interesting generalization of BCP that characterizes metric completeness.

For some other important generalizations of BCP, see (Berinde, 1993), (Berinde, 1996), (Berinde, 1997), (Bhaskar & Lakshmikantham, 2006), (Nieto & Rodríguez-López, 2005), (Nieto & Rodríguez-López, 2007), (Ran & Reurings, 2004) and references therein. A number of fixed point theorems have been obtained in b-metric spaces (Aleksić et al, 2019a), (Aleksić et al, 2019b), (Ali & Abbas, 2017), (An et al, 2015a), (An et al, 2015a), (Ćirić et al, 2012), (Chifu & Petruşel, 2014), (Czerwik, 1993), (Karapinar et al, 2020), (Latif, 2015), (Mitrović, 2019), (Păcurar, 2010).

The development of the metric fixed point theory of multivalued mappings was initiated by (Nadler, 1969). He introduced the concept of set-valued contraction mappings and extended the Banach contraction principle to set-valued mappings by using the Hausdorff metric as follows.

**Theorem 1.10** *Let $(X, d)$ be a complete metric space. If a multivalued mapping $T: X \longrightarrow CB(X)$ satisfies $H(Tu, Tv) \leqslant rd(u, v)$ for all $u, v \in X$ and for some $r \in [0,1)$, then $F(T)$ is nonempty.*

The fixed point theory of multivalued mappings provides a useful machinery to analyze the problems of pure, applied and computational mathematics which can be reformulated in the form of an inclusion for an appropriate multivalued mapping.

For more results in this direction, we refer to (Abbas et al, 2012), (Abbas et al, 2013), (Asl et al, 2012), (Rus et al, 2003), (Mitrović et al, 2020).

Khojasteh et al (2014) proved a new type of the fixed point theorem for multivalued mappings in metric spaces as follows.

**Theorem 1.11** *Let $(X, d)$ be a complete metric space. If a multivalued mapping $T: X \longrightarrow CB(X)$ satisfies*

$$H(Tu, Tv) \leqslant \left(\frac{D(u, Tv) + D(v, Tu)}{1 + \delta(u, Tu) + \delta(v, Tv)}\right) d(u, v) \tag{1.1}$$

*for all $u, v \in X$, then $F(T)$ is nonempty.*

Recently, Rhoades (2015) improved the result of Khojasteh for two multivalued mappings as follows.

**Theorem 1.12** *(Rhoads, 2015) Let $(X, d)$ be a complete metric space. If multivalued mappings $S, T: X \longrightarrow CB(X)$ satisfy $H(Su, Tv) \leqslant n_{S,T}(u, v) m_{S,T}(u, v)$ for all $u, v \in X$, then $F(T) \cap F(S)$ is nonempty, where*

$$\begin{aligned} n_{S,T}(u, v) &= \left(\frac{\max\{d(u, v), D(u, Su) + D(v, Tv), D(u, Tv) + D(v, Su)\}}{1 + \delta(u, Su) + \delta(v, Tv)}\right) \\ m_{S,T}(u, v) &= \max\left\{d(u, v), D(u, Su), D(v, Tv), \frac{D(u, Tv) + D(v, Su)}{2}\right\}. \end{aligned} \tag{1.2}$$

If $S = T$ in above theorem then we get the following result.

**Theorem 1.13,** *(Rhoads, 2015). Let $(X, d)$ be a complete metric space. If a multivalued mapping $T: X \longrightarrow CB(X)$ satisfiesfor all $u, v \in X, H(Tu, Tv) \leqslant n_{T,T}(u, v) m_{T,T}(u, v)$, then $F(T)$ is nonempty.*

Let $\alpha: X \times X \rightarrow \mathbb{R}^+$ and $U, V \in P(X)$. Define $\alpha_*(U, V) = \inf_{u \in U, v \in V} \alpha(u, v)$.

A multivalued mapping $T: X \rightarrow Cl(X)$ is called $\alpha_*-$admissible mapping if for any $u, v \in X, \alpha(u, v) \geqslant 1$ implies that $\alpha_*(Tu, Tv) \geqslant 1$. The concepts of $\alpha_*-$admissible mapping coincides with $\alpha-$admissible mapping in case of a single valued mapping.

Asl et al (2012), Theorem 1, defined $(\alpha_* - \psi) -$ contractive multifunctions and proved the following result.

**Theorem 1.14** Let $(X, d)$ be a complete metric space and $T: X \to Cl(X)$ an $\alpha_* -$ admissible mapping that satisfies $\alpha_*(Tu, Tv)H(Tu, Tv) \leqslant \psi(d(u, v))$ for all $u, v \in X$ and $\psi \in \Psi_2$. Moreover, if there exists a $u_0 \in X$ and $u_1 \in Tu_0$ such that $\alpha(u_0, u_1) \geqslant 1$, then $F(T)$ is nonempty provided that if $\{u_n\}$ is a sequence in $X$ such that $\alpha(u_n, u_{n+1}) \geqslant 1$ for all $n$ and $\lim_{n\to+\infty} u_n = u$, then $\alpha(u_n, u) \geqslant 1$ for all $n$.

In what follows, we assume that a b-metric $d$ is continuous in one variable.

**Definition 1.15,** compare with (Rus et al, 2003). Let $(X, d)$ be a b-metric space. A mapping $T: X \to Cl(X)$ is called a multivalued weakly Picard operator (MWP operator), if for all $u \in X$ and for some $v \in Tu$, there exists a sequence $\{u_n\}$ satisfying ($a_1$) $u_0 = u, u_1 = v$, ($a_2$) $u_{n+1} \in Tu_n$ for all $n \geqslant 0$, ($a_3$) $\{u_n\}$ converges to some $z \in F(T)$. The sequence $\{u_n\}$ satisfying ($a_1$) and ($a_2$) is called a sequence of successive approximations (ssa) of $T$ starting from $(u, v)$. If $T$ is a single-valued mapping, then we call it a Picard operator if it satisfies ($a_1$) to ($a_3$).

Recently Bota et al (2015) proved the fixed point theorem for $(\alpha_* - \psi) -$ contractive multivalued mappings as follows.

**Theorem 1.16** Let $(X, d)$ be a complete b-metric space, $\alpha: X \times X \to \mathbb{R}^+$ and $T: X \to Cl(X)$ an $\alpha_* -$ admissible multivalued operator that satisfies

$$\alpha_*(Tu, Tv)H(Tu, Tv) \leqslant \psi(d(u, v))$$

for all $u, v \in X$ and $\psi \in \Psi_4$. Assume that there exists a $u_0 \in X$ and $u_1 \in Tu_0$ such that $\alpha(u_0, u_1) \geqslant 1$. Then $T$ is a MWP operator provided that if there is a sequence $\{u_n\}$ in $X$ such that $u_n \to u$, then $\alpha(u_n, u) \geqslant 1$ for all $n \in \mathbb{N}_1$.

A mapping $T: X \to Cl(X)$ is called $(\alpha_* - f)$ –admissible mapping if $u, v \in X, \alpha(fu, fv) \geqslant 1$ implies that $\alpha_*(Tu, Tv) \geqslant 1$.

Let $(X, d)$ be a b-metric space, $g: X \to X$ and $T: X \to Cl(X)$. Set

$$N_T(u,v) = \frac{\max\{d(u,v), D(u,Tu) + D(v,Tv), D(u,Tv) + D(v,Tu)\}}{b(\delta(u,Tu) + \delta(v,Tv) + 1)},$$
$$M_T(u,v) = \max\left\{d(u,v), D(u,Tu), D(v,Tv), \frac{D(u,Tv) + D(v,Tu)}{2b}\right\},$$

and

$$N_{g,T}(u,v) = \frac{\max\{d(gu,gv), D(gu,Tu) + D(gv,Tv), D(gu,Tv) + D(gv,Tu)\}}{b(\delta(gu,Tu) + \delta(gv,Tv) + 1)}$$
$$M_{g,T}(u,v) = \max\left\{d(gu,gv), D(gu,Tu), D(gv,Tv), \frac{D(gu,Tv) + D(gv,Tu)}{2b}\right\}.$$

We now give the following definitions.

**Definition 1.17** *Let $(X,d)$ be a b-metric space. A mapping $T: X \to Cl(X)$ is called Suzuki type $(\alpha_* - \psi)$ – multivalued operator if there exists a $\psi \in \Psi_4$ such that*

$$\frac{1}{2}D(u,Tu) \leqslant bd(u,v) \tag{1.3}$$

implies

$$\alpha_*(Tu,Tv)H(Tu,Tv) \leqslant max\{1, N_T(u,v)\}\psi(M_T(u,v)) \tag{1.4}$$

for all $u, v \in X$.

If in the above definition we replace a mapping $T$ by a single valued mapping $f: X \to X$, then we call it a Suzuki type $(\alpha_* - \psi)$ – operator.

**Definition 1.18** *Let $(X,d)$ be a b-metric space. A hybrid pair $(g,T)$ is called a Suzuki type $(\alpha_* - \psi)$ – hybrid pair of operators if there exists a $\psi \in \Psi_4$ such that*

$$\frac{1}{2}D(gu,Tu) \leqslant bd(gu,gv) \tag{1.5}$$

implies

$$\alpha_*(Tu,Tv)H(Tu,Tv) \leqslant \max\{1, N_{g,T}(u,v)\}\psi\big(M_{g,T}(u,v)\big) \tag{1.6}$$

for all $u, v \in X$.

In case $T$ is replaced by a single valued mapping $f: X \to X$, we call it a Suzuki type $(\alpha_* - \psi)$ – pair of operators.

## Fixed point of Suzuki type $(\boldsymbol{\alpha_*} - \boldsymbol{\psi})$ – multivalued operators on b-metric spaces

In this section, we prove that Suzuki type $(\alpha_* - \psi)$ – multivalued operators are MWP operators.

**Theorem 2.1** *Let $(X, d)$ be a complete b-metric space, $\alpha: X \times X \to \mathbb{R}^+$ and $T: X \to Cl(X)$ a Suzuki type $(\alpha_* - \psi)$ –multivalued operator. Further, assume that $T$ is $\alpha_*$ –admissible mapping and there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geqslant 1$. If for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{Z}^+$, then*

**$d_1$-:** *$T$ is an MWP operator.*

**$d_2$-:** *If there is some $u \in F(T)$ such that $u \neq z$ and $\alpha(z, u) \geqslant 1$, then $d(z, u) > \frac{1}{2}$ provided that $1 \leqslant N_T(x, y)$ for all $x, y \in X$.*

*Proof.* ($d_1$) By the given assumption, there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geqslant 1$. If $x_0 = x_1$, then $x_0 \in Tx_0$. Define a sequence $\{x_n\}$ in $X$ by $x_n = x_1 = x_0$ for all $n \in \mathbb{Z}^+$. Thus $x_n \in Tx_n$ for all $n \geqslant 0$ and $\{x_n\}$ converges to $x = x_0 \in F(T)$ and hence $T$ is an MWP operator. Let $x_0 \neq x_1$. Since $T$ is $\alpha_* -$ admissible mapping, $\alpha_*(x_0, x_1) \geqslant 1$ implies that $\alpha_*(Tx_0, Tx_1) \geqslant 1$. As $T$ is a Suzuki type $(\alpha_* - \psi)$ – multivalued operator and

$$\frac{1}{2} D(x_0, Tx_0) \leqslant d(x_0, x_1) \leqslant bd(x_0, x_1), \tag{2.1}$$

so from (1.4), we obtain

$$\begin{aligned}
&< D(x_1, Tx_1) \leqslant H(Tx_0, Tx_1) < \alpha_*(Tx_0, Tx_1) H(Tx_0, Tx_1) \\
&\leqslant \max\{1, N_T(x_0, x_1)\} \psi(M_T(x_0, x_1)) \\
&= \max\left\{1, \left(\frac{\max\{d(x_0, x_1), D(x_0, Tx_0) + D(x_1, Tx_1), D(x_0, Tx_1) + D(x_1, Tx_0)\}}{b(1 + \delta(x_0, Tx_0) + \delta(x_1, Tx_1))}\right)\right\} \\
&\quad \psi\left(\max\left\{d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \frac{D(x_0, Tx_1) + D(x_1, Tx_0)}{2b}\right\}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\max\left\{1,\left(\frac{\max\{d(x_0,x_1),d(x_0,x_1)+D(x_1,Tx_1),bd(x_0,x_1)+bD(x_1,Tx_1)\}}{b(1+d(x_0,x_1)+D(x_1,Tx_1))}\right)\right\}}{\psi\left(\max\left\{d(x_0,x_1),D(x_1,Tx_1),\frac{d(x_0,x_1)+D(x_1,Tx_1)}{2}\right\}\right)} \\
&\leqslant \frac{\max\left\{1,\left(\frac{d(x_0,x_1)+D(x_1,Tx_1)}{1+d(x_0,x_1)+D(x_1,Tx_1)}\right)\right\}}{\psi\left(\max\left\{d(x_0,x_1),D(x_1,Tx_1),\frac{d(x_0,x_1)+D(x_1,Tx_1)}{2}\right\}\right)} \\
&\leqslant \psi(\max\{d(x_0,x_1),D(x_1,Tx_1)\}).
\end{aligned}
$$

That is

$$0 < D(x_1,Tx_1) \leqslant \psi(\max\{d(x_0,x_1),D(x_1,Tx_1)\}). \qquad (2.2)$$

If $\max\{d(x_0,x_1),D(x_1,Tx_1)\} = D(x_1,Tx_1)$, then (2.2) implies that

$$0 < D(x_1,Tx_1) \leqslant \psi(D(x_1,Tx_1)). \qquad (2.3)$$

As $D(x_1,Tx_1) > 0$ and $\psi \in \Psi_4$, (2.3) give

$$0 < D(x_1,Tx_1) \leqslant \psi(D(x_1,Tx_1)) < D(x_1,Tx_1),$$

a contradiction. Hence $\max\{d(x_0,x_1),D(x_1,Tx_1)\} = d(x_0,x_1)$. From (2.3), it follows that

$$0 < D(x_1,Tx_1) \leqslant \psi(d(x_0,x_1)). \qquad (2.4)$$

Let $q > 1$. We may choose $x_2 \in Tx_1$ such that

$$0 < D(x_1,Tx_1) \leqslant d(x_1,x_2) < qD(x_1,Tx_1) \leqslant q\psi(d(x_0,x_1)).$$

That is

$$0 < d(x_1,x_2) < qD(x_1,Tx_1) \leqslant q\psi(d(x_0,x_1)). \qquad (2.5)$$

Since $\alpha(x_1,x_2) \geqslant \alpha(Tx_0,Tx_1) \geqslant 1$, we get $\alpha(Tx_1,Tx_2) \geqslant 1$. Set $c_0 = d(x_0,x_1) > 0$, then from (2.5) we get $x_1 \neq x_2$ and $d(x_1,x_2) < q\psi(c_0)$. As $\psi \in \Psi_4$, so

$$\psi(d(x_1,x_2)) < \psi\big(q\psi(c_0)\big). \qquad (2.6)$$

If $q_1 = \frac{\psi(q\psi(c_0))}{\psi(d(x_1,x_2))}$, then by (2.6) $q_1 > 1$. Now, if $x_2 \in Tx_2$ then the proof is finished. Let $x_2 \notin Tx_2$. Note that

$$\frac{1}{2} D(x_1, Tx_1) \leqslant d(x_1, x_2) \leqslant bd(x_1, x_2).$$

By (1.4)

$$\begin{aligned}
&< D(x_2, Tx_2) \leqslant H(Tx_1, Tx_2) < \alpha_*(Tx_1, Tx_2) H(Tx_1, Tx_2) \\
&\leqslant \max\{1, N_T(x_1, x_2)\}\psi(M_T(x_1, x_2)) \\
&= \max\left\{1, \left(\frac{\max\{d(x_1,x_2), D(x_1,Tx_1) + D(x_2,Tx_2), D(x_1,Tx_2) + D(x_2,Tx_1)\}}{b\big(1 + \delta(x_1,Tx_1) + \delta(x_2,Tx_2)\big)}\right)\right\} \\
&\quad \psi\left(\max\left\{d(x_1,x_2), D(x_1,Tx_1), D(x_2,Tx_2), \frac{D(x_1,Tx_2) + D(x_2,Tx_1)}{2b}\right\}\right) \\
&\leqslant \max\left\{1, \left(\frac{\max\{d(x_1,x_2), d(x_1,x_2) + D(x_2,Tx_2), b(d(x_1,x_2) + D(x_2,Tx_2))\}}{b(1 + d(x_1,x_2) + D(x_2,Tx_2))}\right)\right\} \\
&\quad \psi\left(\max\left\{d(x_1,x_2), d(x_1,x_2), D(x_2,Tx_2), \frac{d(x_1,x_2) + D(x_2,Tx_2)}{2}\right\}\right) \\
&\leqslant \max\left\{1, \left(\frac{d(x_1,x_2) + D(x_2,Tx_2)}{1 + d(x_1,x_2) + D(x_2,Tx_2)}\right)\right\} \\
&\quad \psi\left(\max\left\{d(x_1,x_2), d(x_1,x_2), D(x_2,Tx_2), \frac{d(x_1,x_2) + D(x_2,Tx_2)}{2}\right\}\right) \\
&\leqslant \psi(\max\{d(x_1,x_2), D(x_2,Tx_2)\}).
\end{aligned}$$

That is

$$0 < D(x_2, Tx_2) \leqslant \psi(\max\{d(x_1,x_2), D(x_2,Tx_2)\}). \tag{2.7}$$

If $\max\{d(x_1,x_2), D(x_2,Tx_2)\} = D(x_2,Tx_2)$, then

$$0 < D(x_2, Tx_2) \leqslant \psi(D(x_2,Tx_2)). \tag{2.8}$$

Now $D(x_2,Tx_2) > 0$, $\psi \in \Psi_4$ and (2.8) give

$$0 < D(x_2,Tx_2) \leqslant \psi(D(x_2,Tx_2)) < D(x_2,Tx_2),$$

a contradiction. Hence

$$0 < D(x_2,Tx_2) \leqslant \psi(d(x_1,x_2)). \tag{2.9}$$

We may choose $x_3 \in Tx_2$ such that

$$0 < D(x_2, Tx_2) \leqslant d(x_2, x_3) < q_1 D(x_2, Tx_2) \leqslant q_1 \psi(d(x_1, x_2)) = \psi\big(q\psi(c_0)\big).$$

That is

$$0 < d(x_2, x_3) < q_1 D(x_2, Tx_2) \leqslant q_1 \psi(d(x_1, x_2)) = \psi\big(q\psi(c_0)\big). \qquad (2.10)$$

As $\alpha(x_2, x_3) \geqslant \alpha(Tx_1, Tx_2) \geqslant 1$, so $\alpha(Tx_2, Tx_3) \geqslant 1$. From (2.10), we get $x_2 \neq x_3$ and

$$\psi(d(x_2, x_3)) < \psi^2\big(q\psi(c_0)\big). \qquad (2.11)$$

Set $q_2 = \frac{\psi^2(q\psi(c_0))}{\psi(d(x_2,x_3))} > 1$. If $x_3 \in Tx_3$ then we are done. Suppose that $x_3 \notin Tx_3$. Similarly, we obtain $x_4 \in Tx_3$ such that

$$0 < d(x_3, x_4) < q_1 D(x_3, Tx_3) \leqslant q_2 \psi(d(x_2, x_3)) = \psi^2\big(q\psi(c_0)\big). \qquad (2.12)$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in $X$ such that $x_{n+1} \in Tx_n, x_{n+1} \neq x_n, \alpha(x_{n+1}, x_{n+2}) \geqslant 1$, and it satisfies:

$$0 < D(x_{n+1}, Tx_{n+1}) \leqslant \psi(d(x_n, x_{n+1})) \qquad (2.13)$$

and

$$0 < d(x_{n+1}, x_{n+2}) < \psi^n\big(q\psi(c_0)\big) \qquad (2.14)$$

for all $n \in \mathbb{Z}^+$. From (2.14), for $n, m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leqslant bd(x_n, x_{n+1}) + b^2 d(x_{n+1}, x_{n+2}) + \ldots \\ &\quad + b^{m-n} d(x_{m-2}, x_{m-1}) + b^{m-n} d(x_{m-1}, x_m) \\ &\leqslant b\psi^{n-1}\big(q\psi(c_0)\big) + b^2\psi^n\big(q\psi(c_0)\big) + \ldots \\ &\quad + b^{m-n}\psi^{m-3}\big(q\psi(c_0)\big) + b^{m-n}\psi^{m-2}\big(q\psi(c_0)\big) \\ &= \frac{1}{b^{n-2}}\Big(b^{n-1}\psi^{n-1}\big(q\psi(c_0)\big) + b^n\psi^n\big(q\psi(c_0)\big) + \ldots \\ &\quad + b^{m-2}\psi^{m-2}\big(q\psi(c_0)\big)\Big) \end{aligned}$$

$$= \frac{1}{b^{n-2}} \sum_{i=n-1}^{m-2} b^i \psi^i\big(q\psi(c_0)\big)$$

$$= \frac{1}{b^{n-2}} \left( \sum_{i=0}^{m-2} b^i \psi^i\big(q\psi(c_0)\big) - \sum_{i=0}^{n-2} b^i \psi^i\big(q\psi(c_0)\big) \right).$$

That is

$$d(x_n, x_m) \leqslant \frac{1}{b^{n-2}} \left( \sum_{i=0}^{m-2} b^i \psi^i\big(q\psi(c_0)\big) - \sum_{i=0}^{n-2} b^i \psi^i\big(q\psi(c_0)\big) \right). \qquad (2.15)$$

Set $S_n = \sum_{i=0}^{n} b^i \psi^i\big(q\psi(c_0)\big)$. Then from (2.15) we obtain that

$$d(x_n, x_m) \leqslant \frac{1}{b^{n-2}} (S_{m-2} - S_{n-2}). \qquad (2.16)$$

By Lemma 1.6, $\sum_{i=0}^{\infty} b^i \psi^i(t)$ converges for any $t \geqslant 0$. Hence $\lim_{n\to+\infty} S_{n-2} = S$ for some $S \in \mathbb{R}^+$. If $b = 1$, then from (2.16) we get $\lim_{n\to+\infty} d(x_n, x_m) \leqslant \lim_{n\to+\infty} S_{m-1} - \lim_{n\to+\infty} S_{n-1} = 0$. If $b > 1$, then from (2.16) we have

$$\lim_{n\to+\infty} d(x_n, x_m) \leqslant \lim_{n\to+\infty} \frac{1}{b^{n-1}} (S_{m-1} - S_{n-1}) \leqslant \lim_{n\to+\infty} \frac{S_{m-1}}{b^{n-1}} = 0$$

for all $m, n \in \mathbb{N}$. Hence $\{x_n\}$ is a Cauchy sequence in $X$. There exists $z \in X$ such that

$$\lim_{n\to+\infty} d(x_n, z) = 0. \qquad (2.17)$$

Now we show that $z \in F(T)$. If $D(z, Tz) > 0$, then we claim that one of the following two inequalities

$$\frac{1}{2} D(x_n, Tx_n) \leqslant b d(x_n, z) \qquad (2.18)$$

$$\frac{1}{2}D(x_{n+1},Tx_{n+1}) \leqslant bd(x_{n+1},z) \tag{2.19}$$

holds for all $n \in \mathbb{Z}^+$. Assume on the contrary that there exists an $n_0 \in \mathbb{Z}^+$ such that

$$\frac{1}{2}D(x_{n_0},Tx_{n_0}) > bd(x_{n_0},z) \tag{2.20}$$

and

$$\frac{1}{2}D(x_{n_0+1},Tx_{n_0+1}) > bd(x_{n_0+1},z). \tag{2.21}$$

Now from (2.13), (2.20) and (2.21), we have

$$\begin{aligned}
&d(x_{n_0},x_{n_0+1}) \leqslant bd(x_{n_0},z) + bd(z,x_{n_0+1})\\
&< \tfrac{1}{2}D(x_{n_0},Tx_{n_0}) + \tfrac{1}{2}D(x_{n_0+1},Tx_{n_0+1})\\
&\leqslant \tfrac{1}{2}d(x_{n_0},x_{n_0+1}) + \tfrac{1}{2}\psi\big(d(x_{n_0},x_{n_0+1})\big)\\
&< \tfrac{1}{2}d(x_{n_0},x_{n_0+1}) + \tfrac{1}{2}d(x_{n_0},x_{n_0+1})\\
&= d\big(x_{n_0},x_{n_0+1}\big)
\end{aligned}$$

a contradiction. Hence either (2.18) or (2.19) holds for an infinite subset $N_1$ of $\mathbb{Z}^+$. By the given assumption, it follows that $\alpha(x_n,z) \geqslant 1$. As $T$ is $\alpha_*$ −admissible, $\alpha(Tx_n,Tz) \geqslant 1$. Now if (2.18) holds for all $n \in \mathbb{Z}^+$, then from (1.4) we get

$$\begin{aligned}
D(x_{n+1},Tz) &\leqslant H(Tx_n,Tz) < \alpha_*(Tx_n,Tz)H(Tx_n,Tz)\\
&\leqslant \max\{1, N_T(x_n,z)\}\psi(M_T(x_n,z))\\
&= \max\left\{1,\left(\frac{\max\{d(x_n,z), D(x_n,Tx_n)+D(z,Tz), D(x_n,Tz)+D(z,Tx_n)\}}{b(1+\delta(x_n,Tx_n)+\delta(z,Tz))}\right)\right\}\\
&\quad \psi\left(\max\left\{d(x_n,z), D(x_n,Tx_n), D(z,Tz), \frac{D(x_n,Tz)+D(z,Tx_n)}{2b}\right\}\right)\\
&\leqslant \max\left\{1,\left(\frac{\max\{d(x_n,z), d(x_n,x_{n+1})+D(z,Tz), D(x_n,Tz)+d(z,x_{n+1})\}}{b(1+d(x_n,x_{n+1})+D(z,Tz))}\right)\right\}\\
&\quad \psi\left(\max\left\{d(x_n,z), d(x_n,x_{n+1}), D(z,Tz), \frac{D(x_n,Tz)+d(z,x_{n+1})}{2b}\right\}\right).
\end{aligned}$$

On taking limit as $n \to +\infty$, we have

$$\lim_{n\to+\infty} D(x_{n+1}, Tz) \leqslant \lim_{n\to+\infty} \begin{matrix} \max\left\{1, \left(\frac{\max\{d(x_n,z), d(x_n,x_{n+1})+D(z,Tz), D(x_n,Tz)+d(z,x_{n+1})\}}{b(1+d(x_n,x_{n+1})+D(z,Tz))}\right)\right\} \\ \psi\left(\max\left\{d(x_n,z), d(x_n,x_{n+1}), D(z,Tz), \frac{D(x_n,Tz)+d(z,x_{n+1})}{2b}\right\}\right). \end{matrix}$$

As $\psi$ is continuous at $0, \psi \in \Psi_4$ and $D(z, Tz) > 0$, we have

$$D(z,Tz) \leqslant \max\left\{1, \frac{D(z,Tz)}{b(1+D(z,Tz))}\right\} \psi(D(z,Tz)) = \psi(D(z,Tz)) < D(z,Tz).$$

a contradiction. Consequently, $z \in Tz$. Similarly, we obtain $z \in Tz$ when (2.19) holds for an infinite subset $N_1$ of $\mathbb{Z}^+$.

To prove part (d $_2$), let $u \in F(T)$ such that $u \neq z$ and $\alpha(z,u) \geqslant 1$. Since $T$ is $\alpha_*$ –admissible, $\alpha_*(Tz,Tu) \geqslant 1$. Now $\frac{1}{2}D(z,Tz) = 0 \leqslant d(z,u)$ implies that

$$\begin{aligned} d(z,u) &\leqslant bD(z,Tz) + bD(Tz,u) \leqslant bH(Tz,Tu) \\ &\leqslant b\alpha_*(Tz,Tu)H(H(Tz,Tu)) \end{aligned}$$

$$\leqslant \begin{matrix} b\max\left\{1, \left(\frac{\max\{d(z,u), D(z,Tz)+D(u,Tu), D(z,Tu)+D(u,Tz)\}}{b(1+\delta(z,Tz)+\delta(u,Tu))}\right)\right\} \\ \psi\left(\max\left\{d(z,u), D(z,Tz), D(u,Tu), \frac{D(z,Tu)+D(u,Tz)}{2b}\right\}\right) \end{matrix}$$

$$\leqslant \begin{matrix} b\max\left\{1, \left(\frac{\max\{d(z,u), d(z,z)+d(u,u), d(z,u)+d(u,z)\}}{b(1+d(z,z)+d(u,u))}\right)\right\} \\ \psi\left(\max\left\{d(z,u), d(z,z), d(u,u), \frac{d(z,u)+d(u,z)}{2b}\right\}\right) \end{matrix}$$

$$\leqslant b\max\left\{1, \left(\frac{2d(z,u)}{b}\right)\right\} \psi(d(z,u)).$$

Now, $\max\{1, N_T(x,y)\} = N_T(x,y)$ gives

$$d(z,u) \leqslant 2d(z,u)\psi(d(z,u)) < 2d^2(z,u)$$

and hence $d(z,u) > \frac{1}{2}$.

**Corollary 2.2** *Let* $(X,d)$ *be a complete b-metric space,* $\alpha: X \times X \to \mathbb{R}^+$ *and* $T: X \to Cl(X)$ *an* $\alpha_* -$ *admissible mapping such that*

$$\frac{1}{2}D(x,Tx) \leqslant bd(x,y)$$

*implies that*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leqslant max\{1, N_T(x,y)\}\psi(d(x,y))$$

*for all* $x,y \in X, \psi \in \Psi_4$. *Further, assume that there exists* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0,x_1) \geqslant 1$. *If for any sequence* $\{x_n\}$ *converging to* $x$ *in* $X$, *we have* $\alpha(x_n,x) \geqslant 1$ *for all* $n \in \mathbb{Z}^+$ *then*

**e$_1$-:** $T$ *is an MWP operator*

**e$_2$-:** *If there is some* $u \in F(T)$ *such that* $u \neq z$ *and* $\alpha(z,u) \geqslant 1$, *then* $d(z,u) > \frac{1}{2}$ *provided that*

$$max\{1, N_T(x,y)\} = N_T(x,y) \text{ for all } x,y \in X.$$

**Corollary 2.3** *Let* $(X,d)$ *be a complete b-metric space,* $\alpha: X \times X \to \mathbb{R}^+$ *and* $T: X \to Cl(X)$ *an* $\alpha_* -$ *admissible mapping such that*

$$\frac{1}{2}D(x,Tx) \leqslant bd(x,y)$$

*implies that*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leqslant max\left\{1, \frac{d(x,y)}{b(1+\delta(x,Tx)+\delta(y,Ty))}\right\}\psi(d(x,y))$$

for all $x,y \in X, \psi \in \Psi_4$. Further, assume that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geqslant 1$. If for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n,x) \geqslant 1$ for all $n \in \mathbb{Z}^+$ then

**e$_3$-:** $T$ is an MWP operator.

**e$_4$-:** If there is some $u \in F(T)$ such that $u \neq z$ and $\alpha(z,u) \geqslant 1$, then $d(z,u) > 1$ provided that

$$max\left\{1,\frac{d(x,y)}{b(1+\delta(x,Tx)+\delta(y,Ty))}\right\}=\frac{d(x,y)}{b(1+\delta(x,Tx)+\delta(y,Ty))}$$

for all $x,y\in X$.

Proof. Follows from Corollary 2.3.

**Corollary 2.4** Let $(X,d)$ be a complete b-metric space, $\alpha: X\times X\to\mathbb{R}^+$ and $T: X\to Cl(X)$ an $\alpha_*$ − admissible mapping such that

$$\frac{1}{2}D(x,Tx)\leqslant bd(x,y)$$

implies that

$$\alpha_*(Tx,Ty)H(Tx,Ty)\leqslant N_T(x,y)\psi(M_T(x,y))$$

for all $x,y\in X,\psi\in\Psi_4$. Further, assume that there exists $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geqslant 1$. If for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n,x)\geqslant 1$ for all $n\in\mathbb{Z}^+$ then

**$e_5$-:** $T$ is an MWP operator.

**$e_6$-:** If there is some $u\in F(T)$ such that $u\neq z$ and $\alpha(z,u)\geqslant 1$, then $d(z,u)>\frac{1}{2}$.

Proof. Take $\max\{1,N_T(x,y)\}=N_T(x,y)$ in Theorem 2.1.

**Corollary 2.5** Let $(X,d)$ be a complete b-metric space, $\alpha: X\times X\to\mathbb{R}^+$ and $T: X\to Cl(X)$ an $\alpha_*$ −admissible mapping such that

$$\frac{1}{2}D(x,Tx)\leqslant bd(x,y)\text{impliesthat}\alpha_*(Tx,Ty)H(Tx,Ty)\leqslant N_T(x,y)\psi(d(x,y)) \tag{2.22}$$

for all $x,y\in X,\psi\in\Psi_4$. Further, assume that there exists $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geqslant 1$. If for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n,x)\geqslant 1$ for all $n\in\mathbb{Z}^+$ then

**$e_7$-:** $T$ is an MWP operator.

**$e_8$-:** If there is some $u \in F(T)$ such that $u \neq z$ and $\alpha(z,u) \geqslant 1$, then $d(z,u) > \frac{1}{2}$.

Proof. Take $M_T(x,y) = d(x,y)$ in Corollary 2.4.

The following Corollary is a Suzuki type generalization of (Asl et al, 2012), Theorem 2.1 (Bota et al, 2015), Theorem 1 (Mohammadi, 2013), Theorem 3.1 (Samet et al, 2012), Theorem 2.2 and references therein in the context of b-metric spaces.

**Corollary 2.6** Let $(X,d)$ be a complete b-metric space, $\alpha: X \times X \to \mathbb{R}^+$ and $T: X \to Cl(X)$ an $\alpha_*$ – admissible mapping such that

$$\begin{gathered} \frac{1}{2} D(x,Tx) \leqslant bd(x,y) \text{ implies that} \\ \alpha_*(Tx,Ty)H(Tx,Ty) \leqslant \psi(M_T(x,y)) \end{gathered} \tag{2.23}$$

for all $x,y \in X, \psi \in \Psi_4$. Further, assume that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geqslant 1$. Then $T$ is an MWP operator provided that for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n,x) \geqslant 1$ for all $n \in \mathbb{Z}^+$.

Proof. Take $\max\{1, N_T(x,y)\} = 1$ in Theorem 2.1.

**Corollary 2.7** Let $(X,d)$ be a complete b-metric space, $\alpha: X \times X \to \mathbb{R}^+$ and $T: X \to Cl(X)$ an $\alpha_*$ – admissible mapping such that

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leqslant \psi(M_T(x,y)) \tag{2.24}$$

for all $x,y \in X, \psi \in \Psi_4$. Further, assume that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geqslant 1$. Then $T$ is an MWP operator provided that for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n,x) \geqslant 1$ for all $n \in \mathbb{Z}^+$.

**Corollary 2.8** Let $(X,d)$ be a complete b-metric space, $\alpha: X \times X \to \mathbb{R}^+$ and $T: X \to Cl(X)$ an $\alpha_*$ – admissible mapping such that

$\frac{1}{2} D(x,Tx) \leqslant bd(x,y)$

implies that

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leqslant \psi(d(x,y)) \tag{2.25}$$

for all $x, y \in X, \psi \in \Psi_4$. Further, assume that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geqslant 1$. Then $T$ is an MWP operator provided that for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{Z}^+$.

Proof. Take $M_T(x, y) = d(x, y)$ in Corollary 2.6.

Now we state Theorem 2.1 in the context of single valued mapping $f: X \to X$, where $X$ is a b-metric space. The existence of a fixed point follows immediately from Theorem 2.1. To prove the uniqueness of the fixed point, we need the condition H, given as follows:

(H): for all $x, y \in X$, there exists a $z \in X$ such that $\alpha(x, z) \geqslant 1$ and $\alpha(y, z) \geqslant 1$.

**Corollary 2.9** Let $(X, d)$ be a complete b-metric space and $\alpha: X \times X \to \mathbb{R}^+$ and $f: X \to X$ an $\alpha-$ admissible mapping such that

$$\frac{1}{2} d(x, fx) \leqslant bd(x, y)$$

implies that

$$\alpha(fx, fy) d(fx, fy) \leqslant max\{1, N_f(x, y)\} \psi(M_f(x, y)) \qquad (2.26)$$

for all $x, y \in X, \psi \in \Psi_4$. Further, assume that there exists $x_0 \in X$ and $x_1 = fx_0$ such that $\alpha(x_0, x_1) \geqslant 1$ and for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{Z}^+$. If the condition (H) is satisfied, then $f$ is a Picard operator and for an arbitrary $z \in X$, the sequence $\{f^n z\}$ converges to some $w \in F(f)$ and

**$e_9$-:** $F(f) = \{w\}$ if $max\{1, N_f(x, y)\} = 1$,

**$e_{10}$-:** $d(u, w) > \frac{b}{2}$ for any $u \in F(f)$ such that $u \neq w$ provided that $max\{1, N_f(x, y)\} = N_f(x, y)$.

Proof. Theorem 2.1, $f$ is a Picard operator and $F(f)$ is nonempty. Let $u, v \in F(f)$ such that $u \neq v$. By the condition (H), there exists a $z \in X$ such that $\alpha(u, z) \geqslant 1$ and $\alpha(v, z) \geqslant 1$. Note that $\{f^n z\}$ is a Picard sequence which converges to some $w \in F(f)$. As $f$ is an $\alpha-$admissible mapping, so for all $n \geqslant 1$ we have $\alpha(u, f^n z) \geqslant 1$ and $\alpha(v, f^n z) \geqslant 1$. Since

$$\frac{1}{2}d(u,fu)=0\leqslant bd(u,f^{n-1}z),$$

by (2.26) we have

$$\begin{aligned}
&d(u,f^nz)=d(fu,f^nz)\\
&\leqslant \alpha(fu,ff^{n-1}z)d(fu,ff^{n-1}z)\\
&\leqslant \max\{1,N_f(u,f^{n-1}z)\}\psi(M_f(u,ff^{n-1}z))\\
&\leqslant \max\left\{1,\left(\frac{\max\{d(u,f^{n-1}z),d(u,fu)+d(f^{n-1}z,ff^{n-1}z),d(u,ff^{n-1}z)+d(f^{n-1}z,fu)\}}{b(1+d(u,fu)+d(f^{n-1}z,ff^{n-1}z))}\right)\right\}\\
&\psi\left(\max\left\{d(u,f^{n-1}z),d(u,fu),d(f^{n-1}z,ff^{n-1}z),\frac{d(u,ff^{n-1}z)+d(f^{n-1}z,fu)}{2b}\right\}\right)\\
&\leqslant \max\left\{1,\left(\frac{\max\{d(u,f^{n-1}z),d(u,u)+d(f^{n-1}z,f^nz),d(u,f^nz)+d(f^{n-1}z,u)\}}{b(1+d(u,u)+d(f^{n-1}z,f^nz))}\right)\right\}\\
&\psi\left(\max\left\{d(u,f^{n-1}z),d(u,u),d(f^{n-1}z,f^nz),\frac{d(u,f^nz)+d(f^{n-1}z,u)}{2b}\right\}\right).
\end{aligned}$$

On taking limit as $n\to+\infty$, we obtain that

$$\begin{aligned}
d(u,w)\leqslant\ &\max\left\{1,\left(\frac{\max\{d(u,w),d(u,u)+d(w,w),d(u,w)+d(w,u)\}}{b(1+d(u,u)+d(w,w))}\right)\right\}\\
&\psi\left(\max\left\{d(u,w),d(u,u),d(w,w),\frac{d(u,w)+d(w,u)}{2b}\right\}\right)\\
&=\max\left\{1,\left(\frac{2d(u,w)}{b}\right)\right\}\psi(d(u,w)).
\end{aligned}$$

That is

$$d(u,w)\leqslant \max\left\{1,\left(\frac{2d(u,w)}{b}\right)\right\}\psi(d(u,w)). \tag{2.27}$$

Now, If $\max\{1,N_f(x,y)\}=1$, and $w\neq u$, then we have

$$d(u,w)\leqslant\psi(d(u,w))<d(u,w). \tag{2.28}$$

Also, if $\max\{1,N_f(x,y)\}=1$, and $v\neq w$, then we have

$$d(v,w)\leqslant\psi(d(v,w)). \tag{2.29}$$

A contradiction in both cases. Thus $w = u = v$, and hence $F(f)$ is singleton. If $\max\{1, N_f(x,y)\} = N_f(x,y)$ and $u \neq w$ then from (2.27) we get

$$d(u,w) \leqslant \frac{2d(u,w)}{b}\psi(d(u,w)) < \frac{2}{b}d^2(u,w)$$

and $d(u,w) > \frac{b}{2}$.

**Corollary 2.10** Let $(X,d)$ be a complete b-metric space and $\alpha: X \times X \to \mathbb{R}^+$ and $f: X \to X$ an $\alpha-$admissible mapping such that

$$\frac{1}{2}d(x,fx) \leqslant bd(x,y) \text{ implies that } \alpha(fx,fy)d(fx,fy) \leqslant \psi(M_f(x,y))$$

for all $x,y \in X, \psi \in \Psi_4$. Further, assume that there exists $x_0 \in X$ and $x_1 = fx_0$ such that $\alpha(x_0,x_1) \geqslant 1$ and for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n,x) \geqslant 1$ for all $n \in \mathbb{Z}^+$. If the condition (H) is satisfied, then $f$ is a Picard operator and for an arbitrary $z \in X$, the sequence $\{f^n z\}$ converges to $w \in F(f)$ and $F(f) = \{w\}$.

**Corollary 2.11** Let $(X,d)$ be a complete b-metric space and $\alpha: X \times X \to \mathbb{R}^+$. Let $f: X \to X$ be an $\alpha-$admissible mapping that satisfies

$$\frac{1}{2}d(x,fx) \leqslant bd(x,y)$$

implies that

$$\alpha(fx,fy)d(fx,fy) \leqslant \max\{1, N_f(x,y)\}\psi(d(x,y))$$

for all $x,y \in X, \psi \in \Psi_4$. Moreover, suppose that there exists $x_0 \in X$ and $x_1 = fx_0$ such that $\alpha(x_0,x_1) \geqslant 1$ and if there is a sequence $\{x_n\}$ in $X$ such that $x_n \to x$, then $\alpha(x_n,x) \geqslant 1$ for all $n \in \mathbb{Z}^+$. Further, assume that there exists $x_0 \in X$ and $x_1 = fx_0$ such that $\alpha(x_0,x_1) \geqslant 1$ and for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n,x) \geqslant 1$ for all $n \in \mathbb{Z}^+$. If the condition (H) is satisfied, then $f$ is a Picard operator and for an arbitrary $z \in X$, the sequence $\{f^n z\}$ converges to $w \in F(f)$ and $F(f) = \{w\}$. Also, $d(u,w) > \frac{b}{2}$ for any $u \in F(f)$ such that $u \neq w$ provided that $\max\{1, N_f(x,y)\} = N_f(x,y)$.

**Example 2.12** Let $X=\{x_1,x_2,x_3,x_4,x_5\}$ and $d: X\times X\to\mathbb{R}^+$ be defined as

$$d(x_2,x_5)=d(x_3,x_4)=d(x_3,x_5)=d(x_2,x_4)=6,$$
$$d(x_2,x_3)=9, d(x_1,x_4)=d(x_1,x_5)=10,$$
$$d(x_1,x_2)=d(x_1,x_3)=4, d(x_4,x_5)=1,$$
$$d(x,x)=0 \text{ and } d(x,y)=d(y,x) \text{ for all } x,y\in X.$$

As $9=d(x_2,x_3)\nleq d(x_2,x_1)+d(x_1,x_3)=8$, so $d$ is not a metric on $X$. Indeed,$(X,d)$ is a b-metric space with $b=\frac{9}{8}>1$. Consider a mapping $T: X\longrightarrow Cl(X)$ defined by $Tx_1=Tx_2=Tx_3=\{x_1\}, Tx_4=\{x_2\}$ and $Tx_5=\{x_3\}$. If we take $\psi(t)=\frac{9}{10}t$ for $t\in\mathbb{R}^+$, then $\psi\in\Psi_4$ (see 1.8). If mapping $\alpha: X\times X\to\mathbb{R}^+$ is defined as $\alpha(x_i,x_j)=1$ for all $i,j\in\{1,2,3,4,5\}$, then $T$ is an $\alpha_*$ −admissible mapping. For $x,y\in\{x_1,x_2,x_3\}$, we have $H(Tx,Ty)=0\leqslant\max\{1,N_T(x,y)\}\psi(M_T(x,y))$. For $(x,y)$ when $x\in\{x_1,x_2,x_3\}$ and $y\in\{x_4,x_5\}$, we obtain that

$$\alpha(Tx_1,Tx_4)H(Tx_1,Tx_4)=d(x_1,x_2)=4\leqslant 9=\psi(d(x_1,x_4))$$
$$\leqslant\max\{1,N_T(x_1,x_4)\psi(M_T(x_1,x_4)),$$
$$\alpha(Tx_2,Tx_4)H(Tx_2,Tx_4)=d(x_1,x_2)=4\leqslant\frac{54}{10}=\psi(d(x_2,x_4))$$
$$\leqslant\max\{1,N_T(x_2,x_4)\psi(M_T(x_2,x_4)),$$
$$\alpha(Tx_3,Tx_4)H(Tx_3,Tx_4)=d(x_1,x_2)=4\leqslant\frac{54}{10}=\psi(d(x_3,x_4))$$
$$\leqslant\max\{1,N_T(x_3,x_4)\psi(M_T(x_3,x_4)),$$
$$\alpha(Tx_1,Tx_5)H(Tx_1,Tx_5)=d(x_1,x_3)=4\leqslant 9=\psi(d(x_1,x_5))$$
$$\leqslant\max\{1,N_T(x_1,x_5)\psi(M_T(x_1,x_5)),$$
$$\alpha(Tx_2,Tx_5)H(Tx_2,Tx_5)=d(x_1,x_3)=4\leqslant\frac{54}{10}=\psi(d(x_2,x_5))$$
$$\leqslant\max\{1,N_T(x_2,x_5)\psi(M_T(x_2,x_5)),$$
$$\alpha(Tx_3,Tx_5)H(Tx_3,Tx_5)=d(x_1,x_3)=4\leqslant\frac{54}{10}=\psi(d(x_3,x_5))$$
$$\leqslant\max\{1,N_T(x_3,x_5)\psi(M_T(x_3,x_5)).$$

Note that

$$\frac{1}{2}D(x_4,Tx_4)=\frac{1}{2}d(x_4,x_2)=3>\frac{9}{8}=bd(x_4,x_5),$$

and

$$\frac{1}{2}D(x_5,Tx_5)=\frac{1}{2}d(x_5,x_3)=3>\frac{5}{4}=bd(x_5,x_4).$$

Hence

$$\frac{1}{2}(D(x,Tx) \leqslant bd(x,y)) \leqslant 0$$

implies

$$\alpha(Tx,Ty)H(Tx,Ty) \leqslant \max\{1,N_T(x,y)\}\psi(M_T(x,y))$$

holds for all $x,y \in X$. Thus all the conditions of Theorem 2.1 are satisfied. On the other hand, if we take $x = x_4, y = x_5$, then $\alpha(Tx_4,Tx_5)H(Tx_4,Tx_5) = d(x_2,x_3) = 9 > \psi(d(x_4,x_5)) = \psi(1) = \frac{9}{10}$ and $\alpha(Tx_4,Tx_5)H(Tx_4,Tx_5) \nleq \psi(d(x_4,x_5))$. Consequently, Theorem 1.16 in (Bota et al, 2015) does not hold in this case.

The following example illustrates an assumption $\max\{1,N_T(x,y)\} > 1$.

**Example 2.13** Let $X = \{x_1,x_2,x_3\}$ and $d: X \times X \to \mathbb{R}^+$ be defined as

$$d(x_1,x_2) = 4, d(x_1,x_3) = 1, d(x_2,x_3) = 2,$$
$$d(x,x) = 0 \text{ and } d(x,y) = d(y,x) \text{ for all } x,y \in X.$$

As $4 = d(x_1,x_2) \nleq d(x_1,x_3) + d(x_3,x_2) = 3$, so $d$ is not a metric on $X$. Indeed, $(X,d)$ is a b-metric space with $b = \frac{4}{3} > 1$. Consider a mapping $T: X \longrightarrow Cl(X)$ defined by

$$Tx = \begin{cases} \{x_2\} \text{ if } x = x_1, \\ \{x_1\} \text{ if } x = x_2, \\ \{x_2\} \text{ if } x = x_3. \end{cases}$$

If we take $\psi(t) = \frac{8}{9}t$ for $t \in \mathbb{R}^+$, then $\psi \in \Psi_4$ (see Example 1.8). If $\alpha: X \times X \to \mathbb{R}^+$ is defined as $\alpha(x_i,x_j) = 1$ for all $i,j \in \{1,2,3\}$, then $T$ is $\alpha_* -$ admissible. Note that

$$\begin{aligned} N_T(x_1,x_2) &= \frac{3\max\{d(x_1,x_2), D(x_1,Tx_1) + D(x_2,Tx_2), D(x_1,Tx_2) + D(x_2,Tx_1)\}}{4(1 + \delta(x_1,Tx_2) + \delta(x_2,Tx_1))} \\ &= \frac{3\max\{d(x_1,x_2), d(x_1,x_2) + d(x_2,x_1), d(x_1,x_1) + d(x_2,x_2)\}}{4(1 + d(x_1,x_1) + d(x_2,x_2))} \\ &= \frac{3\max\{4,8,0\}}{4(1+0)} = 6 > 1. \end{aligned}$$

Hence $\max\{1,N_T(x,y)\} = 6 > 1$. Note that

$$N_T(x_1,x_3) = \frac{3\max\{d(x_1,x_3), D(x_1,Tx_1)+D(x_3,Tx_3), D(x_1,Tx_3)+D(x_3,Tx_1)\}}{4(1+\delta(x_1,Tx_3)+\delta(x_3,Tx_1))}$$
$$= \frac{3\max\{d(x_1,x_3), d(x_1,x_2)+d(x_3,x_3), d(x_1,x_3)+d(x_3,x_2)\}}{4(1+d(x_1,x_3)+d(x_3,x_2))}$$
$$= \frac{3\max\{1,4,3\}}{4(1+1+2)} = \frac{3}{4},$$

and

$$N_T(x_2,x_3) = \frac{3\max\{d(x_2,x_3), D(x_2,Tx_2)+D(x_3,Tx_3), D(x_2,Tx_3)+D(x_3,Tx_2)\}}{4(1+\delta(x_2,Tx_3)+\delta(x_3,Tx_2))}$$
$$= \frac{3\max\{d(x_2,x_3), d(x_2,x_1)+d(x_3,x_3), d(x_2,x_3)+d(x_3,x_1)\}}{4(1+d(x_2,x_3)+d(x_3,x_1))}$$
$$= \frac{3\max\{2,4,3\}}{4(1+2+1)} = \frac{3}{4}.$$

Also,

$$\alpha(Tx_1,Tx_2)H(Tx_1,Tx_2) = d(x_2,x_1) = 4 < \frac{64}{3} = N_T(x_1,x_2)\psi(d(x_1,x_2))$$
$$\leqslant \max\{1, N_T(x_1,x_2)\psi(M_T(x_1,x_2)),$$
$$\alpha(Tx_1,Tx_3)H(Tx_1,Tx_3) = d(x_2,x_3) = 2 < \frac{8}{3} = N_T(x_1,x_3)\psi(d(x_1,Tx_1))$$
$$\leqslant \max\{1, N_T(x_1,x_3)\psi(M_T(x_1,x_3)),$$
$$\alpha(Tx_2,Tx_3)H(Tx_2,Tx_3) = d(x_1,x_3) = 1 < \frac{4}{3} = N_T(x_2,x_3)\psi(d(x_2,x_3))$$
$$\leqslant \max\{1, N_T(x_2,x_3)\psi(M_T(x_2,x_3)).$$

Thus all the conditions of Theorem 2.1 are satisfied. On the other hand, if we take $x = x_1, y = x_2$, then $\alpha(Tx_1,Tx_2)H(Tx_1,Tx_2) = d(x_2,x_3) = 4 > \psi(d(x_1,x_2)) = \psi(4) = \frac{32}{9}$. Hence $\alpha(Tx_1,Tx_2)H(Tx_1,Tx_2) \nleq \psi(d(x_1,x_2))$. Consequently Theorem 1.16 in (Bota et al, 2015) is not applicable in this case which is a generalization of Theorems 1.14 and 1.9.

For $b = 1$, Theorem 2.1 reduces to the following important Corollary.

**Corollary 2.14** Let $(X,d)$ be a complete metric space, $\alpha: X \times X \to \mathbb{R}^+$ and $T: X \to Cl(X)$ satisfies the following implication

$$\frac{1}{2}D(x,Tx) \leqslant d(x,y) \text{implies} \max\{1, n_{T,T}(x,y)\}\psi(m_{T,T}(x,y))$$

for all $x, y \in X$ and $\psi \in \Psi_4$ where $n_{T,T}(x,y)$ and $m_{T,T}(x,y)$ are the same as given in (1.2), Further, assume that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geqslant 1$ and for any sequence $\{x_n\}$ converging to $x$ in $X$, we have $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{Z}^+$. Then

**e$_{11}$-:** $T$ is an MWP operator.

**e$_{12}$-:** If there is some $u \in F(T)$ such that $u \neq z$ and $\alpha(z,u) \geqslant 1$, then $d(z,u) > \frac{1}{2}$ provided that

$$\max\{1, n_{T,T}(x,y)\} = n_{T,T}(x,y).$$

Next we present an example which shows that Corollary 2.14 is a potential generalization of Theorems 1.14, 1.9, 1.11, 1.13.

**Example 2.15** Let$X = \{x_1, x_2, x_3, x_4, x_5\}$and$d: X \times X \to \mathbb{R}^+$bedefined by

$$\begin{aligned} &d(x_2,x_5) = d(x_3,x_4) = d(x_3,x_5) = d(x_2,x_4) = 5,\\ &d(x_1,x_4) = d(x_1,x_5) = 9, d(x_1,x_2) = d(x_1,x_3) = 4,\\ &d(x_4,x_5) = 2, d(x_2,x_3) = 8,\\ &d(x,x) = 0 \text{ and } d(x,y) = d(y,x) \text{ for all } x,y \in X. \end{aligned}$$

Note that $d$ is a metric on $X$. Consider a mapping $T: X \longrightarrow Cl(X)$ defined by $Tx_1 = Tx_2 = Tx_3 = \{x_1\}, Tx_4 = \{x_2\}$ and $Tx_5 = \{x_3\}$. If we take $\psi(t) = \frac{8}{9}t$ for $t \in \mathbb{R}^+$, then $\psi \in \Psi_i$ for each $i = 1,2,3,4$ (see Example 1.8). If $\alpha: X \times X \to \mathbb{R}^+$ is defined as $\alpha(x_i, x_j) = 1$ for all $i,j \in \{1,2,3,4,5\}$, then $T$ is $\alpha_*-$admissible mapping. For $x, y \in \{x_1, x_2, x_3\}$, we have $H(Tx,Ty) = 0 \leqslant \max\{1, N_T(x,y)\}\psi(M_T(x,y))$. For $(x,y)$, when $x \in \{x_1,x_2,x_3\}$ and $y \in \{x_4, x_5\}$, we obtain that

$$\begin{aligned} \alpha(Tx_1,Tx_4)H(\alpha(Tx_1,Tx_4)) = d(x_1,x_2) = 4 \leqslant 8 = \psi(d(x_1,x_4))\\ \leqslant \max\{1, n_{T,T}(x_1,x_4)\psi(m_{T,T}(x_1,x_4)),\\ \alpha(Tx_2,Tx_4)H(Tx_2,Tx_4) = d(x_1,x_2) = 4 \leqslant \frac{40}{9} = \psi(d(x_2,x_4))\\ \leqslant \max\{1, n_{T,T}(x_2,x_4)\psi(m_{T,T}(x_2,x_4)),\\ \alpha(Tx_3,Tx_4)H(Tx_3,Tx_4) = d(x_1,x_2) = 4 \leqslant \frac{40}{9} = \psi(d(x_3,x_4)) \end{aligned}$$

$$
\begin{aligned}
&\leqslant \max\{1, n_{T,T}(x_3,x_4)\psi(m_{T,T}(x_3,x_4)),\\
\alpha(Tx_1,Tx_5)H(Tx_1,Tx_5) &= d(x_1,x_3) = 4 \leqslant 8 = \psi(d(x_1,x_5))\\
&\leqslant \max\{1, n_{T,T}(x_1,x_5)\psi(m_{T,T}(x_1,x_5)),\\
\alpha(Tx_2,Tx_5)H(Tx_2,Tx_5) &= d(x_1,x_3) = 4 \leqslant \frac{40}{9} = \psi(d(x_2,x_5))\\
&\leqslant \max\{1, n_{T,T}(x_2,x_5)\psi(m_{T,T}(x_2,x_5)),\\
\alpha(Tx_3,Tx_5)H(Tx_3,Tx_5) &= d(x_1,x_3) = 4 \leqslant \frac{40}{9} = \psi(d(x_3,x_5))\\
&\leqslant \max\{1, n_{T,T}(x_3,x_5)\psi(m_{T,T}(x_3,x_5)).
\end{aligned}
$$

Note that $\frac{1}{2}D(x_4,Tx_4) = \frac{1}{2}d(x_4,x_2) = \frac{5}{2} > 1 = d(x_4,x_5)$, and $\frac{1}{2}D(x_5,Tx_5) = \frac{1}{2}d(x_5,x_3) = \frac{5}{2} > 1 = d(x_5,x_4)$.

Hence

$$\frac{1}{2}(D(x,Tx) \leqslant d(x,y))$$

implies

$$\alpha(Tx,Ty)H(Tx,Ty) \leqslant \max\{1, N_T(x,y)\psi(M_{TT}(x,y))$$

holds for all $x, y \in X$. Thus all the conditions of Corollary 2.14 are satisfied. On the other hand, if we take $x = x_4, y = x_5$,
then

$$\alpha(Tx_4,Tx_5)H(Tx_4,Tx_5) = d(x_2,x_3) = 8 > \psi(d(x_4,x_5)) = \psi(1) = \frac{8}{9}.$$

Hence,

$$\alpha(Tx_4,Tx_5)H(Tx_4,Tx_5) = 8 \nleq \frac{8}{9} = \psi(d(x_4,x_5)).$$

Consequently, Theorem 1.14 is not applicable in this case. Note that Theorem 1.14 is a generalization of Theorem 1.9. Now

$$n_{T,T}(x_4,x_5) = \frac{\max\{d(x_4,x_5), D(x_4,Tx_4)+D(x_5,Tx_5), D(x_4,Tx_5)+D(x_5,Tx_4)\}}{1+\delta(x_4,Tx_4)+\delta(x_5,Tx_5)}$$
$$= \frac{\max\{d(x_4,x_5), d(x_4,x_2)+d(x_5,x_3), d(x_4,x_3)+d(x_5,x_2)\}}{1+d(x_4,x_2)+d(x_5,x_3)}$$
$$= \frac{\max\{1,10,10\}}{11} = \frac{10}{11} \neq 1$$

$$m_{T,T}(x_4,x_5) = \max\left\{d(x_4,x_5), D(x_4,Tx_4), D(x_5,Tx_5), \frac{D(x_4,Tx_5)+D(x_5,Tx_4)}{2}\right\}$$
$$= \max\{1,5,5,5\} = 5$$

implies that $\alpha(Tx_4,Tx_5)H(Tx_4,Tx_5) = 8 \nleq \frac{50}{11} = n_{T,T}(x_4,x_5)m_{T,T}(x_4,x_5)$. Hence, Theorem 1.13 which is a generalization of Theorem 1.11 does not hold in this case.

## Coincidence and common fixed point results in b-metric spaces

As an application of Theorem 2.1, we obtain the existence of coincidence and common fixed point of Suzuki type $(\alpha_* - \psi)-$hybrid pair of operators in b-metric spaces.

**Theorem 3.1** Let $(X,d)$ be a b-metric space and $(g,T)$ a Suzuki type $(\alpha_* - \psi)-$hybrid pair of operators such that $T$ an $(\alpha_* - g)-$admissible mapping. Suppose that there exists $x_0 \in X$ and $gx_1 \in Tx_0$ such that $\alpha(gx_0, gx_1) \geqslant 1$ and for any sequence $\{x_n\}$ in $X$ with $gx_n \to gx$, we have $\alpha(gx_n, gx) \geqslant 1$ for all $n \in \mathbb{Z}^+$. Then there exists $x$ in $X$ such that $gx \in Tx$ provided that $T(X) \subseteq g(X)$ and $g(X)$ is complete. Moreover, if there is some $gy \in Ty$ such that $gx \neq gy$ and $\alpha(gx,gy) \geqslant 1$ then $d(gx,gy) \geqslant \frac{1}{2}$. Further, $F(g,T)$ is nonempty if any of the following conditions hold:

**f$_1$-:** The hybrid pair $(g,T)$ is $w-$compatible, $\lim_{n\to+\infty} g^n(x) = w$ for some $w \in X$ and $x \in C(g,T)$ and $g$ is continuous at $w$.

**f$_2$-:** The mapping $g$ is $T-$weakly commuting at some $x \in C(g,T)$ and $g^2x = gx$.

**f₃-:** The mapping $g$ is continuous at at some $x \in C(g,T)$ and $\lim_{n\to+\infty} g^n(w) = x$ for some $w \in X$.

Proof. Lemma 1.3, there is a set $E \subseteq X$ such that $g: E \to X$ is one-to-one and $g(E) = g(X)$. Define a mapping $\mathcal{T}: g(E) \to CB(X)$ by $\mathcal{T}gx = Tx$ for all $g(x) \in g(E)$. The mapping $\mathcal{T}$ is well defined because $g$ is one–to-one. Since $(g,T)$ is a Suzuki type $(\alpha_* - \psi)$ – hybrid pair of operators, therefore

$$\begin{cases} \frac{1}{2}D(gx,Tx) \leqslant bd(gx,gy)) \\ \text{implies} \\ \alpha_*(Tx,Ty)H(Tx,Ty) \leqslant \max\{1, N_{g,T}(x,y)\}\psi\big(M_{g,T}(x,y)\big) \\ = \begin{array}{l} \max\left\{1, \dfrac{\max\{d(gx,gy), D(gx,Tx)+D(gy,Ty), D(gx,Ty)+D(gy,Tx)\}}{b(1+\delta(gx,Tx)+\delta(gy,Ty))}\right\} \\ \psi\left(\max\left\{d(gx,gy), D(gx,Tx), D(gy,Ty), \dfrac{D(gx,Ty)+D(gy,Tx)}{2b}\right\}\right) \end{array} \end{cases}$$

for all $x, y \in X$ for some $\psi \in \Psi_4$ and $\varphi \in \Phi$. Thus

$$\begin{cases} \frac{1}{2}D(gx,\mathcal{T}gx) \leqslant bd(gx,gy)) \\ \text{implies} \\ \alpha_*(\mathcal{T}gx,\mathcal{T}gy)H(\mathcal{T}gx,\mathcal{T}gy) \leqslant \max\{1, N_{\mathcal{T}}(x,y)\}\psi(M_{\mathcal{T}}(x,y)) \\ = \begin{array}{l} \max\left\{1, \frac{\max\{d(gx,gy), D(gx,\mathcal{T}gx)+D(gy,\mathcal{T}gy), D(gx,\mathcal{T}gy)+D(gy,\mathcal{T}gx)\}}{b(1+\delta(gx,\mathcal{T}gx)+\delta(gy,\mathcal{T}gy))}\right\} \\ \psi\left(\max\left\{d(gx,gy), D(gx,\mathcal{T}gx), D(gy,\mathcal{T}gy), \frac{D(gx,\mathcal{T}gy)+D(gy,\mathcal{T}gx)}{2b}\right\}\right) \end{array} \end{cases}$$

for all $gx, gy \in g(E)$. Since $g(E) = g(X)$ is complete. By the given assumption, there exists $x_0 \in X$ and $gx_1 \in Tx_0$ such that $\alpha(gx_0, gx_1) \geqslant 1$. As $T$ is $(\alpha_* - g)$ – admissible, we have $\alpha(Tx,Ty) \geqslant 1$ which implies that $\alpha(\mathcal{T}gx, \mathcal{T}gy) \geqslant 1$. Thus $\mathcal{T}$ is $\alpha_*$ –admissible. Hence $\mathcal{T}$ satisfies all the conditions of Theorem 2.1. Consequently, $\mathcal{T}$ is an MWP operator on $g(E)$, and we obtain a point $u \in g(E)$ such that $u \in \mathcal{T}u$. Since $u \in g(E)$, there is a point $x$ in $X$ such that $gx = u$. This implies that $gx \in \mathcal{T}gx = Tx$. By Theorem 2.1 if there is some $w \in \mathcal{T}w$ such that $u \neq w$ and $\alpha(u,w) \geqslant 1$, then we have $d(u,w) \geqslant \frac{1}{2}$ if $\max\{1, N_{g,T}(x,y)\} = N_{g,T}(x,y)$. For $w \in \mathcal{T}w$ there is a $y$ in $X$ such that such $gy = w$ and $gy \in \mathcal{T}gy =$

$Ty$. Consequently, $d(gx,gy) \geqslant \frac{1}{2}$. Now we prove that $F(g,T) \neq \emptyset$. First consider the case when ($C_1$) holds. Since the pair $(g,T)$ is $w-$compatible and $\lim_{n\to+\infty} g^n(x) = u$ for some $u \in X$, the continuity of $g$ at $u$ implies that $gu = u$ and $\lim_{n\to+\infty} g^n(x) = gu$. Now $\alpha(g^n(x), gu) \geqslant 1$ and$(\alpha_* - g)-$admissibility of $T$ imply that $\alpha(Tg^{n-1}(x), Tu) \geqslant 1$. By $w-$compatibility of the pair $(g,T)$, we have $g^n(x) \in T(g^{n-1}(x))$, that is $g^n(x) \in C(g,T)$ for all $n \in \mathbb{N}$. Note that

$$\frac{1}{2}D(g^n(x), T(g^{n-1}(x))) \leqslant d(g^n(x), g^n(x)) = 0 \leqslant bd(gg^{n-1}(x), gu).$$

Since $(g,T)$ is a generalized Suzuki type $(\alpha_* - \psi)-$hybrid pair of operators, therefore

$$\begin{aligned}
&D(g^n x, Tu)\\
&\leqslant H(Tg^{n-1}x, Tu) \leqslant \alpha(Tg^{n-1}(x), Tu)H(Tg^{n-1}x, Tu)\\
&\leqslant \max\left\{1, \frac{\max\{d(g^n x, gu), D(g^n x, Tg^{n-1}x) + D(gy, Tu), D(g^n x, Tu) + D(gu, Tg^{n-1}x)\}}{b(1+\delta(g^n x, Tg^{n-1}x) + \delta(gu, Tu))}\right\}\\
&\quad \psi\max\left\{d(g^n x, gu), D(g^n x, Tg^{n-1}x), D(gu, Tu), \frac{D(g^n x, Tu) + D(gu, Tg^{n-1}x)}{2b}\right\}\\
&\leqslant \max\left\{1, \frac{\max\{d(g^n x, gu), d(g^n x, g^n x) + D(gu, Tu), D(g^n x, Tu) + d(gu, g^n x)\}}{b(1+d(g^n x, g^n x) + D(gu, Tu))}\right\}\\
&\quad \psi\max\left\{d(g^n x, gu), d(g^n x, g^n x), D(gu, Tu), \frac{D(g^n x, Tu) + d(gu, g^n x)}{2b}\right\}.
\end{aligned}$$

On taking limit as $n \to +\infty$, we obtain that

$$D(gu, Tu) \leqslant \max\left\{1, \frac{D(gu, Tu)}{b(1 + D(gu, Tu))}\right\}\psi(D(gu, Tu)) = \psi(D(gu, Tu)).$$

If $D(gu,Tu) > 0$, then we have $D(gu,Tu) < D(gu,Tu)$, a contradiction. Consequently, $u = gu \in Tu$. That is $F(g,T) \neq \emptyset$. Now let ($C_2$) hold, then $g^2x = gx$ for some $x \in C(g,T)$. By $T-$weakly commuting of $g$, we have $gx = g^2x \in Tgx$. Hence $gx \in F(g,T)$. In case ($C_3$) holds, $\lim_{n\to+\infty} g^n(u) = x$ for some $u \in X$ and $x \in C(g,T)$. By continuity of $g$, we obtain that $x = gx \in Tx$.

Hence $F(g,T) \neq \emptyset$.

**Corollary 3.2** Let $(X,d)$ be a b-metric space and $(g,T)$ a hybrid pair such that $T$ is an $(\alpha_* - g)$ – admissible. If there exists a $\psi \in \Psi_4$ such that

$$\frac{1}{2}D(gx,Tx) \leqslant bd(gx,gy)$$

implies that

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leqslant \max\{1, N_{g,T}(x,y)\}\psi(d(x,y))$$

for all $x,y \in X$. Suppose that there exists $x_0 \in X$ and $gx_1 \in Tx_0$ such that $\alpha(gx_0,gx_1) \geqslant 1$ and for any sequence $\{x_n\}$ in $X$ such that $gx_n \to gx$, we have $\alpha(gx_n,gx) \geqslant 1$ for all $n \in \mathbb{Z}^+$. Then there exists $x$ in $X$ such that $gx \in Tx$ provided that $T(X) \subseteq g(X)$ and $g(X)$ is complete. Moreover, if there is some $gy \in Ty$ such that $gx \neq gy$ and $\alpha(gx,gy) \geqslant 1$, then $d(gx,gy) \geqslant \frac{1}{2}$. Further, $F(g,T)$ is nonempty if the conditions ($j_1$)-($j_3$) Theorem 3.1 hold.

**Corollary 3.3** Let $(X,d)$ be a b-metric space and $(g,T)$ a hybrid pair such that $T$ is an $(\alpha_* - g)$ – admissible. If there exists a $\psi \in \Psi_4$ such that

$$\frac{1}{2}D(gx,Tx) \leqslant bd(gx,gy)$$

implies

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leqslant \psi(d(x,y))$$

for all $x,y \in X$. Suppose that there exists $x_0 \in X$ and $gx_1 \in Tx_0$ such that $\alpha(gx_0,gx_1) \geqslant 1$ and for any sequence $\{x_n\}$ in $X$ such that $gx_n \to gx$, we have $\alpha(gx_n,gx) \geqslant 1$ for all $n \in \mathbb{Z}^+$. Then there exists $x$ in $X$ such that $gx \in Tx$ provided that $T(X) \subseteq g(X)$ and $g(X)$ is complete. Moreover, if there is some $gy \in Ty$ such that $gx \neq gy$ and $\alpha(gx,gy) \geqslant 1$, then $d(gx,gy) \geqslant \frac{1}{2}$. Further, $F(g,T)$ is nonempty if the conditions ($j_1$)-($j_3$) in Theorem 3.1 hold.

## Data dependence of fixed point sets and Ulam-Hyers stability results

Consider the following class of functions

$$\Theta = \{\xi: \mathbb{R}^+ \to \mathbb{R}^+ \text{such that } \xi \text{ is increasing and continuous at } 0\}.$$

Let $(X,d)$ be a b-metric space and $T: X \to P(X)$. The fixed point problem of $T$ is to find an $x \in X$ such that

$$x \in Tx. \tag{4.1}$$

Inequality (4.1) is also known as fixed point inclusion. The fixed point inclusion is said to be generalized Ulam-Hyers stable if there exists a function $\xi \in \Theta$ such that for each $\varepsilon > 0$ and for each solution $u_*$ of the inequality

$$D(u, Tu) \leqslant \varepsilon \tag{4.2}$$

there exists a solution $z_*$ of the fixed point problem (4.1) such that $d(u_*, z_*) \leqslant \xi(\varepsilon)$.

Further, if there exists a $c > 0$ such that $\xi(t) = ct$ for each $t \in \mathbb{R}^+$, then the fixed point problem (4.1) is said to be Ulam-Hyers stable. Let $F(T)$ and $U$ be the sets of solutions of (4.1) and (4.2), respectively. For more on Ulam-Hyers stability of fixed point problems, we refer the interested reader to (Ulam, 1964), (Lazar, 2012), (Petru et al, 2011), (Rus, 2009), (Hyers, 1941). Let $(X, d)$ be a b-metric space and $T: X \to Cl(X)$ be a multivalued mapping then $E(T) = \{x \in X: \{x\} = Tx\}$.

Define a multivalued operator $T^\infty: G(T) \to P(F(T))$ by

$$T^\infty(x, y) = \{z \in F(T): \text{there is an ssa of } T \text{at } (x, y) \text{converging to } z\}$$

where $G(T) = \{(x, y): x \in X, y \in Tx\}$ is a graph of $T$.

A selection of $T: X \longrightarrow P(X)$ is a single valued mapping $t: X \to X$ such that $tx \in Tx$ for all $x \in X$.

**Definition 4.1** *(Rus et al, 2003). Let $(X, d)$ be a metric space and $c > 0$. An MWP operator $T: X \longrightarrow P(X)$ is called a$c-$ multivalued weakly Picard (briefly $c-$MWP) operator if there exists a selection $t^\infty$ of $T^\infty$ such that $d(x, t^\infty(x, y)) \leqslant cd(x, y)$ for all $(x, y) \in G(T)$.*

One of the main results concerning $c-$MWP operators is the following:

**Theorem 4.2** *(Rus, 2001). Let $(X, d)$ be a metric space and $T_1, T_2: X \to P(X)$. If $T_i$ is a $c_i-$MWP operator for each $i \in \{1,2\}$ and there exists $\lambda > 0$ such that $H(T_1x, T_2x) \leqslant \lambda$, for all $x \in X$. Then $H(F(T_1), F(T_2)) \leqslant \lambda max\{c_1, c_2\}$.*

Now we prove the following result.

***Theorem 4.3*** *Let $(X,d)$ be a complete b-metric space and $\alpha: X \times X \to \mathbb{R}^+$. Suppose that*

**g₁-:** *for each $i \in \{1,2\}, T_i: X \to Cl(X)$ are multivalued operators such that*

$$\frac{1}{2}D(x,T_i x) \leqslant bd(x,y)$$

*implies that*

$$\alpha_*(T_i x, T_i y)H(T_i x, T_i y) \leqslant max\{1, N_{T_i}(x,y)\}\psi_i(d(x,y)) \qquad (4.3)$$

*for all $x, y \in X, \psi_i \in \Psi_4$.*

**g₂-:** *for each $i \in \{1,2\}, T_i$ is $\alpha_* -$ admissible mapping,*

**g₃-:** *there exists $x_0 \in X$ and $x_1 \in T_i x_0$ such that $\alpha(x_0,x_1) \geqslant 1$ for each $i \in \{1,2\}$,*

**g₄-:** *if there is a sequence $\{x_n\}$ in $X$ such that $x_n \to x$, then $\alpha(x_n,x) \geqslant 1$ for all $n \in \mathbb{Z}^+$,*

**g₅-:** *there exists $\lambda > 0$ such that $H(T_1 x, T_2 x) \leqslant \lambda$, for all $x \in X$.*

*Then $Fix(T_i) \in Cl(X), i \in \{1,2\}$ and each $T_i$ is an MWP operator such that*

$$H(Fix(T_1), Fix(T_2)) \leqslant bmax\{\lambda_1, \lambda_2\}$$

*where $\lambda_i = \sum_{k=0}^{\infty} b^k \psi_i^k(\lambda)$ for each $i \in \{1,2\}$.*

*Proof.* From Theorem 2.1, it follows that $Fix(T_i) \neq \emptyset$ for each $i \in \{1,2\}$. Let $\{x_n\}$ be a sequence in $Fix(T_1)$ such that $x_n \to z$ as $n \to +\infty$. This implies that $\alpha(x_n, z) \geqslant 1$. Since $T_1$ is $\alpha_* -$admissible mapping, $\alpha(T_1 x_n, T_1 z) \geqslant 1$. As

$$\frac{1}{2}D(x_n, T_1 x_n) = 0 \leqslant bd(z, x_n),$$

so we get

$$
\begin{aligned}
&D(z,T_1z)\\
&\leqslant bd(z,x_n)+bD(x_n,T_1z)\\
&\leqslant bd(z,x_n)+bH(T_1z,T_1x_n)\leqslant bd(z,x_n)+b\alpha(T_1x_n,T_1z)H(T_1z,T_1x_n)\\
&\leqslant bd(z,x_n)+\max\{1,N_{T_1}(x,y)\}\psi_1(d(x,y))\\
&\leqslant \left(b\max\left\{1,\frac{\max\left\{\begin{array}{l}d(z,x_n),D(z,T_1z)+D(x_n,T_1x_n),\\ D(z,T_1x_n)+D(x_n,T_1z)\end{array}\right\}}{b(1+\delta(z,T_1z)+\delta(x_n,T_1x_n))}\right\}\psi_1(d(z,x_n))\right.\\
&\qquad \left.+bd(z,x_n)\right)\\
&\leqslant \left(b\max\left\{1,\frac{\max\left\{\begin{array}{l}d(z,x_n),D(z,T_1z)+d(x_n,x_n),\\ d(z,x_n)+D(x_n,T_1z)\end{array}\right\}}{b(1+D(z,T_1z)+d(x_n,x_n))}\right\}\psi_1(d(z,x_n))\right.\\
&\qquad \left.+bd(z,x_n)\right)
\end{aligned}
$$

On taking limit as $n\to+\infty$, we obtain that $D(z,T_1z)\leqslant 0$, that is, $z\in T_1z$ and hence $F(T_1)$ is closed.

Similarly, $F(T_2)$ is a closed subset of $X$.

From Corollary 2.2, we conclude that $T_i$ for each $i\in\{1,2\}$ is an MWP operator.

By a similar process as followed in Theorem 2.1 starting from $x_1\in F(T_1)$ and $x_2\in T_2x_1$, we obtain a sequence $\{x_n\}$ such that $x_{n+1}\in T_2x_n$ for all $n\geqslant 1$, $x_{n+1}\neq x_n, \alpha(x_{n+1},x_{n+2})\geqslant 1, 0<D(x_{n+1},Tx_{n+1})\leqslant \psi_2(d(x_n,x_{n+1}))$ and

$$
0<d(x_{n+1},x_{n+2})<\psi_2^n\big(q\psi(c_0)\big) \tag{4.4}
$$

for all $n\geqslant 1$, where $c_0=d(x_1,x_2)$.

Following the arguments similar to those in the proof of Theorem 2.1, we conclude that $\{x_n\}$ is a Cauchy sequence and there is an element $u$ in $X$ such that $x_n\to u$ as $n\to+\infty$ and $u\in T_2u$.

Note that

$$
\begin{aligned}
&d(x_n, x_{n+p}) \\
&\leqslant bd(x_n, x_{n+1}) + b^2 d(x_{n+1}, x_{n+2}) + \ldots + b^{p-1} d(x_{n+p-2}, x_{n+p-1}) \\
&\qquad + b^p d(x_{n+p-1}, x_{n+p}) \\
&\leqslant b\psi_2^{n-1}\big(q\psi_2(c_0)\big) + b^2\psi_2^{n}\big(q\psi_2(c_0)\big) + \ldots + b^{p-1}\psi_2^{n+p-3}\big(q\psi_2(c_0)\big) \\
&\qquad + b^p\psi_2^{n+p-2}\big(q\psi_2(c_0)\big) \\
&\leqslant \frac{1}{b^{n-2}}\Big(b^{n-1}\psi_2^{n-1}\big(q\psi_2(c_0)\big) + b^{n}\psi_2^{n}\big(q\psi_2(c_0)\big) + \ldots + b^{n+p-2}\psi_2^{n+p-2}\big(q\psi_2(c_0)\big)\Big) \\
&= \frac{1}{b^{n-2}} \sum_{k=n-1}^{n+p-2} b^k \psi_2^k\big(q\psi_2(c_0)\big) \leqslant \frac{1}{b^{n-2}} \sum_{k=n-1}^{n+p-2} b^k \psi_2^k\big(q\psi_2(\lambda)\big) \\
&= \frac{1}{b^{n-2}} \left( \sum_{k=0}^{n+p-2} b^k \psi_2^k\big(q\psi_2(\lambda)\big) - \sum_{k=0}^{n-1} b^k \psi_2^k\big(q\psi_2(\lambda)\big) + b^{n-1}\psi_2^{n-1}\big(q\psi_2(\lambda)\big) \right).
\end{aligned}
$$

That is,

$$
\begin{aligned}
d(x_n, x_{n+p}) \leqslant \frac{1}{b^{n-2}} \Bigg( &\sum_{k=0}^{n+p-2} b^k \psi_2^k\big(q\psi_2(\lambda)\big) - \sum_{k=0}^{n-1} b^k \psi_2^k\big(q\psi_2(\lambda)\big) \\
&+ b^{n-1}\psi_2^{n-1}\big(q\psi_2(\lambda)\big) \Bigg).
\end{aligned} \tag{4.5}
$$

On taking limit as $p \to +\infty$, we get

$$
\begin{aligned}
d(x_n, u) \leqslant \frac{1}{b^{n-2}} \Bigg( &\sum_{k=0}^{\infty} b^k \psi_2^k\big(q\psi_2(\lambda)\big) - \sum_{k=0}^{n-1} b^k \psi_2^k\big(q\psi_2(\lambda)\big) \\
&+ b^{n-1}\psi_2^{n-1}\big(q\psi_2(\lambda)\big) \Bigg).
\end{aligned} \tag{4.6}
$$

By Lemma 1.6, $\sum_{k=0}^{\infty} b^k \psi_2^k(t)$ converges for any $t \geqslant 0$, there exists $\lambda_2 \geqslant 0$ such that $\sum_{k=0}^{\infty} b^k \psi_2^k(\lambda) = \lambda_2$ and hence

$$d(x_n, u) \leqslant \frac{1}{b^{n-2}}\left(\lambda_2 - \sum_{k=0}^{n-1} b^k \psi_2^k\big(q\psi(\lambda)\big) + b^{n-1}\psi_2^{n-1}\big(q\psi_2(\lambda)\big)\right). \quad (4.7)$$

For $n = 1$, we get $d(x_1, u) \leqslant b\lambda_2$. Thus for $x_0 \in F(T_1)$, there exists $u \in F(T_2)$ such that $d(x_0, u) \leqslant b\lambda_2$. Similarly for each $z_0 \in F(T_2)$, we get $v \in F(T_1)$ and $\lambda_1 \geqslant 0$ such that $d(z_0, v) \leqslant b\lambda_1$. It follows from Lemma 1.4 that

$$H(F(T_1), F(T_2)) \leqslant b\max\{\lambda_1, \lambda_2\}.$$

Now we discuss the Ulam-Hyers stability results.

**Theorem 4.4** *Let $(X, d)$ be a b-metric space and $T: X \to Cl(X)$. Assume that all the hypotheses of Corollary 2.3 hold. Then we have*

**h$_1$-:** *The fixed point inclusion (4.1) is $\zeta_i^{-1}$ – generalized Ulam-Hyers stable for $i = 1,2$, provided that for each $x \in F(T)$ there exists $z \in U$ such that $\alpha(x, z) \geqslant 1$, where $\zeta_1, \zeta_2: \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\zeta_1(t) = t - b^2\psi(t), \zeta_2(t) = t - bt\psi(t)$ are strictly increasing, onto and continuous at $t = 0$.*

**h$_2$-:** *If $E(T) \neq \emptyset$, then the fixed point inclusion (4.1) is $\zeta_i^{-1}$ – generalized Ulam-Hyers stable for $i = 3,4$, provided that for $x \in E(T)$ there exists $z \in U$ such that $\alpha(x, z) \geqslant 1, \zeta_3, \zeta_4: \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\zeta_3(t) = t - b\psi(t), \zeta_4(t) = t - t\psi(t)$ are strictly increasing, onto and continuous at $t = 0$.*

**h$_3$-:** *(Estimate between the fixed point sets of two multivalued mappings) If $S: X \to Cl(X)$ is such that for $x \in F(S)$ there exists $z \in F(T)$ with $\alpha(x, z) \geqslant 1$ and for $x \in F(T)$ there exists $z \in F(S)$ with $\alpha(x, z) \geqslant 1$, and $H(S(x), T(x)) \leqslant \lambda$ for all $x \in X$, then $H(F(S), F(T)) \leqslant \max_{i=1,2} \zeta_i^{-1}(b^2\lambda)$ where $\zeta_i$ is same as in (h$_1$) for each $i = 1,2$.*

**h$_4$-:** *(Estimate between the fixed point sets of two multivalued mappings) If $S: X \to Cl(X)$ is such that for $x \in F(S)$, there exists $z \in E(T)$ with $\alpha(x, z) \geqslant 1$ and for $x \in E(T)$ there exists $z \in F(S)$ with $\alpha(x, z) \geqslant 1$, and $H(S(x), T(x)) \leqslant \lambda$ for all $x \in X$, then $H(F(S), F(T)) \leqslant \max_{i=3,4} \zeta_i^{-1}(b\lambda)$ where $\zeta_i$ is same as in (h$_2$) for each $i = 3,4$.*

**h$_5$-:** *(Well-posedness of the fixed point problem with respect to b-metric $d$) If for any sequence $\{x_n\}$ in $X$, there exists a unique point $x^* \in E(T)$ such that $\alpha(x_n, x^*) \geqslant 1$ and $\lim_{n\to+\infty} D(x_n, Tx_n) = 0$, then $\lim_{n\to+\infty} d(x_n, x^*) = 0$.*

**h₆-:** *(Well-posedness of the fixed point problem with respect to b-metric $H$) If for any sequence $\{x_n\}$ in $X$, there exists a unique point $x^* \in E(T)$ such that $\alpha(x_n, x^*) \geqslant 1$ and $lim_{n\to+\infty} H(\{x_n\}, Tx_n) = 0$, then $lim_{n\to+\infty} d(x_n, x^*) = 0$.*

**h₇-:** *(Limit shadowing property of the multivalued operators) If for any sequence $\{x_n\}$ in $X$, there exists a unique point $x^* \in E(T)$ with $\alpha(x_n, x^*) \geqslant 1$ and $lim_{n\to+\infty} D(x_n, Tx_n) = 0$, then there exists a sequence of successive approximations $\{y_n\}$ such that $lim_{n\to+\infty} d(x_n, y_n) = 0$.*

*Proof.* ($h_1$) From Corollary 2.3, $T$ is an MWP operator and hence $F(T)$ is nonempty. If $x^* \in F(T)$, then by given condition there exists a $y^* \in U$ such that $\alpha(x^*, y^*) \geqslant 1$. The $\alpha_* -$admissibility of $T$ gives that $\alpha(Tx^*, Ty^*) \geqslant 1$. Since $y^* \in U$, for any given $\varepsilon > 0$, we have $D(y^*, Ty^*) \leqslant \varepsilon$. Note that

$$\frac{1}{2} D(x^*, Tx^*) = 0 \leqslant bd(x^*, y^*).$$

Then

$$\begin{aligned}
d(x^*, y^*) &\leqslant bD(x^*, Tx^*) + bD(Tx^*, y^*) = bD(Tx^*, y^*) \\
&\leqslant b^2(H(Tx^*, Ty^*) + D(Ty^*, y^*)) \\
&\leqslant b^2(\alpha(Tx^*, Ty^*)H(Tx^*, Ty^*) + \varepsilon) \\
\text{(u-3)} \quad &\leqslant b^2\left(\max\left\{1, \frac{d(x^*, y^*)}{b(1 + \delta(x^*, Tx^*) + \delta(y^*, Ty^*))}\right\}\psi(d(x^*, y^*)) + \varepsilon\right) \\
&\leqslant b^2\left(\max\left\{1, \frac{d(x^*, y^*)}{b(1 + d(x^*, x^*) + D(y^*, Ty^*))}\right\}\psi(d(x^*, y^*)) + \varepsilon\right) \\
&\leqslant b^2\left(\max\left\{1, \frac{d(x^*, y^*)}{b}\right\}\psi(d(x^*, y^*)) + \varepsilon\right).
\end{aligned}$$

If $\max\left\{1, \frac{d(x^*, y^*)}{b}\right\} = 1$, then we have $d(x^*, y^*) \leqslant b^2(\psi(d(x^*, y^*)) + \varepsilon)$. If $\zeta_1(d(x^*, y^*)) = d(x^*, y^*) - b^2\psi(d(x^*, y^*))$, then from the above inequality we get $\zeta_1(d(x^*, y^*)) \leqslant b^2\varepsilon$ and hence $d(x^*, y^*) \leqslant \zeta_1^{-1}(b^2\varepsilon)$. Consequently, the fixed point inclusion (4.1) is $\xi -$generalized Ulam-Hyers stable, where $\xi = \zeta_1^{-1}$.

If $\max\left\{1, \frac{d(x^*, y^*)}{b}\right\} = \frac{d(x^*, y^*)}{b}$, then $d(x^*, y^*) \geqslant b$. From (4.1) we obtain

that

$$
\begin{aligned}
d(x^*,y^*) &\leqslant b^2\left(\frac{d(x^*,y^*)}{b}\psi(d(x^*,y^*))+\varepsilon\right)\\
&\leqslant bd(x^*,y^*)\psi(d(x^*,y^*))+b^2\varepsilon\\
&\leqslant bd(x^*,y^*)\psi(d(x^*,y^*))+b^2\varepsilon.
\end{aligned}
$$

Now if $\zeta_2(d(x^*,y^*))=d(x^*,y^*)-bd(x^*,y^*)\psi(d(x^*,y^*))$, then from the above inequality we get $\zeta_2(d(x^*,y^*))\leqslant b^2\varepsilon$ and hence $d(x^*,y^*)\leqslant \zeta_2^{-1}(b^2\varepsilon)$. Consequently, the fixed point inclusion (4.1) is $\xi-$generalized Ulam-Hyers stable, where $\xi=\zeta_2^{-1}$.

($h_2$) Let $E(T)\neq\emptyset$, and $x^*\in E(T)$ then

$$d(x^*,y^*)=D(Tx^*,y^*)\leqslant b(H(Tx^*,Ty^*)+D(Ty^*,y^*)).$$

Following the arguments similar to those in the proof of ($h_1$), the result follows.

($h_3$) Let $x^*\in F(S)$, then there exists a $y^*\in F(T)$ such that $\alpha(x^*,y^*)\geqslant 1$. By $\alpha_*-$admissibility of $T$ we get $\alpha(Tx^*,Ty^*)\geqslant 1$. Note that

$$\frac{1}{2}D(y^*,Ty^*)=0\leqslant bd(x^*,y^*).$$

Then by the given assumption on $T$, we obtain that

$$
\begin{aligned}
d(x^*,y^*) &\leqslant bD(x^*,Sx^*)+bD(Sx^*,y^*)\\
&= bD(Sx^*,y^*)\leqslant b^2(H(Sx^*,Tx^*)+D(Tx^*,y^*))\\
&\leqslant b^2(H(Sx^*,Tx^*)+H(Tx^*,Ty^*))\leqslant b^2(\lambda+H(Tx^*,Ty^*))\\
&\leqslant b^2(\lambda+\alpha(Tx^*,Ty^*)H(Tx^*,Ty^*))\\
&\leqslant b^2\left(\lambda+\max\left\{1,\frac{d(x^*,y^*)}{b(1+\delta(x^*,Tx^*)+\delta(y^*,Ty^*))}\right\}\psi(d(x^*,y^*))\right)\\
&\leqslant b^2\left(\lambda+\max\left\{1,\frac{d(x^*,y^*)}{b(1+d(x^*,x^*)+D(y^*,Ty^*))}\right\}\psi(d(x^*,y^*))\right)\\
&\leqslant b^2\left(\lambda+\max\left\{1,\frac{d(x^*,y^*)}{b(1+d(x^*,x^*)+D(y^*,Ty^*))}\right\}\psi(d(x^*,y^*))\right)\\
&\leqslant b^2(\lambda+\psi(d(x^*,y^*))).
\end{aligned}
$$

If $\max\left\{1,\frac{d(x^*,y^*)}{b}\right\}=1$, and

$$\zeta_1(d(x^*,y^*))=d(x^*,y^*)-b^2(\psi(d(x^*,y^*))),$$

then from the above inequality we get $\zeta_1(d(x^*,y^*))\leqslant b^2\lambda$. Consequently, for every $x^*\in F(S)$, there exists a $y^*\in F(T)$ such that $d(x^*,y^*)\leqslant \zeta_1^{-1}(b^2\lambda)$. Similarly, it can be proved that for every $y^*\in F(T)$, there exists a $x^* \in F(T)$ such that $d(x^*, y^*) \leqslant \zeta_1^{-1}(b^2\lambda)$. Hence by Lemma 1.4, we obtain that

$$H(F(S), F(T)) \leqslant \zeta_1^{-1}(b^2\lambda).$$

If $\max\left\{1, \frac{d(x^*,y^*)}{b}\right\} = \frac{d(x^*,y^*)}{b}$, then for $\zeta_1(d(x^*, y^*)) = d(x^*, y^*) - bd(x^*, y^*)\psi(d(x^*, y^*))$ we get

$$H(F(S), F(T)) \leqslant \zeta_2^{-1}(b^2\lambda).$$

Consequently,

$$H(F(S), F(T)) \leqslant \max_{i=1,2} \zeta_i^{-1}(b^2\lambda).$$

($h_4$) This can be proved on the similar lines as in ($h_3$) using the definition of $E(T)$.

($h_5$) If $\{x_n\}$ is a sequence in $X$, there exists a unique $x^* \in E(T)$ such that $\alpha(x_n, x^*) \geqslant 1$ and $\lim_{n\to+\infty} D(x_n, Tx_n) = 0$. Then there exists $u_n \in Tx_n$ such that $\lim_{n\to+\infty} D(x_n, Tx_n) = \lim_{n\to+\infty} d(x_n, u_n) = 0$. Since $T$ is $\alpha_* -$ admissible, $\alpha(Tx_n, Tx^*) \geqslant 1$. As $\frac{1}{2}D(x^*, Tx^*) = 0 \leqslant bd(x_n, x^*)$, by given assumption we have

$$\begin{aligned}
d(x_n, x^*) &\leqslant b(D(x_n, Tx_n) + D(Tx_n, x^*)) \\
&\leqslant b(D(x_n, Tx_n) + H(Tx_n, Tx^*)) \\
&\leqslant b(D(x_n, Tx_n) + \alpha(Tx_n, Tx^*)H(Tx_n, Tx^*)) \\
&\leqslant b\Big(D(x_n, Tx_n) \\
&\quad + \max\left\{1, \frac{d(x_n,x^*)}{b(1 + D(x_n,Tx_n) + D(x^*,Tx^*))}\right\}\psi(d(x_n, x^*))\Big) \\
&\leqslant b\Big(D(x_n, Tx_n) \\
&\quad + \max\left\{1, \frac{d(x_n,x^*)}{b(1 + D(x_n,Tx_n))}\right\}\psi(d(x_n, x^*))\Big) \\
&\leqslant b\left(D(x_n, Tx_n) + \max\left\{1, \frac{d(x_n,x^*)}{b}\right\}\psi(d(x_n, x^*))\right).
\end{aligned}$$

If $\max\left\{1, \frac{d(x_n,x^*)}{b}\right\} = 1$, then we have

$$d(x_n, x^*) - b\psi(d(x_n, x^*))) \leqslant bD(x_n, Tx_n).$$

That is, $\zeta_3(d(x_n, x^*)) \leqslant bD(x_n, Tx_n)$. Similarly, if $\max\left\{1, \frac{d(x_n,x^*)}{b}\right\} = \frac{d(x_n,x^*)}{b}$, we get $\zeta_4(d(x_n, x^*)) \leqslant bD(x_n, Tx_n)$. This implies for each $i \in \{3,4\}$ we get $d(x_n, x^*) \leqslant \zeta_i^{-1}(bD(x_n, Tx_n))$. On taking limit as $n \to +\infty$ and using the continuity of $\zeta_i$ at 0, for each $i \in \{3,4\}$ we get the desired result.

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(h$_6$) follows from (h$_5$) as $D(x_n, Tx_n) \leqslant H(\{x_n\}, Tx_n)$.

(h$_7$) From (h$_5$) it is clear that $\lim_{n\to+\infty} d(x_n, x^*) = 0$. Since $x^* \in E(T)$, so there exists a sequence of successive approximations defined by $y_n = x^*$ for all $n$ such that $\lim_{n\to+\infty} d(x_n, y_n) = \lim_{n\to+\infty} d(x_n, x^*) = 0$.

## Existence and stability of solutions of differential inclusions

Let $Cl_c(\mathbb{R})$ be collection of nonempty closed and convex subsets of $\mathbb{R}$ and $F: \mathbb{R} \to Cl_c(\mathbb{R})$ a lower semicontinuous multivalued mapping. Consider the initial value problem

$$\begin{aligned} &x'(t) \in F\big(x(t)\big), \text{for} t \in J, \\ &x(t) = x_0 \text{for} t = a_1, \\ &x \in C(J), \end{aligned} \tag{4.8}$$

where $J = [a_1, a_2]$ and $C(J)$ is a Banach space of absolutely continuous real valued functions defined on $J$. Since $\mathbb{R}$ with usual metric is paracompact, $F$ is a lower semicontinuous multivalued mapping with $F(u)$ closed and convex for each $u \in \mathbb{R}$, by Michael's Theorem (Michael, 1956), there exists a continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f(u) \in F(u)$ for all $u \in \mathbb{R}$.

Now consider the following initial value problem

$$\begin{aligned} &x'(t) = f\big(x(t)\big), \text{for} t \in J, \\ &x(t) = x_0 \text{for} t = a_1, \\ &x \in C(J). \end{aligned} \tag{4.9}$$

Note that the solution of problem (4.9) is a solution of problem (4.8). Integrating from $a_1$ to $t$, we obtain

$$\int_{a_1}^{t} x'(s)ds = \int_{a_1}^{t} f\big(x(s)\big)ds,$$

that is,

$$x(t) = x_0 + \int_{a_1}^{t} f(x(s))ds, \text{for} t \in J. \tag{4.10}$$

On the other hand, if (4.10) holds then (4.9) holds. Thus (4.9) and (4.10) are equivalent.

Suppose that $f:\mathbb{R}\to\mathbb{R}$satisfies the following hypotheses:

$\int_{a_1}^{t} f(x(s))ds = 0$, for $t \in J$ if and only if $x(t)=x_0(t)$ for all $t\in J$.

There exists a nonnegative real number $L_f$ such that $L_f(a_2-a_1)<\frac{1}{b^2}$, where $b$ is b-metric constant and for all $u,v\in\mathbb{R}$, the relation $\|f(u)-f(v)\|\leqslant L_f\|u(t)-v(t)\|$ holds.

Define $T: X\to X$, where $X:=C(J)$ by

$$T(x(t)) = x_0 + \int_{a_1}^{t} f(x(s))ds, \text{for} t\in J. \tag{4.11}$$

Let $d: C(J)\times C(J)\to\mathbb{R}^+$ be defined as

$$d(x,y) := \max_{t\in J}\|x(t)-y(t)\|^2.$$

Then $(C(J),d)$ is complete b-metric space. Define $\alpha: C(J)\times C(J)\to\mathbb{R}^+$ by

$$\alpha(x,y)=\begin{cases} k & \text{for} x\neq x_0, y\neq y_0, \text{where} k\geqslant 1,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\psi:\mathbb{R}^+\to\mathbb{R}^+$ be defined as $\psi(t):=L_f^2(a_2-a_1)t$. Clearly, $\psi\in\Psi_4$.

First we show that mapping $T$ is $\alpha_*-$admissible. As $\alpha(x,y)\geqslant 1$ implies that $\alpha(x,y)=k$. For $x\neq x_0$ and $y\neq x_0$, from ($i_1$-) we have $Tx\neq x_0$ and $Ty\neq x_0$ on $J$. It follows that $\alpha_*(Tx,Ty)=k\geqslant 1$ and hence $T$ is $\alpha_*-$admissible. Now, by ($i_2$-) for all $x,y\in X$,

$$\begin{aligned} d(Tx,Ty) &= \max_{t\in J}\left\| \int_{a_1}^{t} f(x(s))ds - \int_{a_1}^{t} f(y(s))ds \right\|^2 \\ &\leqslant \max_{t\in J}\int_{a_1}^{t} \| f(x(s))ds - f(y(s))ds \|^2 \end{aligned}$$

$$\leqslant \max_{t\in J} \int_{a_1}^{t} L_f^2 \parallel x(s) - y(s) \parallel^2 ds$$

$$\leqslant L_f^2 \max_{t\in J} \int_{a_1}^{t} \max_{s\in J} \parallel x(s) - y(s) \parallel^2 ds$$

$$= L_f^2 d(x,y) \max_{t\in J} \int_{a_1}^{t} ds = L_f^2 (a_2 - a_1) d(x,y) = \psi(d(x,y)).$$

By Corollary 2.3, we obtain the solution of problem (4.9) which provides the solution of problem (4.8) as well. Define the mappings $\zeta_1, \zeta_2 : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\zeta_1(t) = t - b^2 L_f^2 (a_2 - a_1)t = t - b^2 \psi(t),$$
$$\zeta_2(t) = t - b L_f^2 (a_2 - a_1)t^2 = t - bt\psi(t),$$

where $\psi(t) = L_f^2(a_2 - a_1)t$. Clearly the mapping $\zeta_1$ is strictly increasing and onto. Consequently, all the axioms of Theorem 4.4 hold with mapping $\zeta_1$. Hence the fixed point inclusion (4.8) is $\zeta_1^{-1}$ – generalized Ulam-Hyers stable. Now $\frac{d}{dt}\zeta_2(t) > 0$ if $1 - 2bL_f^2(a_2 - a_1)t > 0$. As $bL_f^2(a_2 - a_1) < 1$, hence the fixed point inclusion (4.8) is $\zeta_2^{-1}$ – generalized Ulam-Hyers stable if $t < \frac{1}{2}$.

Let $F: J \times \mathbb{R} \times \mathbb{R} \to Cl_c(\mathbb{R})$ be a lower semicontinuous multivalued mapping. Consider the initial value problem

$$\begin{aligned} &x'(t) \in F(t, x(t), x(t-h)), \text{for} t \in J, \\ &x(t) = x_0 \text{for} t \in [a_1 - h, a_1], \\ &x \in C(J), \end{aligned} \tag{4.12}$$

where $h$ is a positive real number. Then there exists a selection $f$ such that $f(s,u,v) \in F(s,u,v)$ for all $u, v \in \mathbb{R}$ and $s \in [a_1 - h, a_1]$, see (Michael, 1956). Note that any solution of the problem

$$\begin{aligned} &x'(t) = f\big(t, x(t), x(t-h)\big), \text{for} t \in J, \\ &x(t) = x_0 \text{for} t \in a_1 - h, a_1], \\ &x \in C(J) \end{aligned} \tag{4.13}$$

is a solution for problem (4.12). Further, (4.13) is equivalent to

$x(t) = x_0 + \int_{a_1}^{t} f(s, x(s), x(s-h)ds, \text{fort} \in J,$
$x(t) = x_0 \text{fort} \in a_1 - h, a_1].$

We suppose that $f$ satisfies the following hypotheses:

$\int_{a_1}^{t} f(s, x(s), x(s-h))ds = 0$, for $t \in J$ if and only if $x = x_0$ on $J$.

$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leqslant L_f(\|u_1 - u_2\| + \|v_1 - v_2\|),$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$, where $L_f(a_2 - a_1) < \frac{1}{4b^2}$ and $b$ is a metricconstant.
Define the operator $T: Y \to Y$, where $Y := C[a_1 - h, a_2] \times \mathbb{R} \times \mathbb{R}$ by

$$T(x(t)) = \begin{cases} x_0 + \int\limits_{a_1}^{t} f(s, x(s), x(s-h))ds, \text{fort} \in J, \\ x(t) = x_0 \text{fort} \in a_1 - h, a_1]. \end{cases} \tag{4.14}$$

From the definition of $\alpha$, the admissibility of $T$ follows. Now by ($i_4$-) for all $x, y \in Y$, we have

$$\begin{aligned} d(Tx, Ty) &= \max_{t \in J} \| \int\limits_{a_1}^{t} f(s, x(s), x(s-h))ds - \int\limits_{a_1}^{t} f(s, x(s), y(s-h))ds \|^2 \\ &\leqslant \max_{t \in J} \int\limits_{a_1}^{t} \| f(s, x(s), x(s-h))ds - f(s, x(s), y(s-h))ds \|^2 \\ &\leqslant \max_{t \in J} \int\limits_{a_1}^{t} L_f^2 \| x(s) - y(s) \|^2 (2^2)ds \\ &= 4L_f^2 \max_{t \in J} \int\limits_{a_1}^{t} \| x(s) - y(s) \|^2 ds \\ &\leqslant 4L_f^2(a_2 - a_1)\psi(d(x, y)), \end{aligned}$$

where $\psi(t) = 4L_f^2(a_2 - a_1)t$. By Corollary 2.3, we obtain the solution of problem (4.13) which is also being the selection is the solution for (4.12). If $\zeta_1(t) = t - 4b^2L_f^2(a_2 - a_1)t$ and $\zeta_2(t) = t - 4bL_f^2(a_2 - a_1)t^2$, then the fixed point inclusion (4.12) is $\zeta_1^{-1}$ – generalized Ulam-Hyers stable. Now $\frac{d}{dt}\zeta_2(t) > 0$ if $1 - 8bL_f^2(a_2 - a_1)t > 0$. As $4bL_f^2(a_2 - a_1) < 1$, hence the fixed point inclusion (4.12) is $\zeta_2^{-1}$ – generalized Ulam-Hyers stable if $t < \frac{1}{2}$.

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РЕШЕНИЯ И УСТОЙЧИВОСТЬ ДИФФИРЕНЦИАЛЬНЫХ ВКЛЮЧЕНИЙ ПО УЛАМУ-ХАЙЕРСУ, ВКЛЮЧАЯ РАЗНОВИДНОСТИ МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ ПО СУДЗУКИ В *b*-МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

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РУБРИКА ГРНТИ: 27.00.00 МАТЕМАТИКА;
27.25.17 Метрическая теория функций,
27.39.27 Нелинейный функциональный анализ
ВИД СТАТЬИ: оригинальная научная статья

*Резюме:*

*Введение/цель: В данной статье представлены совпадения и общие неподвижные точки многозначного отображения типа Судзуки в b-метрических пространствах.*

*Методы: Обсуждаются предельные свойства, корректность и устойчивость решенийзадачс неподвижной точкой таких отображений по методу Улама-Хайерса.*

*Результаты: Получена верхняя граница расстояния Хаусдорфа между неподвижными точками множеств. В качестве доказательства полученных результатов, в статье приведено несколько примеров.*

*Выводы: Применение полученных результатов доказывает существование дифференциальных включений.*

*Ключевые слова: b-метрические пространства, многозначные отображения, неподвижная точка и задачи, Улам-Хайерс стабильность, начальная задача.*

# РЕШЕЊА И УЛАМ-ХИЕРОВА СТАБИЛНОСТ ДИФЕРЕНЦИЈАЛНИХ ИНКЛУЗИЈА, УКЉУЧУЈУЋИ СУЗУКИЈЕВЕ ВРСТЕ ВИШЕЗНАЧНОГ ПРЕСЛИКАВАЊА НА *b*-МЕТРИЧКИМ ПРОСТОРИМА

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ОБЛАСТ: математика
ВРСТА ЧЛАНКА: оригинални научни рад

*Сажетак:*

*Увод/циљ: У раду су представљене коинцидентне и заједничке фиксне тачке Сузукијеве врсте вишезначног пресликавања на b-метричким просторима.*

*Методе: Анализирана су гранична својства, добра постављеност и Улам-Хиерова стабилност решења за фиксни проблем вишезначних пресликавања.*

*Резултати: Добијена је горња граница Хауздорфовог растојања између фиксних тачака скупова. Наведени су примери који подржавају добијене резултате.*

*Закључак: Применом представљених резултата установљена је егзистенција диференцијалне инклузије.*

*Кључне речи: b-метрички простори, вишезначно пресликавање, фиксна тачка и проблеми, Улам-Хиерова стабилност, почетни проблем.*

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