PARTIAL STABILITY OF MULTI ATTRIBUTE DECISION-MAKING SOLUTIONS FOR INTERVAL DETERMINED CRITERIA WEIGHTS - THE PROBLEM OF NONLINEAR PROGRAMMING

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DOI: 10.5937/vojtehg68-27014; https://doi.org/10.5937/vojtehg68-27014

FIELD: Mathematics, Nonlinear programming
ARTICLE TYPE: Original scientific paper

Abstract:

Introduction/purpose: The paper presents a designed procedure for solving a class of nonlinear programming (NLP) tasks with the nonlinear and differentiable objective function, linear natural constraints (intervals of possible arguments values - variables) and the normalization condition for arguments. The procedure was applied to determine the partial stability of the solution of the problem of multi attibute decision-making (MADM).

Methods: The basis of the procedure is to define the nodes of argument pairs and their parameters for the allowable multidimensional points. The parameters are implemented in the gradient method, the favorable directions method and the line search method. In the development of the procedure, the basics of the TOPSIS method for MADM with interval-given criteria weights were used, primarily due to the nonlinearity of the reference function.

Results: The paper elaborates the procedure of determining extreme and other admissible solutions of the reference function (boundary and basic solutions) and all vertices of the convex set of the function definition. This forms a complete graph of the function, i.e. the required solutions from the allowable set can be determined. A procedure for determining a set of solutions for defining a separating hyperplane of a set of function values has been developed; in this way, as a specific case, a set of solutions of partial stability of the variant is defined as MADM solutions. Adequate procedures have been proposed to eliminate the degeneration of the procedure (wedging and oscillation of the solution).

Conclusions: The most significant contribution of the paper is the definition of the nodes of argument pairs and their parameters which ensure the *normalization condition in each node and for each allowable point, non-negativity of variables and independence of argument changes in nodes, within active constraints. An original procedure for determining function graphs has been developed. An appropriate real numerical example is given.*

Keywords: criteria weights, nodes of argument pairs, gradient method, favorable direction method, system of basic solutions, multi attribute decision-making, partial stability of solutions.

Introduction

Problems of nonlinear programming (NLP) with the nonlinear objective function, linear natural constraints of arguments (intervals of possible values of arguments) and the normalization condition for arguments cannot be solved by applying classical NLP methods. The normalization condition implies a constant sum and positive values of arguments in each multidimensional point from the admissible convex set of the function definitions. Relying on the knowledge and procedures from the classical NLP methods (Petrić, 1979), (Hadley, 1964), (Zangville, 1969), (Bazaraa et al, 2006), (Luenberger & Ye, 2016), developed for problems with or without limitations, the procedure developed in this paper can be applied for the development of the procedure for solving this class of NLP tasks. At the same time, the necessary procedures based on the introduced concept of *nodes of arguments* (or *nodes of criterion*) for the problem of multi attribute decision-making (MADM) have been developed, thus transforming the criterion function and constraints and creating a new NLP model based on node parameters. The new model does not contain a singled out normalization condition, because it is built into each feasible point through the node parameters.

As the aim of the paper is conceived on two bases - to show a possible procedure for solving this class of NLP tasks while including consideration of partial stability of MADM solutions - a complex method TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) was chosen as an example (Hwang & Yoon, 1981), (Yoon, 1987). The chosen method is based on multiple distances of quantitative indicators of quality of variants (values according to the established criteria - criteria values from the best and most unfavorable existing ("perceived") criteria values. The method was chosen solely because of the nonlinearity of the reference function, since in most other methods this function is linear (VIKOR, MABAC, COPRAS, AHP), and not because of preference over some other methods. The above procedure can also be applied to these, as well as other methods with a continuous reference function. The function of partial stability of one variant in relation to the another one represents a set of weight points for which the difference of the reference TOPSIS values of these variants is positive. Feasible weight points are given to components whose values are within a certain interval, which can be the result of determining the value of weights using group methods, combining multiple methods, incomplete or unreliable information, uncertainty of decision makers and the like.

By applying the TOPSIS method, a reference nonlinear objective function is obtained, the constraints of the variables are linear, and their values must meet the normalization condition, which limits the application of standard NLP methods for conditional optimization. Based on the constraints and possible changes in the values of the weight components (variables), *the nodes of the pairs of criteria (arguments)* are formed. They ensure the normalization condition, non-negativity of variables and independence of weight changes in one node from changes in other nodes, under active constraints. The Cauchy gradient method of the fastest fall (growth) (Vujičić et al, 1980, pp.89-92) and the line search method, adapted to the conditional optimization and characteristics of the nodes of the pairs of criteria, were applied as a basis for the proposed procedure for solving the NLP problem. The first part of the paper defines the function of similarity of a variant to an ideal solution, the nodes of criteria pairs and their characteristics for one variant, and also presents the procedure for determining the extremum of a function (exact and approximate solutions). The way of solving a possible occurrence of degeneration of the procedure (*wedging or oscillation of the solution* in the "vicinity" of the boundary of the set of feasible solutions) is also shown. In the second part of the paper, the partial stability of one variant in relation to the other ones is defined and the already performed parameter relations for one variant are applied to the partial stability function. Based on the introduced system of basic solutions for the required value of the reference TOPSIS function (separating the hyperplane of the values set of the function), a set of solutions was determined for which one variant is better in relation to the other selected variant. The paper does not explicitly deal with the analysis of the influence of criteria weights on the values of quantitative indicators of variant quality, but with the procedure of determining weight points from the set of admissible points, for which the stability of one variant in relation to the other one from the set of available variants can be determined. As an illustration of the procedure, a corresponding real numerical example is given.

The objective function (the similarity function of the variant to the ideal solution)

An MADM problem: There are *m* variants $V_i ; i=\overline{1,m} \in I$ available and each of them is described with *n* attributes that are used as criteria $K_j ; j=\overline{1,n} \in J$ in the decision-making process: the MADM problem is defined as a requirement to determine the variant $V_p ; p \in I$ that is best according to all criteria K_j, as well as a ranking list of all variants. In the decision matrix $C=\{ c_{ij} ; i=\overline{1,m}; j=\overline{1,n} \}$ ($c_{ij} \in R$ are criteria values), the criteria are associated with numerical values of weights $w_j \in (0,1)$ with the normalization condition $\sum_{j=1}^{j=n} w_j = 1$ and the operators - min/max criteria: $L_j = -1$ (min) or $L_j = +1$ (max).

The TOPSIS method: It is based on compromise decision making and Lp metrics (Hwang & Yoon, 1981), (Zeleny, 1982), (Yoon, 1987) and can be displayed in several steps, when determining:

- Normalized criteria values:

a) $a_{ij} = c_{ij} \Big/ \sqrt{\sum_{i=1}^{i=m} c_{ij}^2}$, for *za* $L_j = +1$ (maximum),

b) $a_{ij} = c_{ij}^{/} \Big/ \sqrt{\sum_{i=1}^{i=m} c_{ij}^{/\,2}}$, $c_{ij}^{/} = c_j^* + c_j^- - c_{ij}$,), for $L_j = -1$ (minimum), (1)

where c_j^* are the best and c_j^- are the most unfavorable criteria ("perceived" ideal and anti-ideal)[1], when all criteria become maximizing $(L_j = +1)$.

- Distances of *Lp* metrics for $p = 1,2,\infty$ according to the normalized values of the "perceived" ideal $V^* = (a_j^* = max_j \{ a_{ij} \})$ and the anti-ideal $V^- = (a_j^- = min_j \{ a_{ij} \})$:

a) $t_{p,i}^* = L_p (V^* ; V_i) = [\sum_{j=1}^{j=n} w_j^p (a_j^* - a_{ij})^p]^{1/p}$,

b) $t_{p,i}^- = L_p (V^- ; V_i) = [\sum_{j=1}^{j=n} w_j^p (a_{ij} - a_j^-)^p]^{1/p}$, $p = 1,2,\infty$. (2)

- Unified distances of variants from ideals and anti-ideals:

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[1] It is possible to apply linear normalization $a_{ij} = (c_{ij} - c_j^-) \big/ (c_j^* - c_j^-)$ which increases the range of normalized criteria values $0 \le a_{ij} \le 1$. The decision maker can determine both the absolute best (desirable) and the worst (undesirable or critical) values of the criteria functions that are outside the perceived best and worst criterion values, thus forming a secondary ideal and anti-ideal.

$$t_i^* = \sum_p \chi_{p,\gamma} t_{p,i}^*;\ \ t_i^- = \sum_p \chi_{p,\gamma} t_{p,i}^-(i);\ \ p = 1,2,\infty; \quad (3)$$

where $\chi_{p,\gamma}$ i $\sum_p \chi_{p,\gamma} = 1$ are the coefficients of the linear combination in the system of three metrics (t_1, t_2, t_∞), which represent the relative reliability of the function t_p for the dimension γ (e.g. γ is the number of criteria, variants, class, rankings, etc.) (Yoon, 1987, pp.283-284) or are chosen depending on the nature of the problem i.e. on what is required: a greater overall benefit $(p = 1)$, geometric proximity to the ideal $(p = 2)$ or smaller individual maximum deviations of the criteria values $(p = \infty)$ (Opricović, 1986, pp.45-48).

- Ideal similarity vector $S = \{s_i\}$ - similarity (closeness) of the variant V_i to the ideal solution V^* with elements (ideal similarity coefficients):

$$s_i = t_i^- / (t_i^* + t_i^-);\ 0 \le s_i \le 1. \quad (4)$$

- Rank of variants according to the criterion:

$$R(i) = max_i\{s_i\}. \quad (5)$$

The coefficient of similarity of ideal (4) is a quantitative indicator of the quality of the variant V_i at the same time according to all criteria and in relation to the ideal and the anti-ideal (or the degree of "goodness" of the variant). For $s_i > 0{,}5$ (when i $t_i^- > t_i^*$) the variant V_i has a greater influence on the variant and the variant is considered to be under the "control" of the ideal (the opposite is also true for $t_i^- < t_i^*$ the anti-ideal).

The similarity function of the variant to the ideal (similarity function) (4), for the constant criterion values $c_{ij} \in R$ (1) and one variant V_i (hereinafter the index "i" is implied), can be represented as a real function of *n* variables - weight $w_j \in (0;1)$ for each $j \in J$:

$$s(\underline{w}) = s_i(\underline{w});\ \underline{w} = \{w_j \in (0;1)\};\ \sum_{j=1}^{j=n} w_j = 1;\ \ for\ \ i \in I\ \ i\ \ j \in J\ , \quad (6)$$

where $\sum_{j=1}^{j=n} w_j = 1$ is the normalization condition. For the solved MADM problem, the weight components $\underline{w}$ are given in the intervals $\underline{w} = (w_j) \in [w_j^A, w_j^B]$, where w_j^A are the lower limits and w_j^B the upper limits of the interval of the values of the weight components (some components can be specified as discrete values). Interval weight estimates can be obtained in the process of group decision making on weights, when applying several objective methods of determining weights, as a consequence of incomplete information or uncertainty of decision-makers and the like. A set of initial weights is formed for each criterion w_j^P and the lower w_j^{AP} and upper limit values w_j^{BP} are separated. The starting point of the weight $\underline{w}^{0P} = (w_j^{0P})$ has components $w_j^{AP} \le w_j^{0P} \le w_j^{BP}$ that can be the arithmetic means of several obtained weights (either modal or medial values) or at will chosen weight and for which it is generally $\sum_{j=1}^{j-n} w_j^{0P} \ne 1$. By normalizing these values, *the basic point of weights* $\underline{w}^0 = (w_j^0) \in [w_j^A , w_j^B]$ and $\sum_{j=1}^{j=n} w_j^0 = 1$ is obtained, with the limit weights w_j^A and w_j^B whose sums are $\sum_{j=1}^{j=n} w_j^A < 1$ and $\sum_{j=1}^{j=n} w_j^B > 1$. According to the basic point of weights $\underline{w}^0 = (w_j^0)$ and expressions (1-5), a *basic TOPSIS solution of the MADM problem* $(\underline{w}^0 ; s_p^0 = s_p (\underline{w}^0))$ is obtained or a variant $V_p , p \in I$ for $s_p (\underline{w}^0) = max_i \{ s_i (\underline{w}^0); i = \overline{1,m} \}$.

The function definition set $s(\underline{w})$ is a compact (closed and bounded) and convex set of points $\underline{w} \in \overline{E} \subseteq \Re^n$, such that $\overline{E} \subseteq \overline{F} \subseteq \Re^n$. A set $\overline{F}$ is an *n*-dimensional set bounded by *2n* hyperplanes, w_j^A and w_j^B, and $\underline{w} = (w_j) \in [w_j^A , w_j^B]$ is an *n*-dimensional point. The point $\underline{w} \in \overline{F}$, with the components $w_j > 0$ for each $j \in J$, not connected by the normalization condition $\sum_{j=1}^{j=n} w_j^0 = 1$, is the vertex of the set $\overline{F}$ only if each component has a value of $w_j = w_j^A$ or $w_j = w_j^B$. The vertex of the set $\overline{F}$ must contain *n* components: *p* components $w_{j_A}^A > 0$ for $j_A \in J_A \subset J$ and *q* components $w_{j_B}^B > 0$ for $j_B \in J_B \subset J$, so that $0 \le p \le n$, $0 \le q \le n$, $p + q = n$, $J_A \cup J_B = J$, and $J_A \cap J_B = \varnothing$; the total number of vertices is 2^n (variations with repetition). Since it is $\sum_{j_A} w_{j_A}^A + \sum_{j_B} w_{j_B}^B \ne 1$ in the general case and due to the normalization condition $\sum_{j=1}^{j=n} w_j = 1$, the set $\overline{E} \subseteq \overline{F}$ does not contain vertices, and thus not all points of boundaries (edges) and sides of the set $\overline{F}$.

A point $\underline{w} \in \overline{E} \setminus E$ is a *boundary point* of a set $\overline{E} \subseteq \Re^n$ only if there is $w_j = w_j^A$ or $w_j = w_j^B$ for at least one $j \in J$, and an *inner point* $\underline{w} \in E$ - only if there is $w_j^A < w_j < w_j^B$ for every $j \in J$, where E is the interior of the set $\overline{E}$. Each vertex of the set $\overline{E}$ contains *n* components: *p* components $w_{j_A}^A$, *q* components $w_{j_B}^B$ (where in $0 \le p \le n-1$, $0 \le q \le n-1$ and $p+q=n-1$) and a w_r component:

$$w_r = 1-(\sum_{j_A} w_{j_A}^A + \sum_{j_B} w_{j_B}^B), \; w_r^A \le w_r \le w_r^B ; \qquad (7)$$

which is a condition for some combinations of the values $w_{j_A}^A$ and $w_{j_B}^B$ to form the vertex of the set $\overline{E}$. As it is $w_r^A \le (1-\sum_{j_A} w_{j_A}^A - \sum_{j_B} w_{j_B}^B) \le w_r^B$, condition (7) in the general case cannot be fulfilled for all $2^{(n-1)}$ possible combinations of the values $w_{j_A}^A$ and $w_{j_B}^B$.

A set of function values $s(\underline{w})$ is a set $\overline{S} = \{ s \in [s^m ; s^M] \} \subseteq \Re$ limited with the values of the function for extreme solutions: *minimum* $(\underline{w}^m ; s^m = s(\underline{w}^m))$ and *maximum* $(\underline{w}^M ; s^M = s(\underline{w}^M))$. *Mapping* $s : \overline{E} \to \overline{S} \; (\Re^n \to \Re)$ is a surjection: there is at least one point $\underline{w} \in \overline{E}$ for which there is $s(\underline{w}) = C \in \overline{S}$ and for each point $\underline{w} \in \overline{E}$ there is only one value $s(\underline{w}) = C \in \overline{S}$: $(\forall s(\underline{w}) \in \overline{S})\; (\exists \underline{w} \in \overline{E})$; $(\forall \underline{w} \in \overline{E})\; (\exists! \, s(\underline{w}) = C \in \overline{S})$. The set of all solutions forms the *graph of the function* $\Gamma_s = \{ (\underline{w}; s) \in \Re^{n+1} \mid \underline{w} \in \overline{E}, s = s(\underline{w}) \in \overline{S} \}$.

The extremes of the function $s(\underline{w})$ are obtained as solutions of the NLP problem with a nonlinear objective function, *2n* linear constraints, the normalization condition and positive values of the variables:

$$(min/\,max) \;\; s(\underline{w});$$

$$w_j \ge w_j^A ; \; w_j \le w_j^B ; \; \sum_{j=1}^{j=n} w_j = 1 ; \; w_j > 0 ; \; j = \overline{1,n} . \qquad (8)$$

The function $s(\underline{w})$ on a convex set $\overline{E}$ is continuous and twice differentiable, but it is not possible to unambiguously determine the convexity or concavity of the function on the whole set $\overline{E}$. For a special case and for a constant value $s(\underline{w}) = C$, function (4) can be written as $C = t^- / (t^* + t^-)$ or $C t^* - (1-C) t^- = 0$. Hence the assertion (Yoon, 1987, p.280) that a function $s(\underline{w})$ is convex for the subsets of points $\underline{w} \in \overline{E_1} \subseteq \overline{E}$ in which it is $s(\underline{w}) \ge 0.5$, and concave for subsets of points $\underline{w} \in E_2 = \overline{E} \setminus \overline{E_1}$ in which it is $s(\underline{w}) < 0.5$. Therefore, each *local extremum* is also a *global extremum*, that is, a function has extreme solutions at the boundary of the set $\overline{E}$, which are unique in terms of the values of the function and arguments. The convex function ($s(\underline{w}) \geq 0.5$) has a maximum, and the concave ($s(\underline{w}) < 0.5$) has a minimum at the vertex of the set $\overline{E}$ (Martić, 1978, pp.144-145) and these are *exact extreme solutions* that meet the *optimality criterion.* The minimum convex and maximum concave functions are at the boundary of the set (they can also be the vertices of the set $\overline{E}$), and if they are not the vertices of the set $\overline{E}$, then they are determined as *approximate extreme solutions* (incorrect, acceptable) according to the predefined criteria for function values and/or arguments according to real (*exact*) extreme solutions.

Nodes of the pairs of arguments (criteria)

Defining *the nodes of the pairs of arguments (criteria)* and their basic parameters is the most important phase of the presented method of solving the NLP problem. Nodes are formed for one variant V_i and each pair of criteria $r,t \in J; r \neq t$ and one point $\underline{w}^k$. In a narrower sense, *the node of the pair (two) of criteria (r,t) is the node of two different components w_j^k for $j = r,t$ of one point of weight $\underline{w}^k$: it represents a qualitatively new set of parameters arising from the mutual relations of characteristics (parameters) of current components w_r^k and w_t^k.*

The transition from the initial to a new solution is done by changing the starting point $\underline{w}^k$ to a new point of weights $\underline{w}^{k+1}$, where $k = 0,1,2,\ldots,$ is the mark of the iterative solution. The basic parameters of the point $\underline{w}^k$ are:

<u>*Active constraints of the criteria*</u> d_j^{Ak} and d_j^{Bk} (possible changes in weight components) at the point $\underline{w}^k$ for the intervals $[w_j^A, w_j^B]$:

$$d_j^{Ak} = w_j^k - w_j^A \geq 0;\ d_j^{Bk} = w_j^B - w_j^k \geq 0;\ d_j^{AB} = d_j^{Ak} + d_j^{Bk} = w_j^B - w_j^A,\ j \in J, \quad (9)$$

where $d_j^{Ak} \geq 0$ is the largest possible decrease, $d_j^{Bk} \geq 0$ is the largest possible increase in weight w_j^k, and $d_j^{AB} \geq 0$ is the size of the interval $[\underline{w}^A, \underline{w}^B]$. The values d_j^{Ak} and $d_j^{Ak} \geq 0$ and their sums $\sum_{j=1}^{j=n} d_j^{Ak}$ and $\sum_{j=1}^{j=n} d_j^{Bk}$ can be related by any sign (<, =,>).

<u>*The vector of change (increment) is the weight*</u> $\underline{v}^k$: when changing the point $\underline{w}^k$ to the point $\underline{w}^{k+1}$, the new point $\underline{w}^{k+1} = \underline{w}^k + \underline{v}^k$, where the

$\underline{v}^k=(v_j^k)$ *vector of change (increment) is the weight*. The values v_j^k can be $\neq 0$ or $=0$, which is why non-negative values are introduced $v_j^{Ak}, v_j^{Bk} \geq 0$ ($v_j^{Ak}>0$ reduction and $v_j^{Bk}>0$ increase weight w_j^k):

$$a)\ v_j^{Ak}=-v_j^k \geq 0,\ v_j^{Ak} \in [0, d_j^{Ak}];\ \text{za}\ v_j^k \leq 0;$$

$$b)\ v_j^{Bk}=v_j^k \geq 0,\ v_j^{Bk} \in [0, d_j^{Bk}],\ \text{za}\ v_j^k \geq 0. \quad (10)$$

The values of the weight components at the new point $\underline{w}^{k+1}$ are[2]:

$$w_j^{k+1}=w_j^k+v_j^k=w_j^k+v_j^{Bk}-v_j^{Ak}. \quad (11)$$

where for each component w_j^k (11) at least one of the values v_j^{Ak} or v_j^{Bk} is equal to 0.

The gradient of function: from the development of function (6) into Taylor's polynomial of the first degree:

$$s(\underline{w}^{k+1})=s(\underline{w}^k)+\nabla s(\underline{w}^k)(\underline{w}^{k+1}-\underline{w}^k)+R_1=s(\underline{w}^k)+\sigma(\underline{w}^k)+R_1 \quad (12)$$

and for the value of the remainder $R_1 \approx 0$, the auxiliary function $\sigma^k=\sigma(\underline{w}^k)$ is equal to:

$$\sigma(\underline{w}^k)=\nabla s(\underline{w}^k)(\underline{w}^{k+1}-\underline{w}^k)=\nabla s(\underline{w}^k)(\underline{v}^{Bk}-\underline{v}^{Ak}); \quad (13)$$

where $\nabla s(\underline{w}^k)=\{\partial s(\underline{w}^k)/\partial w_j^k\}_{j=\overline{1,n}}$ the gradient of the function $s(\underline{w})$ is at the point $\underline{w}^k$. *Approximate values of the gradient components* $g_j^k=g_j(\underline{w}^k)\approx \partial s(\underline{w}^k)/\partial w_j^k$ can be calculated by the *method of double increment of variables* (Milovanović & Stanimirović, 2002, p.114):

$$g_j^k=[s(\underline{w}^{j,k+})-s(\underline{w}^{j,k-})]/2\delta;\ \ \delta>0;\ \ j\in J, \quad (14)$$

where points $\underline{w}^{j,k+}=(w_1^k,\cdots,w_j^k+\delta,\cdots,w_n^k)$ and $\underline{w}^{j,k-}=(w_1^k,\cdots,w_j^k-\delta,\cdots,w_n^k)$, and $\delta>0$ the increment is small (e.g. $\delta=10^{-6}$ or less).

Node parameters

For the transition from the point $\underline{w}^k$ to the point $\underline{w}^{k+1}$ and with the normalization condition $\sum_{j=1}^{j=n} w_j^k=1$, it is necessary to change the weights of at least two criteria $r,t\in J$ that form a *node* (r,t) with a unique value

[2] The index $k=0,1,2,\ldots$, indicates the quantities in the point $\underline{w}^k$ (eg: s^k, σ^k, d_j^{Ak}, $d_{(r,t)}^k$, g_j^k, $g_{(r,t)}^k$) and the quantities that "come out" of it (eg: v_j^k, v_j^{Ak}, $z_{(r,t)}^k$, $\tau_{(r,t)}^k$).

of weight changes $v^{Ak}_{r(t)} = v^{Bk}_{t(r)} \geq 0$ and the direction of changes: $v^{Ak}_{r(t)}$ is the reduction of the weight w^k_r, and $v^{Bk}_{t(r)}$ is the increase of the weight w^k_t. The components $r,t \in J$ of the point $\underline{w}^{k+1}$ are equal to:

$$w^{k+1}_r = w^k_r - v^{Ak}_{r(t)};\;\; w^{k+1}_t = w^k_t + v^{Bk}_{t(r)};\;\; v^{Ak}_{r(t)} = v^{Bk}_{r(t)} \geq 0;\;\; r,t \in J\,, \quad (15)$$

while the other components are unchanged $w^{k+1}_j = w^k_j,\;\; j \in J \setminus \{r,t\}$. A set of nodes is formed for a known solution

$$\Theta^k = \{(r,t) \,|\, r,t \in J, r \neq t\} \quad (16)$$

with $n(n-1)$ elements in total. The basic characteristics of the nodes are:

<u>*Active constraints of nodes*</u> $d^k_{(r,t)} = d_{(r,t)}(\underline{w}^k)$ depend on a possible decrease in the *r*-component and on an increase of the *t*-component weight in nodes (9) (available resources):

$$d^k_{(r,t)} = \begin{cases} min(d^{Ak}_r; d^{Bk}_t) \geq 0;\;\; (r,t) \in \Theta \\ 0;\;\; r = t,\;\; [(r,t) \notin \Theta] \end{cases}, \quad (17)$$

where the matrix $D^k = (d^k_{(r,t)})_{nxn}$ for $r,t \in J$.

<u>*Variables*</u> $z^k_{(r,t)}$: variables $z^k_{(r,t)} \geq 0$ are introduced, whose values show the change of the weight components in the nodes, whereby a square matrix $Z^k = (z^k_{(r,t)})_{nxn}$ is formed, with the following elements:

$$a)\;\; z^k_{(r,t)} \begin{cases} \leq d^k_{(r,t)} \geq 0,\;\; (r,t) \in \Theta \\ = 0;\;\; r = t,\;\; [(r,t) \notin \Theta] \end{cases},$$

$$b)\;\; z^k_{(r,t)} = v^{Ak}_{r(t)} = v^{Bk}_{t(r)} \geq 0\,. \quad (18)$$

Since $v^{Ak}_{r(t)} = v^{Bk}_{t(r)}$, the values of the variables $z^k_{(r,t)} > 0$ are conditional and compensatory values of the weight changes in the node (r,t), and their summation in the nodes $(r,t) \in \Theta$ gives the total increase v^{Ak}_r or decrease v^{Bk}_t of each weight component:

$$a)\;\; j = r \rightarrow v^{Ak}_r = v^{Ak}_{r(1)} + \cdots + v^{Ak}_{r(n)} = \sum_{t=1}^{t=n} v^{Ak}_{r(t)} = \sum_{t=1}^{t=n} z^k_{(r,t)} \leq d^{Ak}_r\,,$$

$$b)\;\; j = t \rightarrow v^{Bk}_t = v^{Bk}_{t(1)} + \cdots + v^{Bk}_{t(n)} = \sum_{r=1}^{r=n} v^{Bk}_{t(r)} = \sum_{r=1}^{r=n} z^k_{(r,t)} \leq d^{Bk}_t\,,$$

$$c)\;\; \sum_{j=1}^{j=n} v^{Bk}_j - \sum_{j=1}^{j=n} v^{Ak}_j = 0\,. \quad (19)$$

The normalization condition is provided in each node and does not need to be considered further. The condition from the starting point of weights $\sum_{j=1}^{j=n} w_j^0 = 1$ (11) is also contained in the point $\underline{w}^1$ because of (19c): $\sum_{j=1}^{j=n} w_j^1 = \sum_{j=1}^{j=n} w_j^0 + \sum_{j=1}^{j=n} v_j^{B0} - \sum_{j=1}^{j=n} v_j^{A0} = 1$. The normalization condition is transformed into $\sum_{j=1}^{j=n} v_j^{Bk} - \sum_{j=1}^{j=n} v_j^{Ak} = 0$ or $\sum_{j=1}^{j=n} v_j^k = 0$ and is contained in each node (r,t) and for each point $\underline{w}^k \in \overline{E}$ in any iteration $k = 0,1,2,\ldots,$.

Gradient function in the node $g_{(r,t)}^k$: the increment of the value of the auxiliary function (13) for the node (r,t) is:

$$\sigma_{(r,t)}^k = (\partial s(\underline{w}^k)/\partial w_t^k) \cdot v_{t(r)}^{Bk} - (\partial s(\underline{w}^k)/\partial w_r^k) \cdot v_{r(t)}^{Ak}. \tag{20}$$

By changing $g_j^k \approx \partial s(\underline{w}^k)/\partial w_j^k$ for g_r^k and g_t^k (14) and $z_{(r,t)}^k = v_{r(t)}^{Ak} = v_{t(r)}^{Bk} \geq 0$ (19a, b) expression (20) becomes:

$$\sigma(z_{(r,t)}^k) \approx z_{(r,t)}^k (g_t^k - g_r^k) = z_{(r,t)}^k \cdot g_{(r,t)}^k, \ g_{(r,t)}^k = g(z_{(r,t)}^k), \tag{21}$$

where $g_{(r,t)}^k = g_{(r,t)}(\underline{w}^k)$ is the approximate value of the gradient function component $s(\underline{w})$ in the node $(r,t) \in \Theta^k$. The values $g_{(r,t)}^k$ for all nodes (r,t) form a square antisymmetric matrix $G^k = (g_{(r,t)}^k)_{nxn}$ with the elements:

$$g_{(r,t)}^k = \begin{cases} g_t^k - g_r^k; \ (r,t) \in \Theta^k \\ 0; \ r = t, \ [(r,t) \notin \Theta^k] \end{cases}, \tag{22}$$

where $g_{(r,t)}^k = -g_{(t,r)}^k$. The approximate total change in the value of the function $s(\underline{w}^k)$ for all nodes $(r,t) \in \Theta^k$, according to (21), is equal to:

$$\sigma(Z^k) = \sum_{(r,t)} g(z_{(r,t)}^k) z_{(r,t)}^k, \ (r,t) \in \Theta^k. \tag{23}$$

Possible changes in the function values by the nodes $\tau_{(r,t)}^k$ are derived values based on the values of the elements of the matrix D^k (18) and G^k (22). A matrix $T^k = (\tau_{(r,t)}^k)_{nxn}$ is formed, with the elements:

$$\tau_{(r,t)}^k = \begin{cases} g_{(r,t)}^k d_{(r,t)}^k \neq 0; \ za \ \ g_{(r,t)}^k \neq 0 \ \ i \ \ d_{(r,t)}^k > 0 \\ 0; \ za \ \ g_{(r,t)}^k = 0 \ \ ili \ \ d_{(r,t)}^k = 0 \end{cases}, \ (r,t) \in \Theta^k \tag{24}$$

of which there may be at most *n (n-1)* elements $\tau^k_{(r,t)} \neq 0$. This defines the basic characteristics of the nodes (r,t): $d^k_{(r,t)}$, $g^k_{(r,t)}$ and $\tau^k_{(r,t)}$ and the variables $z^k_{(r,t)}$, shown in Table 1.

Active nodes

The characteristic $\tau^k_{(r,t)} \neq 0$, as a derived quantity, is the most significant indicator of the possibility of changing the value of the function in the node (r,t) for the current point $\underline{w}^k$. Based on the values $\tau^k_{(r,t)} \neq 0$, the matrix T^k_- with the elements that are 0 or $\tau^k_{(r,t)} < 0$ and the matrix T^k_+ with the elements that are 0 or $\tau^k_{(r,t)} > 0$; this determines the subsets of *active nodes* Θ^k_- and Θ^k_+ at the point $\underline{w}^k$:

a) $\Theta^k_- = \{(r,t) \mid \tau^k_{(r,t)} < 0\} \subset \Theta^k$;

b) $\Theta^k_+ = \{(r,t) \mid \tau^k_{(r,t)} > 0\} \subset \Theta^k$. (25)

The active nodes in the point $\underline{w}^k$ are the nodes $(r,t) \in \Theta^k_- \cup \Theta^k_+$, and the *active gradient components* are only the components $g^k_{(r,t)} \neq 0$ in the active nodes (the nodes in which there are $g^k_{(r,t)} \neq 0$ and $d^k_{(r,t)} = 0$ are not active nodes). According to the influence on the value of the function (increase or decrease), i.e. for determining the extreme of the function (minimum, maximum), *active nodes* are the nodes $(r,t) \in \Theta^k_-$ for decreasing the value or determining the minimum of the function and the nodes $(r,t) \in \Theta^k_+$ for increasing the value or determining the maximum of the function. *Active gradient components* are only the components $g^k_{(r,t)} < 0$ (min) or $g^k_{(r,t)} > 0$ (max) in the active nodes ($\tau^k_{(r,t)} \neq 0$). The sum $\sum_{(r,t)} |\tau^k_{(r,t)}| = \sum_{(r,t)} |g^k_{(r,t)}| d^k_{(r,t)} \geq 0$ for the node $(r,t) \in \Theta^k_-$ is the largest possible decrease, and for the node $(r,t) \in \Theta^k_+$ the largest possible increase of the value of the function $s(\underline{w})$ is at the point $\underline{w}^k$. The value $\sum_{(r,t)} |\tau^k_{(r,t)}|$ can also be a criterion for accepting the achieved solution as an approximate extreme solution and for interrupting the iterative procedure if $\sum_{(r,t)} |\tau^k_{(r,t)}| \leq \varepsilon_\tau$ for $(r,t) \in \Theta^k_-$ (min) or $(r,t) \in \Theta^k_+$ (max), because the improvement of the function value in the next iteration cannot be greater than the defined value ε_τ (e.g. $\varepsilon_\tau \leq 5 \cdot 10^{-5}$).

The formation of the nodes $(r,t) \in \Theta^k$ ensures that the normalization condition $\sum_{j=1}^{j=n} w_j^k = 1$ (or $\sum_{j=1}^{j=n} v_j^k = 0$) is contained in each point $\underline{w}^k \in \overline{E}$ the non-negativity of the variables $z_{(r,t)}^k \geq 0$ and the independence of weight changes in a particular node $(r,t) \in \Theta^k$ from changes in the other nodes, with active limitations of criteria (18, 19).

Extreme solutions of the objective function (similarity functions)

The extremes of the function $s = s(\underline{w})$ (8) are determined by an iterative procedure starting from some feasible solution $(\underline{w}^k; s^k)$ which, in general, is not extreme. Improving the initial solution is possible only if there is at least one node $(r,t) \in \Theta_-^k$ (25a) (decrease in the value s^k) or only if there is at least one node $(r,t) \in \Theta_+^k$ (25b) (increase in the value s^k). According to expression (13), the current solution $(\underline{w}^k; s^k)$ is improved by increasing (decreasing) the value of the auxiliary function $\sigma(Z^k)$ (13, 23), when the nodes taken into account are only the nodes $(r,t) \in \Theta^k$ in which weight changes contribute to the improvement of the value of the auxiliary function $\sigma(Z^k)$.

The iterative procedure determines the boundary solutions $(\underline{w}^{k+1} \in \overline{E} \setminus E; s^{k+1})$ with the improved function $s(\underline{w})$ values, i.e. the boundary points $(\underline{w}^{k+1} \equiv \underline{w}^{k*}) \in \overline{E} \setminus E$ that will give an improved TOPSIS solution (2-4), while active constraints allow it. At the end of the procedure, a solution will be obtained at the point at the vertex of the set $\overline{E}$, which will be the exact extreme solution $\underline{w}^{k*} = (\underline{w}^m \vee \underline{w}^M) \in \overline{E} \setminus E$ or the initial solution for a further procedure and determination of the approximate extreme solution. In both cases, there is a single iterative procedure by which an admissible solution is obtained at the vertex of the set $\overline{E}$, from which no better solution can be obtained at any vertex of the set $\overline{E}$.

Exact extreme solutions

For the initial solution $(\underline{w}^k; s^k)$*,* usually k=0, it is convenient to form a table similar to Table 1 which contains the values of the active constraints d_j^{Ak} and d_j^{Bk} (9), the node characteristics (r,t) - the elements of the matrices D^k (17), G^k (22) and T^k (24) at the point $\underline{w}^k$, the partial sums $g_{(r,t)}^k$ and $\tau_{(r,t)}^k$ dependent on the active nodes Θ_-^k or Θ_-^k, the space for writing variables $z_{(r,t)}^k$ and their sums as the components of the vector $\underline{v}^k = (v_j^k)$.

The extremes of the function $s(\underline{w})$ (8) are determined as the solutions of the NLP problem with restrictions on the weight changes in the nodes $z_{(r,t)}^k \leq d_{(r,t)}^k > 0$ (18) and according to the criteria $\sum_{t=1}^{t=n} z_{(r,t)}^k \leq d_r^{Ak} \geq 0$ and $\sum_{r=1}^{r=n} z_{(r,t)}^k \leq d_t^{Bk} \geq 0$ (19), that is, from the condition that the points of extremes are admissible, and, at the same time, the boundary points $\underline{w^{k+1} \in \overline{E} \setminus E}$.

The mathematical model NLP (8) was transformed according to the node parameters and a new model was formed containing the nonlinear objective function (due to the multiple differentiability of the function $s(\underline{w})$ and the nonlinearity of the function $g(z_{(r,t)}^k)$), $n(n+1)$ linear constraints (the normalization condition $\sum_{j=1}^{j=n} w_j = 1$ is contained in the nodes parameters) and for $n(n-1)$ variables $z_{(r,t)}^k \geq 0$, with indices as in Table 1:

$$
\begin{aligned}
&(\min/\max)\ \sigma(Z^k) = \textstyle\sum_{(r,t)} g(z_{(r,t)}^k)\, z_{(r,t)}^k, \\
&z_{(r,t)}^k \leq d_{(r,t)}^k, \quad n(n-1) \text{ constraints}, \\
&\textstyle\sum_{t=1}^{t=n} z_{(r,t)}^k \leq d_r^{Ak}, \quad n \text{ constraints}, \\
&\textstyle\sum_{r=1}^{r=n} z_{(r,t)}^k \leq d_t^{Bk}, \quad n \text{ constraints}, \\
&z_{(r,t)}^k \geq 0;\ (r,t) \in \Theta_-^k\ (\min) \text{ or } (r,t) \in \Theta_+^k\ (\max).
\end{aligned}
\tag{26}
$$

The NLP task (8,26) is solved by applying an iterative procedure based on the first-order gradient method or the Cauchy method of the fastest drop (growth) of the value of the function $s(\underline{w})^3$ adapted for the application of node parameters. The direction of the antigradient $-\nabla s(\underline{w}^k)$ is also the direction of the fastest decrease in the value of the function $s(\underline{w})$ at the point $\underline{w}^k$, that is, it is the most favorable direction from the point $\underline{w}^k$ for determining the minimum (Vujičić et al, 1980, p.89); the direction of the gradient $\nabla s(\underline{w}^k)$ is the most favorable direction for determining the maximum.

Through the starting point $\underline{w}^k$, in addition to the most favorable direction, countless other favorable directions can be drawn that will contain the characteristics of one or more active nodes[4]. The aim is to determine the point $\underline{w}^{k+1} \in \overline{E}$ at which the TOPSIS value of the function s^{k+1} is better than the value s^k in the chosen favorable direction and in accordance with the limitations in model (26).

Solving problem (26) requires at least one known feasible solution $(\underline{w}^k; s^k)$ (basic TOPSIS solution $(\underline{w}^0; s^0)$ or any other feasible solution), for which there is at least one node $(r,t) \in \Theta_-^k$ (minimum) or at least one node $(r,t) \in \Theta_+^k$ (maximum) (25). Based on the values of all active components of the gradient at the point $\underline{w}^k$, the most favorable direction or the direction of the fastest fall (growth) of the objective function is set through it. Active nodes are determined depending on the required extreme: $(r,t) \in \Theta_-^k$ (min) or $(r,t) \in \Theta_+^k$ (max). The most $n(n-1)/2$ active nodes are possible for each required extremum, that is, it is the largest number of elements of the sets Θ_-^k and Θ_+^k for the inner point $\underline{w} \in E$. In the intersection of the most favorable direction and some, unknown in advance, hyperplane of the set $\overline{E}$ - w_j^A or w_j^B, there is the boundary point $(\underline{w}^{k*} \equiv \underline{w}^{k+1}) \in \overline{E} \setminus E$:

$$\underline{w}^{k*} = \underline{w}^k + \underline{v}^k = (w_j^k + v_j^k) = (w_j^k + v_j^{Bk} - v_j^{Ak}); \quad j \in J . \qquad (27)$$

---

[3] Based on the gradient method of the fastest fall, several procedures and their modifications have been developed, which are not listed here, and some of them have been treated in the cited literature.

[4] Other favorable directions are applied in eliminating the degeneration of the procedure known as wedging and in determining the *basic solutions* for the required values of the function $s(\underline{w}) = C$ (shown below).

Table 1 – Parameters of the point $\underline{w}^k$ and the nodes (r, t) for the NLP model

Таблица 1 – Параметры точки $\underline{w}^k$ и узлов (r, t) для модели НЛП

Табела 1 – Параметри тачке $\underline{w}^k$ и чворова (r,t) за модел НЛП

$j \equiv t$		1	$\dots$	n	sums*
$j \equiv r$	$d_r^{Ak} \diagdown d_t^{Bk}$	d_1^{Bk}	$\dots$	d_n^{Bk}	sums*
	$g_r^k \diagdown g_t^k$	g_1^k	$\dots$	g_n^k	
1	d_1^{Ak}	0	$\dots$	$d_{(1,n)}^k$	
	g_1^k	0	$\dots$	$g_{(1,n)}^k$	$\sum_{t=1}^{t=n} \lvert g_{(1,t)}^k \rvert$
	$\tau_{(r,t)}^k$	0	$\dots$	$\tau_{(1,n)}^k$	$\sum_{t=1}^{t=n} \lvert \tau_{(1,t)}^k \rvert$
	$z_{(r,t)}^k$	0	$\dots$	$z_{(1,n)}^k$	$\sum_{t=1}^{t=n} z_{(1,t)}^k = v_1^{Ak}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
n	d_n^{Ak}	$d_{(n,1)}^k$	$\dots$	0	
	g_n^k	$g_{(n,1)}^k$	$\dots$	0	$\sum_{t=1}^{t=n} \lvert g_{(n,t)}^k \rvert$
	$\tau_{(r,t)}^k$	$\tau_{(n,1)}^k$	$\dots$	0	$\sum_{t=1}^{t=n} \lvert \tau_{(n,t)}^k \rvert$
	$z_{(r,t)}^k$	$z_{(n,1)}^k$	$\dots$	0	$\sum_{t=1}^{t=n} z_{(n,t)}^k = v_n^{Ak}$
sums*		$\sum_{r=1}^{r=n} \lvert g_{(r,1)}^k \rvert$	$\dots$	$\sum_{r=1}^{r=n} \lvert g_{(r,n)}^k \rvert$	
		$\sum_{r=1}^{r=n} \lvert \tau_{(r,1)}^k \rvert$	$\dots$	$\sum_{r=1}^{r=n} \lvert \tau_{(r,n)}^k \rvert$	$\sum_{(r,t)} \lvert \tau_{(r,t)}^k \rvert$
		$\sum_{r=1}^{r=n} z_{(r,1)}^k = v_1^{Bk}$	$\dots$	$\sum_{r=1}^{r=n} z_{(r,n)}^k = v_n^{Bk}$	

*sums $g_{(r,t)}^k$ and $\tau_{(r,t)}^k$ are determined in relation to $(r,t) \in \Theta_-^k$ or $(r,t) \in \Theta_+^k$

<u>Minimum problem:</u> The direction of the fastest fall is the direction of the antigradient $-\nabla s(\underline{w}^k)$, that is, the direction of the antigradient vector in the active nodes for the point $\underline{w}^k$ (the gradient vector is the sum of the gradient vectors $g_{(r,t)}^k < 0; (r,t) \in \Theta_-^k$ of the active components); the values of the variables $z_{(r,t)}^k > 0, (r,t) \in \Theta_-^k$ (18,19) have the same

interrelationship (proportionality) as the values of the active components of the gradient[5]:

$$z^k_{(r,t)} = \mu^k |g^k_{(r,t)}|;(r,t) \in \Theta^k_-, \quad (28)$$

where $\mu^k > 0$, *the unique coefficient (proportionality) of weight increment for all active nodes,* depends on the active constraints in the nodes ($d^k_{(r,t)} > 0$) and the criteria ($d^{Ak}_j > 0$ and $d^{Bk}_j > 0$); it is necessary to ensure, when passing the point $\underline{w}^k \in \overline{E}$ to the boundary point $\underline{w}^{k*} \in \overline{E} \setminus E$: a) that the point $\underline{w}^{k*}$ (27) is a feasible point; b) that $\underline{w}^{k*}$ is in the most favorable direction; and, c) that the *active constraints* of the nodes and the criteria are met to the maximum (to achieve the maximum possible changes in the weight components v^k_j). The values of the coefficient μ^k are obtained on the basis of the following considerations:

1) Active constraints in the nodes $z^k_{(r,t)} \le d^k_{(r,t)} > 0, (r,t) \in \Theta^k_-$ (18): for the boundary case $z^k_{(r,t)} = d^k_{(r,t)} > 0$ and $z^k_{(r,t)} = \xi^k_{(r,t)} |g^k_{(r,t)}| = d^k_{(r,t)}; (r,t) \in \Theta^k_-$, where $\xi^k_{(r,t)} > 0$ is *the node coefficient*. The lowest value $\xi^k_{(r,t)} > 0$ in all nodes allows at least one resource $d^k_{(r,t)}$ - active constraint to be fully utilized and that $d^{k*}_{(r,t)} = 0$, based on which the *coefficient of active nodes* is determined $\xi^k > 0$:

a) $\xi^k_{(r,t)} = \begin{cases} d^k_{(r,t)} / |g^k_{(r,t)}| > 0; (r,t) \in \Theta^k_- \\ 0; \ (r,t) \notin \Theta^k_- \end{cases}$;

b) $\xi^k = min_{(r,t)} \{ \xi^k_{(r,t)} > 0; (r,t) \in \Theta^k_- \}$. (29)

2) Active constraints of arguments (criteria) $\sum_{t=1}^{t=n} z^k_{(j,t)} \le d^{Ak}_j > 0$ and $\sum_{r=1}^{r=n} z^k_{(r,j)} \le d^{Bk}_j > 0$ for $(j = r; j = t) \in J$ (19a,b): to move to the point $\underline{w}^{k*} \in \overline{E} \setminus E$ at least one of the active constraints to the criteria, which have a positive value $d^{Ak}_j > 0$ or $d^{Bk}_j > 0$ at the point $\underline{w}^k$, should be fully

[5] In the following text, for a simpler presentation, the components of the gradient $g^k_{(r,t)} < 0; (r,t) \in \Theta^k_-$ and $g^k_{(r,t)} > 0; (r,t) \in \Theta^k_+$ are replaced by $|g^k_{(r,t)}| > 0$ for $(r,t) \in \Theta^k_-$ *(min)* or $(r,t) \in \Theta^k_+$ *(max)*.

utilized and there should be $d_j^{Ak*}=0$ or $d_j^{Bk*}=0$ at least for one $j=J$. The weight components for one criterion (27) are $v_j^k=v_j^{Bk}-v_j^{Ak}$, where in the final outcome $v_j^{Ak}=0$ or $v_j^{Bk}=0$. The change of the component v_j^k depends on partial changes in the row and column of the same index $j=r=t$ in the matrix G^k: changes in the row $j=r$ are $\sum_{t=1}^{t=n} z_{(r,t)}^k = v_j^{Ak} = \psi_j^k \sum_{t=1}^{t=n} |g_{(r,t)}^k| > 0$, and in the column $j=t$ they are $\sum_{r=1}^{r=n} z_{(r,t)}^k = v_j^{Bk} = \psi_j^k [\sum_{r=1}^{r=n} |g_{(r,t)}^k| > 0$, where $\psi_j^k > 0$ is the coefficient of proportionality for the *jth*-criterion. It follows that $v_j^k = \psi_j^k [\sum_{r=1}^{r=n} |g_{(r,j)}^k| - \sum_{t=1}^{t=n} |g_{(j,t)}^k|]$, where the sums in the parentheses [.] can be connected by any sign (<, =,>), when three cases are possible: a) $v_j^k < 0 v_j^{Ak}$, followed by $v_j^k = -v_j^{Ak}$ and $v_j^{Bk}=0$; b) $v_j^k>0$, followed by $v_j^k=v_j^{Bk}$ and $v_j^{Ak}=0$; and, c) $v_j^k=v_j^{Bk}=v_j^{Ak}=0$. For boundary cases, when $v_j^{Bk}=d_j^{Bk}>0$ or $v_j^{Ak}=d_j^{Ak}>0$, the *coefficients of the criteria* (ψ_j^k) and the *coefficient of all criteria* (ψ^k) are obtained:

$$a)\ \psi_j^k = \begin{cases} d_j^{Ak} \big/ (\sum_{t=1}^{t=n} |g_{(j,t)}^k| - \sum_{r=1}^{r=n} |g_{(r,j)}^k|) > 0; & for\ \sum_{t=1}^{t=n} |g_{(j,t)}^k| > \sum_{r=1}^{r=n} |g_{(r,j)}^k|; \\ d_j^{Bk} \big/ (\sum_{r=1}^{r=n} |g_{(r,j)}^k| - \sum_{t=1}^{t=n} |g_{(j,t)}^k|) > 0; & for\ \sum_{r=1}^{r=n} |g_{(r,j)}^k| > \sum_{t=1}^{t=n} |g_{(j,t)}^k|; \\ \to \infty; \ \ for\ \sum_{t=1}^{t=n} |g_{(j,t)}^k| = \sum_{r=1}^{r=n} |g_{(r,j)}^k|; \ (r,j),(j,t) \in \Theta_-^k; \end{cases}$$

$$b)\ \psi^k = min_j \{ \psi_j^k > 0;\ j \in J \}. \qquad (30)$$

The transition from the point $\underline{w}^k \in \overline{E}$, which can be a boundary or an inner point, to the boundary point $\underline{w}^{k*} \in \overline{E} \setminus E$ in the direction of the fastest fall, is realized for the value of the *unique coefficient:*

$$\mu^k = min\{ \xi^k > 0; \psi^k > 0 \}, \quad for\ \ \Theta_-^k, \qquad (31)$$

where the values ξ^k and ψ^k can be associated with any sign (<,=,>). Expressions (27-31) are key to determining exact extreme solutions using active node parameters.

In the further procedure, the values of the variables $z_{(r,t)}^k \geq 0$ are calculated (28) as well as the values of the weight increments $v_j^{Ak} = \sum_{t=1}^{t=n} z_{(j,t)}^k \geq 0$ (the sum $z_{(j,t)}^k$ in the rows of the matrix Z^k) and $v_j^{Bk} = \sum_{r=1}^{r=n} z_{(r,j)}^k \geq 0$ (the sum $z_{(r,t)}^k$ in the columns of the matrix Z^k) (19), together with the components of the point weight $\underline{w}^{k*}$ (27), and then the TOPSIS solution $(\underline{w}^{k*};s^{k*})$(2-4) is determined.

Maximum problem: Expressions (27-31) and (2-4) are used to determine the solution $(\underline{w}^{k*};s^{k*})$, so that the nodes $(r,t)\in\Theta_{-}^{k}$ (25a) are replaced by the nodes $(r,t)\in\Theta_{+}^{k}$ (25b), and the matrices T_{-}^{k} - by the matrices T_{+}^{k}.

Optimality solution: In the continuation of the examination procedure, the obtained solution $(\underline{w}^{k*};s^{k*})$ is obtained according to two criteria:

- *basic criterion:*

a) $s^{k*}<s^{k}$ *(min); b)* $s^{k*}>s^{k}$ *(max);* (32)

- *optimality criterion:*

a) $\tau_{(r,t)}^{k*}=g_{(r,t)}^{k*}d_{(r,t)}^{k*}\geq 0;\ \ for\ \ every\ \ (r,t)\in\Theta^{k*}$ *(min)*, or

b) $\tau_{(r,t)}^{k*}=g_{(r,t)}^{k*}d_{(r,t)}^{k*}\leq 0;\ \ for\ \ every\ \ (r,t)\in\Theta^{k*}$ *(max),* (33)

when three cases can occur:

1) criterion (32) is met and criterion (33) is not met: the obtained solution is not extreme, but it is the initial solution $(\underline{w}^{k*};s^{k*})\equiv(\underline{w}^{k+1};s^{k+1})$ for the *(k+1)* iteration, when the node parameters in Table 1 are calculated and the procedure is repeated (27-33); the procedure is repeated until solutions are obtained according to cases 2 or 3;

2) criteria (32) and (33) are met: the exact extreme solution was reached at the vertex of the set $\overline{E}$;

3) criteria (32) and (33) are not met: the solutions $(\underline{w}^{k};s^{k})$ and $(\underline{w}^{k*};s^{k*})$ are the initial solutions for the procedure of determining the approximate extreme solution (the case when (32) is not fulfilled, and when it is fulfilled, (33) is impossible).

For the starting inner point of the weights $w^{k=0}\in E$, the number of active nodes $(r,t)\in\Theta_{-}^{0}$ and $(r,t)\in\Theta_{+}^{0}$ is equal to $n(n-1)/2$. If there is an exact extreme solution $(\underline{w}^{m};s^{m})$ or $(\underline{w}^{M};s^{M})$ (33) and, during the procedure, in each iteration, $\xi^{k}\geq\psi^{k}>0$ and $\mu^{k}=\psi^{k}$ (31), and there is no degeneration - oscillation, then the solution is achieved after the iteration $k=n-1$. After each $k=1,2,\cdots$ iteration, the number of active nodes decreases and is equal to $(n-k)(n-k-1)/2$. At the end of the procedure and after iterations, there is only one active node, and at the point $\underline{w}^{k*}$ after $k = n-1$ iterations, there are no active nodes: $\Theta_{-}^{k} = \varnothing$ (min) or $\Theta_{+}^{k} = \varnothing$ (max). From the beginning of the procedure in each subsequent iteration, the value of the possible total weight change $\sum_{(r,t)} |\tau_{(r,t)}^{k}|$ in the active nodes decreases; for the exact extreme solution is $\sum_{(r,t)} |\tau_{(r,t)}^{k*}| = 0$ and there is no possibility of further improvement of the function value. The number of iterations increases if oscillation or wedging (discussed below) appear "near" the hyperplane w_j^A or w_j^B.

Approximate extreme solutions

Determining approximate extreme solutions is necessary in the case when the extreme solution is not at the vertex of the set $\overline{E}$ and when criteria (32) and (33) are not met. One of possible solutions is the line search procedure[6], adjusted to node parameters, where the number of iterations should be as small as possible. The absence of an exact extreme solution is manifested in the iteration by which it would be obtained, when there is only one active node and when the value of the function s^{k*} is not better than the value s^{k}. Instead of $\sum_{(r,t)} |\tau_{(r,t)}^{k*}| = 0$, new active nodes appear that did not exist previously[7]; this means that on the direction $[\underline{w}^{k}, \underline{w}^{k*}]$ there is a value of the function that is better than s^{k} and s^{k*} (it does not have to be an extreme value, because it can be located at the point $\underline{w}$ which is not on the current direction). First, the known segment of the direction $[\underline{w}^{k}, \underline{w}^{k*}]$ is examined, and if a solution

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[6] Inaccurate linear search methods are widely discussed in the literature for nonlinear unconditional optimization problems (Zangvill, 1973; Bazaraa, et al., 2006; Luenberger & Ye, 2016) and they can also be applied to constraint problems or their adjustment is required. Although the procedures are known, due to the specific characteristics of the nodes, the whole procedure is given here.

[7] If, from the solution $(\underline{w}^{k*}; s^{k*})$, which is not better than the solution $(\underline{w}^{k}; s^{k})$, the already described procedure is continued by choosing the most favorable direction (27-31), due to the change of the gradient sign in previously active nodes, they will become inactive, and new active nodes will appear that did not exist at the beginning of the procedure - for the initial solution $(\underline{w}^{k}; s^{k})$. For the obtained new solution, the value of the function may be even better than the value s^{k*}, but criterion (33) will not be met; then, a new solution is obtained from this solution, etc., until after several iterations, the solution $(\underline{w}^{k*}; s^{k*})$ is obtained again, when the procedure begins to "circle" over the already obtained solutions. There does not have to be a solution $(\underline{w}^{k}; s^{k})$ among these solutions.

that meets the set criteria is not found on it, then, by applying the procedure based on the direction of the fastest fall (growth), a new search segment is determined.

The segment of the direction $[\underline{w}^k, \underline{w}^{k*}]$ is divided into several equal parts (subsegments): $[\underline{w}^k \equiv \underline{w}^{k,l=0}, \underline{w}^{k*} \equiv \underline{w}^{k,l=n^k}]$, where $l = 0,1,2,..,n^k$ is the mark of the points on the segment of the direction, and $n^k \geq 4$ is both the *number of equal search steps* and the number of equal subsegments (integer) which provides search for a sufficient number of points for smaller subsegments. The following points are generated on a segment of the direction $[\underline{w}^{k,0}, \underline{w}^{k,n^k}]$:

$$\underline{w}^{k,l} = \underline{w}^{k,0} + l\,\alpha^k\,\underline{v}^k; \quad for\ \ l = 1,2,...,n_k - 1; \tag{34}$$

where $\alpha^k \in (0;1/4)$ is the constant *size of the weight change step* in each of $n^k \geq 4$ equal steps in total; expression (34) is a linear combination of the points $\underline{w}^{k,0}$ and $\underline{w}^{k,n^k}$: $\underline{w}^{k,l} = (1 - l\,\alpha^k)\underline{w}^{k,0} + l\,\alpha^k\,\underline{w}^{k,n^k}$.

The criteria for accepting the approximate solution $(\underline{w}^{k,l}; s^{k,l})$ as an extreme solution and for stopping the iterative procedure are defined here in relation to the values of the function and the values of the arguments (criteria weights) for three consecutive iterative solutions:

- basic criterion:

$$a)\ \ s^{k,l-1} > s^{k,l} < s^{k,l+1};\ (min),$$

$$b)\ \ s^{k,l-1} < s^{k,l} > s^{k,l+1};\ (max); \tag{35}$$

- argument value criterion:

$$max_j\{|\,w_j^{k,l} - w_j^{k,l-1}|;|\,w_j^{k,l+1} - w_j^{k,l}\,|\} \leq \varepsilon_w\,; \tag{36}$$

- function value criterion:

$$max\{|\,s^{k,l} - s^{k,l-1}|;|\,s^{k,l+1} - s^{k,l}\,|\} \leq \varepsilon_s\,; \tag{37}$$

where $l = 1,2,...,n^k - 1$, and $\varepsilon_w, \varepsilon_s > 0$ are the parameters of small values.

The number of steps n^k and the size of the steps α^k are determined depending on the selected parameter ε_w (36):

$$n^k \geq (max_j\{|\,w_j^{k,n^k} - w_j^{k,0}\,|\}/\varepsilon_w) + 3\ \ (n^k \text{ - the first major integer}); \tag{38}$$

$$\alpha^k = 1/n^k\,,\quad \alpha^k \in (0;1/4)\,. \tag{39}$$

whereby criterion (36) is met. The increments of the weight components $(\alpha^k v_j^k)$ are constant, and, with $n^k \geq 4$, it is ensured that at least three points $\underline{w^{k,l}}$ are determined on each segment $[\underline{w^{k,0}},\underline{w^{k,n^k}}]$ and the value of the limit parameter is achieved:

$$\varepsilon_w^k = max_j\{|w_j^{k,n^k} - w_j^{k,0}|\}/n^k) \leq \varepsilon_w . \quad (40)$$

The choice of parameter values ε_w and ε_s can be based on the sensitivity of the value of the function to changes in the value of arguments in the extremum environment. Although the criteria for stopping the optimization process can be based on the norms of the arguments $||\underline{w^{k+1}} - \underline{w^k}||/||\underline{w^k}|| \leq \varepsilon_w$ and the function values $|s(\underline{w^{k+1}}) - s(\underline{w^k})|/s(\underline{w^k}) \leq \varepsilon_s$ in two consecutive iterations, for the current NLP problem, the parallel application of criteria (36,37) in three iterations is favorable. Criterion (36) limits the largest individual weight changes via the parameter ε_w (40), so that the largest weight increment is $max_j \alpha^k v_j^k = \varepsilon_w^k \leq \varepsilon_w$. *The sensitivity of the value of a function* (o_s^k), as a criterion for the selection of parameters ε_w and ε_s, can be defined as the ratio of changes in the value of the function and the maximum change of individual weights, i.e. as the ratio of the realized parameters $\varepsilon_w^k \leq \varepsilon_w$ and ε_s^k for a certain subsegment: $o_s^k = \varepsilon_s^k/\varepsilon_w^k$. Numerical results for TOPSIS solutions show greater sensitivity of argument values (weight) than function values, because it is $o_s^k = \varepsilon_s^k/\varepsilon_w^k < 1$ or $\varepsilon_s^k < \varepsilon_w^k$, which should focus on the choice of the parameter ε_w. The parameters ε_w and ε_s can also be determined depending on the required accuracy of the values of arguments and functions: in order to round the values of weights and the function of the target to four exact decimal digits, it is enough to set $\varepsilon_w = 5 \cdot 10^{-5}$, which ensures $\varepsilon_s^k < 5 \cdot 10^{-5}$.

In the set of solutions $(\underline{w^{k,l}}, s^{k,l}; l = 1,2,...,n_k - 1)$, there does not have to be a solution for which $s^{k,l} < s^{k,0}$ *(min)* or $s^{k,l} > s^{k,0}$ *(max)*, because such a solution can also be in the first subsegment $[\underline{w^{k,0}}; \underline{w^{k,l=1}}]$. Therefore, for the sake of generality of the procedure, a point $\underline{w^{k,a}}$ is determined on the segment $[\underline{w^{k,0}},\underline{w^{k,n^k}}]$ in which the function has the value:

$$a)\ s^{k,a} = min_l\{s^k;\ l = 1,2,....,a-1,a,a+1,...,n^k - 1\}\ \textit{(min)},$$

$$b)\ s^{k,a} = max_l\{s^k;\ l = 1,2,....,a-1,a,a+1,...,n^k - 1\}\ \textit{(max)}. \quad (41)$$

There is an improved solution on the subsegment $[\underline{w^{k,a-1}};\underline{w^{k,a+1}}]$: according to the selected ε_w and for $n^k \geq 4$ which provides at least four new intervals, determine the required parameters (38-40), search the subsegment and determine the TOPSIS solution according to the criteria (35,37).

The disadvantage of the procedure based on fulfilling criterion (36) is a large number of generated $\underline{w^{k,l}}$ points on the segment $[\underline{w^{k,0}},\underline{w^{k,n^k}}]$, especially for larger interval widths, when the values of the $s^{k,l}$ function and other required quantities need to be calculated for several hundred generated $\underline{w^{k,l}}$ points.

For the solved MADM problem procedure will satisfy criteria (35-37), knowing that due to the nonlinearity of the function gradient in the nodes $g(z^k_{(r,t)})$ and the appearance of new active nodes, the exact solution will be outside the direction $[\underline{w^k};\underline{w^{k*}}]$. For the sake of generality of the procedure and reduction of error according to criteria (36,37), a two-phase procedure can be applied: a) linear search and determination of the best solution in the segment $[\underline{w^k};\underline{w^{k*}}]$, it can also be the solution $[\underline{w^{k+1}}=\underline{w^{k,a}};s^{k,a}]$; and, b) determining a new segment between the obtained solution and the solution on the hyperplanes w_j^A or w_j^B. If the point $\underline{w^{k+1}}$ is where the best solution is achieved on the segment $[\underline{w^k};\underline{w^{k*}}]$, a new direction of the fastest fall (growth) is set through it and a point $\underline{w^{(k+1)*}}$ is determined. On the new segment $[\underline{w^{k+1}};\underline{w^{(k+1)*}}]$, the value of the function is calculated at the newly generated points, etc. The two-phase procedure is repeated until criteria (35,37) are met. In general, an approximately extreme solution was obtained in two steps with a smaller error, but the procedure is much longer.

Procedure degenerations: wedging and oscillation

The presented procedure for determining a point on one hyperplane requires that in each iteration at least one component of weight w_j^k, which is different from the limit values $w_j^A < w_j^k < w_j^B$, have a value $w_j^{k*} = w_j^A$ or $w_j^{k*} = w_j^B$ at the new point $\underline{w^{k*}}$ and is in the direction of the fastest fall (growth). These requirements cannot be met if wedging occurs: starting from some *k*-th iteration and a point $\underline{w}^k$ that can be an inner or boundary point, through all subsequent iterations, the points "accumulate" "near" the hyperplane w_j^A or w_j^B and some expected boundary point $\underline{w}^{k*}$ which cannot be reached in a finite number of iterations. The expected point $\underline{w}^{k*}$ cannot be an extreme point, because wedging does not appear in the iteration in which the extreme solution is obtained; this iteration is preceded by only one active node, and for the occurrence of wedging there must be two or more active nodes, which facilitates the elimination of the problem. The possibility of wedging cannot be established at the initial stage of the proceedings.

Wedging occurs (but not necessarily) when the coefficient of active nodes ξ^k (29) is smaller than the coefficient of all criteria ψ^k (30): $\mu^k = \xi^k < \psi^k$ (31). As a consequence, none of the active constraints under criteria (19a, b) that had a positive value $d_j^{Ak} > 0$ or $d_j^{Bk} > 0$ in the point $\underline{w}^k$, will be fully utilized to achieve that $d_j^{Ak*} = 0$ or $d_j^{Bk*} = 0$ for at least one $j = J$. Through several iterations, the unique coefficient $\mu^k = \xi^k < \psi^k$ (31) decreases and $\mu^k \underset{k\to\infty}{\to} 0$, due to successive reduction $\xi^k = \xi^k_{(r,t)} > 0$. In each subsequent iteration, the values of the weight changes per node $z^k_{(r,t)} = \mu^k |g^k_{(r,t)}|$ decrease as well as the increment of the value of the objective function. When the process enters wedging, the number and indices of active nodes do not change, so that all the components of the direction vector that were active at the starting point $\underline{w}^k$ exist.

The $\mu^k = \xi^k < \psi^k$ disorder can disappear by the algorithm: if in some subsequent iteration in the second node (p,q) $\xi^k = \xi^k_{(p,q)} < \xi^k_{(r,t)}$ and $\xi^k_{(p,q)} \geq \psi^k_j$ are achieved, then the most favorable direction towards $\xi^k_{(p,q)}$ will be chosen, and the active node (r,t) will have no effect or will become an inactive node.

Unlike some possible ways to solve the problem of wedging, e.g. ε-*approximation* (Zangville, 1969) for the current problem of NLP and based on relative independence of active nodes, a procedure based on changing the direction can be applied: a new direction is chosen from a set of favorable directions which exclude individual active nodes. In any iteration after the occurrence of wedging, at the obtained point $\underline{w}^k$ close to the expected boundary point $\underline{w}^{k*}$, a new and less favorable direction is selected based on the characteristics of all active nodes except the node *(r, t)* which is the cause of wedging and is found to be $\mu^k = \xi^k_{(r,t)}$. By setting $g^k_{(r,t)} = 0$ for that node (current iteration only), the node (r,t) is excluded from the set of active nodes (25) and the new direction is selected in accordance with the characteristics of the remaining active nodes using expression (29-31). This does not disturb the procedure, because a favorable direction is chosen, but after obtaining a new boundary point $\underline{w}^{k*}$, the current value $g^{k+1}_{(r,t)} \neq 0$ must be returned to the procedure for the node (r,t), so that the node (r,t) can become an active node again if it satisfies conditions (25), when again the most favorable direction is chosen (29-31). There is a possibility that in several consecutive iterations the selected favorable directions must be changed and the number of active nodes successively reduced, until at some step it is obtained that it is $\mu^k = \psi^k \leq \xi^k > 0$, when the procedure continues by determining the $(\underline{w}^{k*}; s^{k*})$ solution, and the omitted nodes return to the procedure.

As the extreme solution does not depend on the initial solution or on the secondary solutions that are a consequence of choosing one of the favorable directions (based on the characteristics of all or only some active nodes), it gives the possibility to simplify the problem of wedging by: *always when* $\mu^k = \xi^k = \xi^k_{(r,t)} < \psi^k$, set the $g^k_{(r,t)} = 0$ for the node (r,t) and select another favorable direction, and in the *(k+1)* iteration, include the calculated value of $g^{k+1}_{(r,t)} \neq 0$ in the procedure of selecting the most favorable direction. This prevents the occurrence of wedging, that is, automates its removal, and the whole procedure is extended by several iterations at the most.

A special form of degeneration can be described as an *oscillation* of the value of the w^k_r component in the vicinity of a hyperplane. For example: in the *k*-th iteration for the *r*-th component of the point $\underline{w}^k$, the value $w^k_r = w^A_r$ is reached. In the normal course of the procedure, the already reached values on the hyperplane w_j^A or w_j^B do not change until the end of the procedure, which is not the case with oscillations: instead of $w_r^k = w_r^{k+1} = w_r^A$, the $w_r^{k+1} > w_r^A$ value close to w_r^A is obtained in the *(k+1)* iteration, while for some other component the boundary value $w_t^{k+1} = w_t^A$ or $w_t^{k+1} = w_t^B$ is reached. In one of the following iterations, the value of w_j^A is reached again for the r-th component, but wedging can also occur. The oscillation can also be repeated on the same hyperplane, when the $w_r^{k+1} - w_r^A$ differences also decrease. Apart from increasing the number of iterations, the oscillations do not affect the final solution, and the wedging is removed in the described way.

Partial stability of solutions

The stability of the MADM solution – the variant $V_p|(\underline{w}; s_p(\underline{w})); p \in I$ *is defined in relation to all other variants* $V_q|(\underline{w}; s_q(\underline{w}); q \in I \setminus \{p\}$ *and represents the set of points* $\underline{w} \in E_p \subseteq \overline{E}$ *for which it is* $s_p(\underline{w}) > s_q(\underline{w}); p \in I; q \in I \setminus \{p\}$.

The partial stability of the solution $V_p|(\underline{w}; s_p(\underline{w}))$ is determined in relation to one of the other variants $V_q; q \in I \setminus \{p\}$: *The solution* $V_p|(\underline{w}; s_p(\underline{w}))$ *is stable in relation to the solution* $V_q|(\underline{w}; s_q(\underline{w}))$ *for all points* $\underline{w} \in E_{pq} \subseteq \overline{E}$ *for which the function of partial stability is:*

$$h_{pq}(\underline{w}) = s_p(\underline{w}) - s_q(\underline{w}) > 0; \quad p,q \in I, p \neq q, \tag{42}$$

that is, if it is $s_p(\underline{w}) / s_q(\underline{w}) > 1$. For $s_p^m > s_q^M$, the solution $(\underline{w}; s_p(\underline{w}))$ is stable in relation to the solution $(\underline{w}; s_q(\underline{w}))$ at each point $\underline{w} \in \overline{E}$ (Figure 1a) and therefore the determination of partial stability makes sense only if $\overline{S_P} \cap \overline{S_q} \neq \varnothing$, ie, if $s_p^M > s_q^m \wedge s_p^m < s_q^M$ (Figure 1: b,c,d,e). The function $h_{pq}(\underline{w})$ is concave or convex, continuous and differentiable on the convex set $\overline{E}$, and the local extreme is also the global extreme.

The set of values of the function is $\overline{S_{pq}} = \{h_{pq} \in [h_{pq}^m; h_{pq}^M]\} \subseteq \Re$ for $p,q \in I$ and $p \neq q$ or the line segment $\overline{h_{pq}^m; h_{pq}^M}$.

The extremes of the function $h_{pq}(\underline{w})$ follow from the function development $h_{pq}(\underline{w^{k+1}})=h_{pq}^{k+1}$ into the Taylor's polynomial of the first degree (12): $h_{pq}^{k+1}=(s_p^k-s_q^k)+\sigma_{pq}^k$, where $s_p^k-s_q^k=C$. The auxiliary function of σ_{pq}^k, according to (13,21), is equal to:

$$\sigma_{pq}^k=\sum_{(r,t)}(g_{p(r,t)}^k-g_{q(r,t)}^k)z_{(r,t)}^k=\sum_{(r,t)}g_{pq(r,t)}^k z_{(r,t)}^k;\ (r,t)\in\Theta. \quad (43)$$

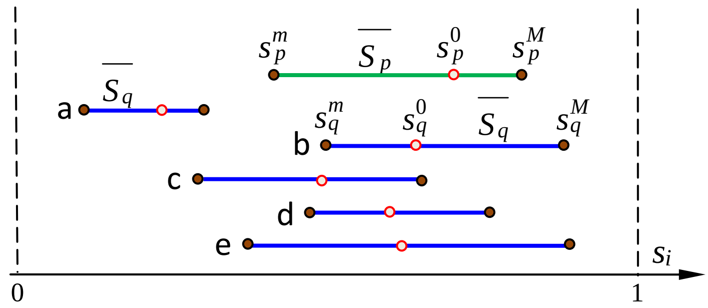

Figure 1 – Possible relations of the set of function values $s_i(\underline{w})$ of two variants

Рис. 1 – Возможные отношения множеств значений функций $s_i(\underline{w})$ двух вариантов

Слика 1 – Могући односи скупова вредности функција $s_i(\underline{w})$ две варијанте

The values of the function gradient components $h_{pq}(\underline{w})$ in the nodes $(r,t)\in\Theta$ are:

$$g_{pq(r,t)}^k=\begin{cases}g_{p(r,t)}^k-g_{q(r,t)}^k;\ (r,t)\in\Theta^k\\ 0;\ \ r=t,\ [(r,t)\notin\Theta^k]\end{cases} \quad (44)$$

and the elements are antisymmetric matrices $G_{pq}(\underline{w}^k)=G_{pq}^k=(g_{pq(r,t)}^k)_{nxn}$. To determine the extremes, it is sufficient to determine the points of weight $\underline{w}$ in which the nonlinear auxiliary function $\sigma_{pq}(\underline{w})$ (43) has extreme values, and then the corresponding TOPSIS solutions. Starting from an admissible solution (let that be the solution at the starting point $(\underline{w}^0;h_{pq}^0)$), by applying the presented procedure for determining the extremum of the function $s_i(\underline{w})$ (27-41) and by calculating the values $g_{pq(r,t)}^k$ (22, 44), d_r^{Ak}, d_t^{Bk} and $d_{(r,t)}^k$ (9, 17) in each iteration, the extreme solutions of $(\underline{w}^m; h^m_{pq} < 0)$ and $(\underline{w}^M; h^M_{pq} > 0)$ are obtained.

The separating hyperplane of the value set of the function $h_{pq} = C^R$ is defined by any value[8] of the $h = C^R \mid h^m < C^R < h^M$ function and divides the value set of the function $\overline{S} = \{h \in [h^m; h^M]\} \subseteq \Re$ into two subsets: $\overline{S^-} = \{h \mid h^m \le h \le C^R\} \subseteq \overline{S}$ and $S^+ = \{h \mid h^M \ge h > C^R\} \subseteq \overline{S}$, where $\overline{S^-} \cup S^+ = \overline{S}$ and $\overline{S^-} \cap S^+ = \varnothing$. An acceptable approximate value of the function in the vicinity of the separating hyperplane of $h = C^R$ is:

$$h^C = C \in [C^R \pm \varepsilon_C]; \quad k = 0,1,2\ldots, \tag{45}$$

where ε_C is a low value parameter (e.g. $\varepsilon_C = 5 \cdot 10^{-5}$ or less). The procedure for determining the set of solutions on the separating hyperplane of $(\underline{w}; h^C = C)$ has several phases, where it is determined: the set of *2n* boundary solutions and their extremes; basic boundary solutions for $h^C = C$ (if any) or basic solutions that are closest to the current hyperplane w^A_j or w^B_j; and, a set of solutions for $h^C = C$ on the set $\overline{E_C} \subseteq \overline{E}$. Accordingly, the partial stability of the variant V_p with respect to V_q is achieved for all points of $\underline{w} \in \overline{E}$ with the corresponding $(\underline{w}; (h > 0) \in S^+))$ solutions.

Boundary solutions

Boundary solutions are a set of solutions on one hyperplane of the set $\overline{E}$ (w^A_j or w^B_j). On the hyperplane w^A_j, these are $(\underline{w}^{a\lambda}; h^{a\lambda})$ solutions for the points $\underline{w}^{a\lambda} = (w^{a\lambda}_1, \cdots, w^{a\lambda}_\lambda = w^A_j, \cdots, w^{a\lambda}_n)$ (due to unambiguous indexing, additional $\lambda = \overline{1,n} \in J$ marks are introduced). If condition (7) is met, then there is at least one boundary point with the component $w^{a\lambda}_\lambda = w^A_\lambda$.

Based on the relative independence of the variables $z^0_{(\lambda,j)} \ge 0$ (18) and the built-in normalization condition (19) in each node, the $w^{a\lambda}_\lambda$

[8] In the following text, the *"pq"* indices were used only if it was necessary due to unambiguity.

component is determined on the basis of any starting point $\underline{w} \in \overline{E}$ - let that be the point $\underline{w}^0$: the choice of variables $z^0_{(\lambda,j)} \leq d^0_{(\lambda,j)} > 0$ eliminates the active constraint of the λ-criterion $d^{A0}_{\lambda} > 0 \rightarrow d^{A1}_{\lambda} = 0$ and achieves the required condition: $\sum_{j=1}^{j=n} z^0_{(\lambda,j)} = d^{A0}_{\lambda}; (\lambda, j) \in \Theta^0$. Nodes (λ, j) and the corresponding variables $z^0_{(\lambda,j)} \leq d^0_{(\lambda,j)} > 0$ (the λ row of the matrix T^0) can be chosen at will until the specified condition is met, or preference can be given to a node in which $z^0_{(\lambda,j)} = d^0_{(\lambda,j)} = max_j\{d^0_{(\lambda,j)}\}$; if it is fulfilled that $z^0_{(\lambda,j)} = d^{A0}_{\lambda}$, then also $w^{a\lambda}_{\lambda} = w^A_{\lambda}$, otherwise the procedure should be continued by selecting a next node from the λ row of the matrix T^0 and by adding the necessary difference, until it is fulfilled that $\sum_{j=1}^{j=n} z^0_{(\lambda,j)} = d^{A0}_{\lambda}$.

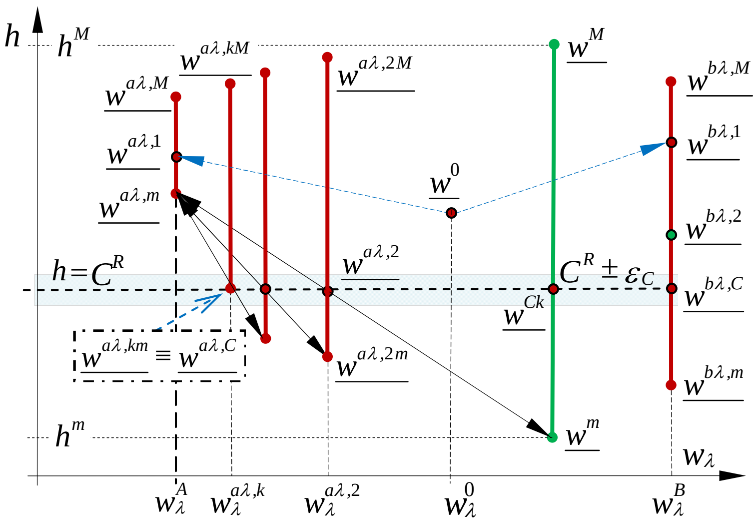

Figure 2 – Overview of the procedure for determining the basic solutions on hyperplanes

Рис. 2 – Обзор процедуры определения базовых решений на гиперплоскостях

Слика 2 – Приказ поступка одређивања основних решења на хиперравнима

Practically, in all rows of the matrix G^0 except in the λ row, $g^0_{(j1,j)} = 0; j \in J; j1 \in J \setminus \{\lambda\}$ should be set and in the active nodes (λ, j) the values of $z^0_{(\lambda,j)} \leq d^0_{(\lambda,j)} > 0$ should be determined until the condition

$\sum_{j=1}^{j=n} z^0_{(\lambda,j)} = d^{A0}_\lambda$ is fulfilled. By applying expression (2-4) for the point $\underline{w^{a\lambda}} = \underline{w^{a\lambda,1}}$, the TOPSIS solution $(\underline{w^{a\lambda,1}}; h^{a\lambda,1})$ is obtained. In the same way, the solution of $(\underline{w^{b\lambda,1}}; h^{b\lambda,1})$ on the hyperplane w^B_j was obtained by eliminating the active constraint $d^{B0}_\lambda > 0 \rightarrow d^{B1}_\lambda = 0$ based on the choice of the variable $z^0_{(j,\lambda,} \leq d^0_{(j,\lambda)} > 0$ from which the $\sum_{j=1}^{j=n} z^0_{(j,\lambda)} = d^{B0}_\lambda$ (the λ column of the matrix T^0) was obtained.

Extreme solutions on the hyperplanes w^A_j and w^B_j (extreme boundary solutions) are the exact solutions at the vertex of the set $\overline{E}$ and are determined by applying the presented method for the extremes of the function $s(\underline{w})$. The starting point is the obtained solutions on the hyperplanes: e.g. for the hyperplane w^A_j, the initial solution is $(\underline{w^{a\lambda,1}}; h^{a\lambda,1})$; the component $w^{a\lambda,1}_\lambda = w^A_\lambda$ retains its value, which requires that the nodes that affect its value must become inactive.

From the starting point $\underline{w^{a\lambda,1}}$, the most favorable direction is chosen in accordance with the characteristics of the active nodes (the set Θ^1_- for the minimum or the set Θ^1_+ for the maximum) that are not in the λ row and the λ column of the matrix T^1: set $g^1_{(\lambda,j)} = g^1_{(j,\lambda)} = 0$ for all nodes *(r,t)* in the row of λ and the column of λ, and the further procedure for determining the extremes $(\underline{w^{a\lambda,m}}; h^{a\lambda,m})$(min) and $(\underline{w^{a\lambda,M}}; h^{a\lambda,M})$(max) is identical to the procedure shown for determining the exact extreme solutions of the function $s(\underline{w})$ (27-31). The interval of the value of the function $[h^{a\lambda,m}; h^{a\lambda,M}]$ on the hyperplane w^A_λ cannot be greater than the interval of the extremum of the function $h(\underline{w})$: $h^{a\lambda,M} - h^{a\lambda,m} \leq h^M - h^m$, Figure 2. In the same way, the extreme solutions on the hyperplane w^B_λ are determined. Extreme boundary solutions (maximum *4n* solutions), among which are the extreme solutions $h(\underline{w})$ of the function, are unique in terms of the corresponding points $\underline{w}$ and the values of the function $h(\underline{w})$. Not all extreme points ($\underline{w^{a\lambda,m}}$ and $\underline{w^{a\lambda,M}}$) are linearly independent; by eliminating the linearly dependent points, a set of points $\underline{w^v}$ is obtained which are the vertices of the set $\overline{E}$.

System of basic solutions for the separating hyperplane $h=C^R$

A *system of basic solutions* (SBS_C) is established for the value of the function $h=C$, which contains *2n* basic solutions: *n* solutions $(\underline{w^{a\lambda,C}}; h^{a\lambda,C}=C)$ and *n* solutions $(\underline{w^{b\lambda,C}}; h^{b\lambda,C}=C)$. These are the *basic boundary solutions* for the point $\underline{w^{a\lambda,C}}$ if $w_\lambda^{a\lambda,C}=w_\lambda^A$ (or $\underline{w^{b\lambda,C}}$ for $w_\lambda^{b\lambda,C}=w_\lambda^B$) or the *basic non-boundary solutions* that are closest to the current hyperplane w_λ^A or w_λ^B with $w_\lambda^{a\lambda,C}>w_\lambda^A$ or $w_\lambda^{b\lambda,C}<w_\lambda^B$:

a) On the hyperplane w_λ^B (or w_λ^A), there is an edge solution $(\underline{w^{b\lambda}}; h^{b\lambda}=C^R)$ (Figure 2, hyperplane w_λ^B) because it is $h^{b\lambda,m}\le C^R\le h^{b\lambda,M}$ and it is determined by repeatedly halving the segment $[\underline{w^{b\lambda,m}}; \underline{w^{b\lambda,M}}]$ until it is achieved that it is $h^{b\lambda,C}=C$. The segment $[\underline{w^{b\lambda,m}}; \underline{w^{b\lambda,M}}]$ is halved for the point $\underline{w^{b\lambda,2}}=0.5\underline{w^{b\lambda,m}}+0.5\underline{w^{b\lambda,M}}$ as well. The TOPSIS solution, in general, has the value $h^{b\lambda,2}\ne C$; in the next iteration, the segment of the interval in which $h=C^R$ ($[\underline{w^{b\lambda,m}}; \underline{w^{b\lambda,2}}]$ or $[\underline{w^{b\lambda,2}}; \underline{w^{b\lambda,M}}]$) is halved and the TOPSIS value is calculated, etc. The procedure is interrupted when $h^{b\lambda,k}=C^k\in[C^R\pm\varepsilon_C]$ is obtained in the k-th iteration and that solution is accepted as *the basic boundary solution* $[\underline{w^{b\lambda,C}}; h^{b\lambda,C}]$; for a larger number of iterations, a smaller absolute error $\varepsilon_C^k=|C^R-h^{b\lambda,C}|<\varepsilon_C$ was obtained.

b) On the hyperplane w_λ^A (or w_λ^B), there is no solution $(\underline{w^{a\lambda}}; h^{a\lambda}=C^R)$ because $C^R\notin[h^{a\lambda,m}; h^{a\lambda,M}]$. To achieve the $h=C$ value, the point $\underline{w^{a\lambda}}$ cannot have a value of $w_\lambda^{a\lambda}=w_\lambda^A$, but a value of $w_\lambda^{a\lambda}>w_\lambda^A$. These solutions are not boundary based on $w_\lambda^{a\lambda}=w_\lambda^A$, but are closest to the hyperplane w_λ^A (basic non-boundary solutions). The position of C^R can be $C^R<h^{a\lambda,m}$ (as in Figure 2) or $C^R>h^{a\lambda,M}$. For the situation in Figure 2, in order to determine the $w_\lambda^{a\lambda}>w_\lambda^A$ component, the point $w_\lambda^{a\lambda,2}>w_\lambda^A$ is first determined by iterative halving of the segment between two known points at which there is a solution with $h=C^R$ until the value $h^{a\lambda,k}=C$ is reached: for $h^{a\lambda,m}>C^R+\varepsilon_C$ value $h^{a\lambda,2}=C^R$ it is on the segment $[\underline{w^m};\underline{w^{a\lambda,m}}]$, and for $h^{a\lambda,M} < C^R - \varepsilon_C$, on the segment $[\underline{w^{a\lambda,M}};\underline{w^M}]$. From the point $\underline{w^{a\lambda,2}}$, for the constant value $w_\lambda^{a\lambda,2}$ (in the row of λ and the column of λ the matrix G^2, set $g^2_{(\lambda,j)} = g^2_{(j,\lambda)} = 0$) and determine the minimum $(\underline{w^{a\lambda,2m}};h^{a\lambda,2m})$. If the obtained solution does not satisfy condition (45) and if it is $h^{a\lambda,2m} < C^R - \varepsilon_C$, by repeatedly halving the new segment $[\underline{w^{a\lambda,2m}};\underline{w^{a\lambda,m}}]$, a new point $\underline{w^{a\lambda,3}}$ and a minimum $(\underline{w^{a\lambda,3m}};h^{a\lambda,3m} < C^S - \varepsilon_C)$ are obtained, etc. The procedure ends when the solution $(\underline{w^{a\lambda,km}};h^{a\lambda,km} = C) \equiv (\underline{w^{a\lambda,C}};h^{a\lambda,C} = C)$ is obtained in the *k*-th iteration, from which it is not possible to further move the point $\underline{w^{a\lambda,km}}$ towards the hyperplane w_λ^A provided that $h = C$, which determines the basic non-boundary solution $(\underline{w^{a\lambda,km}};h^{a\lambda,km} = C) \equiv (\underline{w^{a\lambda,C}};h^{a\lambda,C})$ (based on other weight components $w_j;\, j \in J \setminus \{\lambda\}$, these solutions can also be boundary) [9].

On the hyperplane w_λ^A, if $h^{a\lambda,M} < C^R - \varepsilon_C$, the procedure for determining the solution is similar: the segment $[\underline{w^{a\lambda,M}};\underline{w^M}]$ is considered; the relevant points are $\underline{w^M}$ and $\underline{w^{a\lambda,kM}}$; the non-boundary solution is $(\underline{w^{a\lambda,kM}};h^{a\lambda,kM} = C) \equiv (\underline{w^{a\lambda,C}};h^{a\lambda,C})$.

The presented procedure yields *n* solutions $(\underline{w^{a\lambda,C}};h^{a\lambda,C})$ and *n* solutions $(\underline{w^{b\lambda,C}};h^{b\lambda,C})$ that make up the SBS_C for the value of the function $h = C^R \pm \varepsilon_C$. Due to the components λ of the points $\underline{w^{a\lambda,C}}$ and $\underline{w^{b\lambda,C}}$, the points of solution $\underline{w}$ in SBS_C are mutually linearly independent.

A set of solutions for the separating hyperplane $h = C^R$: From SBS_C linear combinations of the weight points $\underline{w^{a\lambda,C}}$ and $\underline{w^{b\lambda,C}}$, countless new

[9] In the numerical example, the solution $(\underline{w^{b4}};h^{b4,0} = 0)$ is the basic non-boundary solution because $(w_4^{b4} = 0.1681) < (w_4^B = 0.1830)$ although $\underline{w^{b4}} \in \overline{E} \setminus E$ is a boundary point due to $w_1^{b4} = w_1^A = 0.0990$.

weight points can be obtained based both on them and on the solutions with $h=C$:

$$\underline{w^{ab}}=\sum_{j=1}^{j=n}\beta_j^a \underline{w^{aj}}+\sum_{j=1}^{j=n}\beta_j^b \underline{w^{bj}};\ \ \sum_{j=1}^{j=n}\beta_j^a+\sum_{j=1}^{j=n}\beta_j^b=1;\ \ j\in J\ , \qquad (46)$$

where all coefficients $\beta_j^a,\beta_j^b\geq 0$ are not equal to 0, and can be selected according to different criteria. Due to the nonlinearity of the $h(\underline{w^{ab}})$ function, by using the points $\underline{w^m}$ and $\underline{w^M}$ and/or other known points, a satisfactory solution $(\underline{w^{ab,C}};h^{ab,C})$ can be determined by the line search in the vicinity of the point $\underline{w^{ab}}$.

The procedure completely defines the set of all solutions of the function $h(\underline{w})$ (graph of the function): based on the most *4n* extreme boundary solutions (whose points $\underline{w}$ are not linearly independent), a set of linearly independent points $\underline{w^v}$ is singled out, which are also all vertices of the set $\overline{E}$. Other TOPSIS solutions can be determined on the basis of linear combinations of points on the vertices of the set $\overline{E}$. For each criterion $j\in J$ and any value $w_j\in[w_j^A,w_j^B]$, weight points can be determined for which the function $h_{pq}(\underline{w})$ has a maximum and minimum value, as well as points $\underline{w}$ for all values of the function from that interval. If some values of the weight point components are set to a predetermined and allowable value (maximum *n-2* values), it is possible to determine the solution for the required value of $h_{pq}(\underline{w})=C$ based on the parameters of the remaining active nodes. By combining multiple SBS_C solutions for different $C^{\Re}\in[h_{pq}^m;h_{pq}^M]$ values, solutions with a range of $h_{pq}\in[C_1^{\Re};C_2^{\Re}]$ values and stricter criteria for weight component values can be determined.

Numerical example

By applying the TOPSIS method to the MADM problem given by the initial matrix $C=\{c_{ij};\ i=\overline{1,5},\ j=\overline{1,6}\}$, for the weight point $\underline{w^0}$ and the coefficients of the linear combination $\chi_{p,6}=(0.5717;\ 0.2647;\ 0.1636)$ for $p=1,2,\infty$, the basic solution was obtained: the variant V_2 $(\underline{w^0};s_2^0=0.6209)$ and the rank $V_2-V_3-V_5-V_1-V_4$ (1-5) (Table 2).

Table 2 – Criteria Matrix, basic solution and extreme solutions
Таблица 2 – Матрица критериальных значений, базовые решение и экстремальные решения
Табела 2 – Матрица критеријумских вредности, основно решење и екстремна решења

j / i		criteria						$s_i(\underline{w}^0)$ (rank)	$s_i(\underline{w}^m)$	$s_i(\underline{w}^M)$
		$K_1(+)$	$K_2(-)$	$K_3(+)$	$K_4(-)$	$K_5(+)$	$K_6(-)$			
variants	V_1	415	85	1112	60	1.42	11.9	0.4348 (4)	0.4107	0.4645
	V_2	432	94	970	35	1.71	15.2	0.6209 (1)	0.5846	0.6518
	V_3	405	77	1015	55	1.88	14.6	0.6058 (2)	0.5812	0.6366
	V_4	352	62	1055	54	1.06	13.8	0.3522 (5)	0.3248	0.3838
	V_5	328	78	1045	38	1.43	17.5	0.4997 (3)	0.4717	0.5214
$\underline{w}^A$		0.099	0.132	0.237	0.147	0.208	0.088	Set of function values $s_i(w)$		
$\underline{w}^B$		0.134	0.161	0.273	0.183	0.241	0.105			
$\underline{w}^0$		0.112	0.144	0.258	0.167	0.223	0.096			
d_j^{A0}		0.013	0.012	0.021	0.020	0.015	0.008			
d_j^{B0}		0.022	0.017	0.015	0.016	0.018	0.009			
d_j^{AB}		0.035	0.029	0.036	0.036	0.033	0.017			

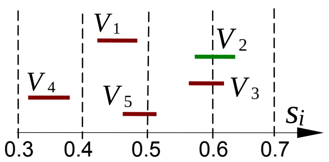

Table 2 provides data for the weight range limits w_j^A and w_j^B, the initial active limits d_j^{A0} and d_j^{B0} (9) for $k=0$, and the extreme values of $s_i(\underline{w}^m)$ and $s_i(\underline{w}^M)$. Based on the characteristics of the formed nodes (Table 1), exact extreme solutions were obtained at the vertices of the set $\overline{E}$ (27-31, 2-4) when the optimality criterion (33) was met, regardless of the convexity or concavity of the function. The variant V_2 is slightly better than the variant V_3, but in the conditions of interval given weights, the ratio of their extreme values is $(s_2^M=0.6518)>(s_3^m=0.5812)$ and $(s_2^m=0.5846)<(s_3^M=0.6366)$, which shows that the sets of values of functions partially overlap and require testing the stability of solution $V_2|(\underline{w};s_2(\underline{w}))$ in relation to the solution $V_3|(\underline{w};s_3(\underline{w}))$.

The solution V_2 is stable with respect to V_3 for $h_{2,3}(\underline{w})=s_2(\underline{w})-s_3(\underline{w})>0$. For the function $h(w)$ (42), extreme solutions are determined (which are on the vertices of the set $\overline{E}$, in accordance with (7)), where, except for the component w_3, all other components w_j have values of one of the limits of the weight interval, Table 3. The set of values of the function is $\overline{S_{2,3}}=\{h_{2,3}\in[-0.0298;\ 0.0557]\}$, in the point $\underline{w}^M$ the largest difference of TOPSIS values of the variants V_2 and V_3 $(h_{2,3}(\underline{w}^M)=0.0557)$ is achieved, while in the point $\underline{w}^m$ the variant V_3 is "better" than the variant V_2 $(h_{2,3}(\underline{w}^m)=-0.0298)$.

Table 3 – Extreme solutions of the function $h_{2,3}(\underline{w})$
Таблица 3 – Экстремальные решения функции $h_{2,3}(\underline{w})$
Табела 3 – Екстремна решења функције $h_{2,3}(\underline{w})$

	w_1^0	w_2^0	w_3^0	w_4^0	w_5^0	w_6^0	$s_2(\underline{w})$	$s_3(\underline{w})$	$h_{2,3}(w)$
$\underline{w}^m$	0.0990	0.1610	0.2640	0.1470	0.2410	0.0880	0.6009	0.6307	-0.0298
$\underline{w}^M$	0.1340	0.1320	0.2550	0.1830	0.2080	0.0880	0.6425	0.5868	0.0557

For each hyperplane $w_j^A=w_\lambda^{a\lambda}$ and $w_j^B=w_\lambda^{b\lambda}$, boundary solutions and extreme boundary solutions ($4n=24$ solutions) are determined, whereby the individual extreme boundary solutions are identical to each other or identical to the extreme solutions of the function $h(\underline{w})$. On the segments $[\underline{w}^{a\lambda,m};\underline{w}^{a\lambda,M}]$ and $[\underline{w}^{b\lambda,m};\underline{w}^{b\lambda,M}]$, the basic boundary solutions for the separating hyperplane $h^C=C\in[C^R+\varepsilon_C]$ (45) and for the reference value of the function $C^R=0$ (if any) are determined, or the basic non-boundary solutions are determined. In order to determine only solutions with positive values of the function $h>0$, due to partial stability and $S^+=\{h\,|\,h^M\geq h>C^R=0\}\subseteq\overline{S}$, a modified expression (45) was applied: $h^C=C\in[C^R+\varepsilon_C]$.

The obtained basic solutions are also boundary solutions based on the current hyperplane because $w_\lambda^{a\lambda,0}=w_\lambda^A$ or $w_\lambda^{b\lambda,0}=w_\lambda^B$, except for the solution for the point $\underline{w}^{b4,0}$ which is not a boundary solution based on the hyperplane w_4^B because $w_\lambda^{b\lambda,C}\neq w_\lambda^B$ (non-boundary basic solution) and $w_4^{b4,0}=0.1681<w_4^B=0.1830$. In general, according to the definition of the

boundary solution, the solution $\underline{w}^{b4,0}$ is a boundary solution based on other hyperplanes, because $w_1^{b4,0} = w_1^A = 0.990$ and $w_2^{b4,0} = w_2^B = 0.1610$ (Table 4)[10].

Table 4 – System of basic solutions for the separating hyperplane $h(\underline{w}) = 0$
Таблица 4 – Система базовых решений для разделения гиперплоскостей $h(\underline{w}) = 0$
Табела 4 – Систем основних решења за хиперраван раздвајања $h(\underline{w}) = 0$

$\underline{w}$ / j	$\underline{w}^{a1,0}$	$\underline{w}^{b1,0}$	$\underline{w}^{a2,0}$	$\underline{w}^{b2,0}$	$\underline{w}^{a3,0}$	$\underline{w}^{b3,0}$	$\underline{w}^{a4,0}$	$\underline{w}^{b4,0}$	$\underline{w}^{a5,0}$	$\underline{w}^{b5,0}$	$\underline{w}^{a6,0}$	$\underline{w}^{b6,0}$
1	0.0990	0.1340	0.1075	0.1106	0.1175	0.1053	0.1300	0.0990	0.1140	0.1073	0.1117	0.1116
2	0.1495	0.1532	0.1320	0.1610	0.1511	0.1503	0.1353	0.1610	0.1527	0.1489	0.1505	0.1505
3	0.2681	0.2418	0.2699	0.2509	0.2370	0.2730	0.2719	0.2482	0.2679	0.2528	0.2607	0.2438
4	0.1613	0.1567	0.1532	0.1644	0.1593	0.1603	0.1470	0.1681	0.1573	0.1620	0.1600	0.1600
5	0.2251	0.2263	0.2353	0.2250	0.2301	0.2232	0.2127	0.2404	0.2080	0.2410	0.2291	0.2291
6	0.0971	0.0880	0.1021	0.0880	0.1050	0.0880	0.1031	0.0881	0.1002	0.0880	0.0880	0.1050
$h(\underline{w})$	0.00002	0.00001	0.00004	0.00000	0.00003	0.00001	0.00003	0.00003	0.00004	0.00004	0.00000	0.00004
s_2=s_3	0.6091	0.6188	0.6238	0.6089	0.6174	0.6092	0.6139	0.6122	0.6007	0.6222	0.6150	0.6153

It is shown that the function $h(\underline{w})$ between the points of the weight of the basic solutions is concave or convex: between the points $\underline{w}^{a1,0}$ and $\underline{w}^{b1,0}$ the function is *convex*, and between the points $\underline{w}^{a2,0}$ and $\underline{w}^{b2,0}$ the function is *concave*. Due to the values of the λ weight components ($w_\lambda^{a\lambda}$ and $w_\lambda^{b\lambda}$), all points of the basic solutions $\underline{w}^{a\lambda,0}$ and $\underline{w}^{b\lambda,0}$ are linearly independent and their linear combinations give innumerable new weight points in the environment of the separating hyperplane. By applying expression (46) for $\beta_j^a = \beta_j^b = 1/2n = 1/12$, the solution is obtained:

$\underline{w}^{ab} = (0.1123;\ 0.1497;\ 0.2572;\ 0.1591;\ 0.2266;\ 0.0951)$;

$h_{2,3}^0(\underline{w}^{ab}) = 0.0003 < \varepsilon_C = 0.00005$ and $s_2(\underline{w}^{ab}) = s_3(\underline{w}^{ab}) = 0.6139$. The solution does not need to be corrected in accordance with other known

[10] For $\varepsilon_C = 5 \cdot 10^{-5}$ and the calculation of one basic boundary solution for which $w_j^A = w_\lambda^{a\lambda}$ or $w_j^B = w_\lambda^{b\lambda}$ about twenty iterations were required, and for the non-edge solution $\underline{w}^{b4}$ - about forty iterations.

points, for example in accordance with the extremes of the function or the boundary extreme solutions, because $h_{2,3}(\underline{w}^{ab}) < \varepsilon_C$ and the solution $(\underline{w}^{ab}; h_{2,3}^{ab})$ can be accepted in accordance with condition (45). The fulfilled condition (45) can also be a consequence of the concavity or convexity of the function on the segments between the points of the basic solutions.

Table 5 – Vertices of the set $\overline{E}$
Таблица 5 – Вершины множества $\overline{E}$
Табела 5 – Врхови скупа $\overline{E}$

$\underline{w}^v$	w_1	w_2	w_3	w_4	w_5	w_6	$h(\underline{w}^v)$
$\underline{w}^1$	0.0990	0.1610	0.2640	0.1470	0.2410	0.0880	-0.0298
$\underline{w}^2$	0.0990	0.1610	0.2470	0.1470	0.2410	0.1050	-0.0294
$\underline{w}^3$	0.0990	0.1610	0.2730	0.1470	0.2320	0.0880	-0.0283
$\underline{w}^4$	0.1090	0.1610	0.2370	0.1470	0.2410	0.1050	-0.0269
$\underline{w}^5$	0.1060	0.1610	0.2730	0.1470	0.2080	0.1050	-0.0216
$\underline{w}^6$	0.1340	0.1610	0.2370	0.1470	0.2330	0.0880	-0.0194
$\underline{w}^7$	0.1020	0.1320	0.2730	0.1470	0.2410	0.1050	-0.0111
$\underline{w}^8$	0.1340	0.1320	0.2730	0.1470	0.2090	0.1050	0.0040
$\underline{w}^9$	0.0990	0.1610	0.2370	0.1830	0.2303	0.0897	0.0207
$\underline{w}^{10}$	0.1230	0.1610	0.2370	0.1830	0.2080	0.0880	0.0337
$\underline{w}^{11}$	0.1190	0.1320	0.2370	0.1830	0.2410	0.0880	0.0416
$\underline{w}^{12}$	0.0990	0.1320	0.2730	0.1830	0.2080	0.1050	0.0467
$\underline{w}^{13}$	0.1160	0.1320	0.2730	0.1830	0.2080	0.0880	0.0512
$\underline{w}^{14}$	0.1340	0.1320	0.2370	0.1830	0.2090	0.1050	0.0550
$\underline{w}^{15}$	0.1340	0.1320	0.2380	0.1830	0.2080	0.1050	0.0554
$\underline{w}^{16}$	0.1340	0.1320	0.2550	0.1830	0.2080	0.0880	0.0557

Some of the weight components do not have to be given intervally but as discrete values (maximum *n-2* components), which also enables the determination of a set of solutions for a certain value of the function and the definition of the separating hyperplane. For example, if the weight point is $\underline{w} = (w_j)$ with the components $w_j = w_j^0$ for $j = 1,2,3$ and the components $w_j \in [w_j^A, w_j^B]$ for $j = 4,5,6$ (according to Table 2), and the required value is $h_{2,3}(\underline{w})=0.0500$ (to ensure a significant "advantage" of the variant V_2 over V_3), althhough $h_{2,3}(\underline{w})<h_{2,3}^{M}=0.0557$, such a solution does not exit because the maximum possible value of the function for these conditions is equal to $h_{2,3}(\underline{w})=0.0421$. For a smaller value, for example for $h_{2,3}(\underline{w})=0.0400$, there is a set of basic boundary solutions, and linear combinations of points of difficulty of these solutions determine other solutions that meet condition (45). One of these solutions is: $\underline{w^{ab}}=(0.1120;\ 0.1440;\ 0.2580;\ 0.1828;\ 0.2132;\ 0.0900)$; $h_{2,3}(\underline{w^{ab}})=0.0400$; $s_2(\underline{w^{ab}})=0.6278$ and $s_3(\underline{w^{ab}})=0.5878$.

From the set of points $\underline{w}$ for the extreme boundary solutions ($\underline{w^{a\lambda,m}}$, $\underline{w^{b\lambda,m}}$, $\underline{w^{a\lambda,M}}$ and $\underline{w^{b\lambda,M}}$), which are not all linearly independent, linearly independent points $\underline{w^{v}}$ are singled out and all vertices of the set $\overline{E}$ are determined by them (16 vertices of the set $\underline{w^{v}}\in\overline{E}$ are obtained, Table 5). This completely describes the set of definitions of the function $\overline{E}$, which with TOPSIS values of the function, represents a complete graph of the function $\Gamma_h=\{(\underline{w};h_{2,3})\in\Re^7\,|\,\underline{w}\in\overline{E},h_{2,3}(\underline{w})\in\overline{S_{2,3}}\}$. Knowledge of function graphs enables determination of sets of solutions complying with specific requirements in accordance with the stated limitations, which exceeds the goal and scope of this work.

Conclusion

The initial idea of developing a concept for testing the stability of the solution of the MADM problem (best variant and the corresponding quantitative indicator of the quality of the variant according to the chosen MADM method) in relation to other solutions (variants) and variable criteria weights was operationalized only through the examination of partial stability in relation to some other solution - one variant. The problem of determining the set of solutions of partial stability is set as a problem of NLP with the aim of finding feasible solutions that meet the conditions from the definition of partial stability. The TOPSIS method with parameters and interval-given criteria weights was considered as a basis, which defined the reference function as nonlinear and differentiable, in the presence of a normalization condition for arguments (weight components).

The created NLP model contains a nonlinear objective function, linear constraints based on the nature of the arguments (values: from - to) and the normalization condition for the arguments. An appropriate method was not known for solving the set NLP task, and therefore an attempt was made to solve the problem by introducing the nodes of argument (criteria) pairs and by defining their parameters. This ensures the normalization condition in each node and for each feasible point, non-negativity of variables and independence of variables in nodes, within the limits of active constraints. Node parameters were applied to determine the extremes of the function, the extremes on the hyperplanes of the set of arguments and other feasible solutions needed to determine the partial stability of the MADM solution, as well as to eliminate the consequences of accompanying degeneration (wedging and oscillation of the solution).

The presented procedure for determining the extremes of a given NLP problem differs from the basic gradient method in applying nodes parameters, choosing favorable directions, determining improved solutions, as well as in the procedure of linear search for the point of difficulty for an improved TOPSIS solution. The well-known and applied line search procedure can be replaced by another, for example, the "golden ratio" procedure, if this would contribute to the reduction of the procedure.

The procedure can be applied to other MADM methods with a nonlinear reference function, as well as to the class of NLP problems with conditional optimization, in which the mathematical model contains a nonlinear and on the whole set of arguments differentiable objective function, natural linear constraints and the normalization condition for variables. The procedure is robust and requires a larger number of calculations, so adequate software support would increase the possibilities of application.

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ЧАСТИЧНАЯ УСТОЙЧИВОСТЬ МНОГОАТРИБУТИВНОГО ПРИНЯТИЯ РЕШЕНИЙ ПО ИНТЕРВАЛЬНО ЗАДАННОМУ ВЕСУ КРИТЕРИЯ − ПРОБЛЕМА НЕЛИНЕЙНОГО ПРОГРАММИРОВАНИЯ

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РУБРИКА ГРНТИ: 27.00.00 МАТЕМАТИКА;
27.47.19 Исследование операций
ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: В статье представлена разработанная процедура для решения класса задач нелинейного программирования (НЛП) с нелинейной и дифференцируемой целевой функцией, линейными естественными ограничениями и условием нормализации переменных (аргументов). Процедура была применена для определения частичной устойчивости решения задач многоатрибутивного принятия решений.

Методы: Основой процедуры является определение узлов пар аргументов и их параметров для допустимых многомерных точек. Параметры внедрены в примененном градиентном методе, методе возможных направлений и методе линейного поиска. При разработке процедуры были использованы основы метода TOPSIS как метода для многоатрибутивного принятия решений с интервально заданными критериями веса, в первую очередь из-за нелинейности в вызове функций.

Результаты: Также разработана процедура определения экстремальных и других допустимых решений при вызове функций (маргинальные и базовые решения) и всех вершин выпуклого множества определения функции. Таким образом сформирован полный график функции, т.е. определены требуемые решения из допустимого множества. Разработана процедура установления множества решений для определения разделяющей гиперплоскости множества значений функции; благодаря чему в отдельных случаях множество решений частичной устойчивости варианта определяется как решение многоатрибутивного принятия решений. Были предложены соответствующие процедуры для устранения отклонений в процедуре (заклинивание и колебание решений).

Выводы: Данное исследование является значительным вкладом в определение узлов аргументов и их параметров, которые обеспечивают условия нормализации в каждом узле и для каждой допустимой точки, неотрицательность переменных и независимость изменений аргументов в узлах в рамках активных ограничений. Разработана оригинальная процедура определения графов функций. Приведены соответствующие реальные числовые примеры.

Ключевые слова: веса критериев, узлы пар аргументов, градиентный метод, метод возможных направлений, система базовых решений, метод многоатрибутивного принятия решений, частичная устойчивость решений.

ПАРЦИЈАЛНА СТАБИЛНОСТ РЕШЕЊА ВИШЕАТРИБУТНОГ ОДЛУЧИВАЊА ЗА ИНТЕРВАЛНО ЗАДАТЕ ТЕЖИНЕ КРИТЕРИЈУМА – ПРОБЛЕМ НЕЛИНЕАРНОГ ПРОГРАМИРАЊА

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Сажетак:

Увод/циљ: У раду је приказан пројектовани поступак за решавање класе задатака нелинеарног програмирања (НЛП) са нелинеарном и диференцијабилном функцијом циља, линеарним природним ограничењима и нормирајућим условом за променљиве (аргументе). Поступак је примењен за одређивање парцијалне стабилности решења проблема вишеатрибутног одлучивања (ВАО).

Методи: *Основ поступка представља дефинисање чворова парова аргумената и њихових параметара за допустиве вишедимензионалне тачке. Параметри се имлементирају у градијентни метод, метод повољних праваца и метод линијског тражења. У развоју поступка коришћени су основи метода ТОПСИС за ВАО са интервално задатим тежинама критеријума, првенствено због нелинеарности референтне функције.*

Резултати: *Разрађен је поступак одређивања екстремних и других допустивих решења референтне функције (рубна и основна решења) и свих врхова конвексног скупа дефинисаности функције. Тиме је формиран потпуни график функције, на основу којег се могу одредити захтевана решења из допустивог скупа. Развијен је поступак одређивања скупа решења за дефинисање хиперравани раздвајања скупа вредности функције. На тај начин се, као специфичан случај, дефинише и скуп решења парцијалне стабилности варијанте као решења ВАО. За отклањање дегенерације поступка (заклињавање и осциловање решења) предложене су адекватне процедуре.*

Закључак: *Најзначајнији допринос овог рада јесте дефинисање чворова аргумената и њихових параметара којима се осигурава нормирајући услов у сваком чвору и за сваку допустиву тачку, ненегативност променљивих и независност промена аргумената у чворовима, у границама активних ограничења. Такође, развијен је оригиналан поступак за одређивање графика функције и представљен одговарајући реалан нумерички пример.*

Кључне речи: *тежине критеријума, чворови парова аргумената, градијентни метод, метод повољних праваца, систем основних решења, вишеатрибутно одлучивање, парцијална стабилност решења.*

Paper received on / Дата получения работы / Датум пријема чланка: 10.06.2020.
Manuscript corrections submitted on / Дата получения исправленной версии работы / Датум достављања исправки рукописа: 06.07.2020.
Paper accepted for publishing on / Дата окончательного согласования работы / Датум коначног прихватања чланка за објављивање: 08.07.2020.