



ON SOME F – CONTRACTION OF PIRI–KUMAM–DUNG–TYPE MAPPINGS IN METRIC SPACES

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Abstract:

Introduction/purpose: This paper establishes some new results of Piri–Kumam–Dung-type mappings in a complete metric space. The goal was to improve the already published results.

Methods: Using the property of a strictly increasing function as well as the known Lemma formulated in (Radenović et al, 2017), the authors have proved that a Picard sequence is a Cauchy sequence.

Results: New results were obtained concerning the F – contraction mappings of S in a complete metric space. To prove it, the authors used only property (W_1) .

Conclusion: The authors believe that the obtained results represent a significant improvement of many known results in the existing literature.

Key words: Banach principle, F –contractive mapping, metric space, fixed point.

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Introduction and preliminaries

It is well known that the Banach fixed point theorem is the most celebrated result in Nonlinear analysis, Functional analysis, Mathematical Analysis, Topology and other mathematical disciplines. It shows that in a complete metric space, each contractive mapping has a unique fixed point. This result is used as a main tool for the existence of solutions of many non-linear problems. A great number of generalizations of this famous results appear in the literature. On the one hand, the usual contractive condition is replaced by a weakly contractive condition, while, on the other hand, the action space is replaced by some generalization of a standard metric space (as b -metric space, partial metric space, partial b -metric space, b -metric like space, G -metric space, G_b -metric space, etc).

A fundamental role in the foundations of the constructions is played by the fixed point theorems in metric spaces. They have been intensively studied for quite some time. The Banach fixed point theorem (proved in 1922 (Banach, 1922)), provides a technique for solving a variety of problems in mathematical science and engineering.

In recent investigations, Wardowski (Wardowski, 2012) described a new contraction where the author proved fixed point results in a very general setting in a so-called F -contraction. Piri and Kumam (Piri & Kumam, 2014) refined his results by launching some weaker conditions on self-mapping regarding a complete metric space and over the mapping F . For more details see (Chen et al, 2016), (Cosentino & Vetro, 2015), (Cosentino et al, 2015), (Ćirić, 2003), (Dey et al, 2019), (Dung et al, 2015), (Goswami et al, 2019), (Gubran et al, 2021), (Kadelburg & Radenović, 2018), (Lukacs & Kajanto, 2018), (Luambano et al, 2019), (Minak et al, 2014), (Piri & Kumam, 2014, 2016), (Popescu & Stan, 2020); (Secelean, 2013, 2016), (Shukla et al, 2014), (Wardowski, 2012, 2018), (Wardowski & Dung, 2014).

In the further work we need the following notation.

Definition 1.1 (Wardowski, 2012) Let (Λ, ρ) be a metric space and \mathfrak{F} be a family of mappings $F : (0, +\infty) \rightarrow (-\infty, +\infty)$ satisfying conditions:

(W1) F is strictly increasing, i.e. for all $a, b \in (0, +\infty)$, $a < b$ implies $F(a) < F(b)$;

(W2) for each sequence $\{a_n\} \subset (0, +\infty)$, $n \in \mathbb{N}$, $\lim_{n \rightarrow +\infty} a_n = 0$ if and only if $\lim_{n \rightarrow +\infty} F(a_n) = -\infty$;

(W3) there exists $s \in (0, 1)$ such that $\lim_{a \rightarrow 0^+} a^s F(a) = 0$.

A mapping $S : \Lambda \rightarrow \Lambda$ is said to be an F -contraction (or Wardowski function) on (Λ, ρ) if there exists $\theta > 0$ such that for all $u, v \in \Lambda$, $\rho(Su, Sv) > 0$ implies

$$\theta + F(\rho(Su, Sv)) \leq F(\rho(u, v)). \tag{1.1}$$

Some examples of an F -contraction can be found, for instance, in (Wardowski, 2012). It is obvious, based on condition **(W1)** and inequality (1.1), that every F -contraction is necessarily continuous.

Remark 1.1 Let $F : (0, +\infty) \rightarrow (-\infty, +\infty)$ be a strictly increasing function. Then there are two possibilities:

1. $F(0+0) = \lim_{x \rightarrow 0^+} F(x) = m, m \in \mathbb{R}$;
2. $F(0+0) = \lim_{x \rightarrow 0^+} F(x) = -\infty$.

Hence, each strictly increasing function $F : (0, +\infty) \rightarrow (-\infty, +\infty)$ satisfies either 1. or 2. For more details, see (Aljančić, 1969, Proposition 1, Section 8). In the first case, we have that F satisfies **(W3)**. Indeed, $\lim_{x \rightarrow 0^+} x^k F(x) = 0^k \cdot m = 0$ for all $k \in (0, 1)$. It is clear that in the second case F satisfies condition **(W2)**.

Further, Wardowski in (Wardowski, 2012) proved some new fixed point result which was a proper generalization of the Banach contraction principle.

Theorem 1.1 (Wardowski, 2012) Let (Λ, ρ) be a complete metric space and $S : \Lambda \rightarrow \Lambda$ be an F -contraction. Then S has a unique fixed point $u^* \in \Lambda$ and for every $u \in \Lambda$ the sequence $\{S^n u\}, n \in \mathbb{N}$ converges to u^* .

This theorem played a significant role in the further research in the metric fixed point theory. Several authors, see (Chen et al, 2016), (Cosentino & Vetro, 2015), (Cosentino et al, 2015), (Ćirić, 2003), (Dey et al, 2019), (Dung et al, 2015), (Goswami et al, 2019), (Gubran et al, 2021), (Kadelburg & Radenović, 2018), (Lukacs & Kajanto, 2018), (Miculescu & Mihail, 2017), (Piri & Kumam, 2014, 2016), (Popescu & Stan, 2020), (Secelean, 2013, 2016), (Shukla et al, 2014), (Wardowski, 2012, 2018), (Wardowski & Dung, 2014) generalized it by introducing the various type of F -contractions in other general metric spaces, especially in b -metric spaces, partial metric spaces, etc. Others have considered Wardowski's approach in a multi-valued case for metric spaces and its generalizations.

In 2013, Secelean (Secelean, 2013) proved that condition **(W2)** in Definition 1.1 can be replaced by an equivalent condition

(A1) $\inf F = -\infty$, or

(A2) there exists a sequence $\{a_n\} \subset (0, +\infty)$, $n \in \mathbb{N}$ such that

$$\lim_{n \rightarrow +\infty} F(a_n) = -\infty.$$

Considering Theorem 1.1, with conditions **(A1)**, **(A2)** and a new condition

(PK1), F is continuous on $(0, +\infty)$,

Piri and Kumam (Piri & Kumam, 2014) introduced a new type of F -contraction, a so-called F -Suzuki contraction. With this notion, they generalized and extended the well-known fixed point results of Wardowski (Wardowski, 2012) and Secelean (Secelean, 2013).

Definition 1.2 (Piri & Kumam, 2014) Let (Λ, ρ) be a metric space. A mapping $S: \Lambda \rightarrow \Lambda$ is said to be an F -Suzuki contraction if there exists

$\theta > 0$ such that for all $u, v \in \Lambda$ with $\rho(Su, Sv) > 0$, $\frac{1}{2}\rho(u, Su) < \rho(u, v)$

implies

$$\theta + F(\rho(Su, Sv)) \leq F(\rho(u, v)),$$

where F satisfies **(W1)**, **(A1)** and **(PK1)**.

Theorem 1.2 (Piri & Kumam, 2014) Let $S: \Lambda \rightarrow \Lambda$ be a given mapping in a complete metric space (Λ, ρ) . Suppose that $F: (0, +\infty) \rightarrow (-\infty, +\infty)$

satisfies conditions **(W1)**, **(A1)** and **(PK1)**, and let there exist $\theta > 0$ such that for all $u, v \in \Lambda$ with $\rho(Su, Sv) > 0$ holds

$$\theta + F(\rho(Su, Sv)) \leq F(\rho(u, v)).$$

Then, S has a unique fixed point $u^* \in \Lambda$ and for every $u \in \Lambda$ the sequences $\{S^n u\}$, $n \in \mathbb{N}$ converges to u^* .

Theorem 1.3 (Piri & Kumam, 2014) Let (Λ, ρ) be a complete metric space and $S: \Lambda \rightarrow \Lambda$ be an F -Suzuki contraction. Then S has a unique fixed point $u^* \in \Lambda$ and for every $u \in \Lambda$ the sequence $\{S^n u\}$, $n \in \mathbb{N}$ converges to u^* .

It is well known (see, for instance, (Cosentino & Vetro, 2014)) that the contraction conditions for the mappings $S: \Lambda \rightarrow \Lambda$ on the metric space (Λ, ρ) containing mainly the elements $\rho(u, v)$, $\rho(u, Su)$, $\rho(v, Sv)$, $\rho(u, Sv)$, $\rho(v, Su)$ can be complemented with $\rho(S^2u, v)$, $\rho(S^2u, Sv)$, $\rho(S^2u, u)$ and $\rho(S^2u, Su)$. This fact inspired Dung and Hang (Dung & Hang, 2015) to introduce a new concept, a generalized F -contraction, and to prove several fixed point theorems for such mapping.

Definition 1.3 (Dung & Hang, 2015) A mapping S of the metric space (Λ, ρ) into itself is said to be a generalized F -contraction on (Λ, ρ) if there exists $F: (0, +\infty) \rightarrow (-\infty, +\infty)$ satisfying conditions **(W1)**, **(W2)**, **(W3)** and $\theta > 0$ such that, for all $u, v \in \Lambda$, with $\rho(Su, Sv) > 0$ follows

$$\theta + F(\rho(Su, Sv)) \leq F(\Pi(u, v)),$$

where

$$\Pi(u, v) = \max \left\{ \rho(u, v), \rho(u, Su), \rho(v, Sv), \frac{\rho(u, Sv) + \rho(v, Su)}{2}, \frac{\rho(S^2u, u) + \rho(S^2u, Sv)}{2}, \rho(S^2u, Su), \rho(S^2u, v), \rho(S^2u, Sv) \right\}.$$

Theorem 1.4 (Dung & Hang, 2015) Let (Λ, ρ) be a complete metric space and let $S : \Lambda \rightarrow \Lambda$ be a generalized F -contraction. If S or F is continuous, then S has a unique fixed point $u^* \in \Lambda$ and for every $u \in \Lambda$ the sequence $\{S^n u\}, n \in \mathbb{N}$ converges to u^* .

In this article, we will prove Theorem 1.2, Theorem 1.3 and Theorem 1.4 in the easier way: using only condition **(W1)** and the following Lemma.

Lemma 1.1 (Radenović et al, 2017) Let $\{u_n\}, n \in \mathbb{N}$ be a sequence in the metric space (Λ, ρ) such that $\lim_{n \rightarrow +\infty} \rho(u_n, u_{n+1}) = 0$. If $\{u_n\}$ is not a Cauchy in (Λ, ρ) , then there exists $\xi > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences

$\{\rho(u_{n_k}, u_{m_k})\}, \{\rho(u_{n_k+1}, u_{m_k})\}, \{\rho(u_{n_k}, u_{m_k-1})\}, \{\rho(u_{n_k+1}, u_{m_k-1})\}, \{\rho(u_{n_k+1}, u_{m_k+1})\}, \dots$ tend to ξ^+ as $k \rightarrow +\infty$.

Remark 1.2 Notice, that if the condition of Lemma 1.1 is satisfied, then the sequences $\{\rho(u_{n_k+s}, u_{m_k})\}$ and $\{\rho(u_{n_k+s}, u_{m_k+1})\}$ also converge to ξ^+ when $k \rightarrow +\infty$, where $s \in \mathbb{N}$.

Results

We begin this section with the theorem which generalizes and improves Theorem 1.2. In our result, the function $F : (0, +\infty) \rightarrow (-\infty, +\infty)$ satisfies only condition **(W1)**.

Theorem 2.1 Let (Λ, ρ) be a complete metric space and $S : \Lambda \rightarrow \Lambda$ be a F -contraction mapping with property **(W1)**, that is, let there exist $\theta > 0$ such that

$$\theta + F(\rho(Su, Sv)) \leq F(\rho(u, v)), \quad (2.1)$$

for all $u, v \in \Lambda$ with $Su \neq Sv$.

Then S has a unique fixed point, say, u^* and for all $u \in \Lambda$ the sequence $\{S^n u\}, n \in \mathbb{N}$ converges to u^* .

Proof. Firstly, from condition **(W1)**, there are both $\lim_{c \rightarrow d^-} F(c) = F(d-0)$ and $\lim_{c \rightarrow d^+} F(c) = F(d+0)$ for all $d \in (0, +\infty)$, because it is known from mathematical analysis that the following is true

$$F(d-0) \leq F(d) \leq F(d+0), \quad d \in (0, +\infty). \quad (2.2)$$

Further, from the assumption that S is an F -contraction, it follows that S is contractive ($u \neq v$ implies $\rho(Su, Sv) < \rho(u, v)$). This means that the mapping S is continuous. Besides, F -contractive condition **(W1)** implies the uniqueness of the fixed point if it exists.

We will show that S has a fixed point. Let u_0 be an arbitrary point in Λ . Consider the sequence $\{u_n\}, n \in \mathbb{N} \cup \{0\}$ with $u_{n+1} = Su_n$. If $u_k = u_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$ then u_k is a unique fixed point of S and the conclusion of the Theorem follows. Therefore, suppose that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Based on F -contractive condition (2.1) of the mapping S , we get

$$F(\rho(u_n, u_{n+1})) < \theta + F(\rho(u_n, u_{n+1})) \leq F(\rho(u_{n-1}, u_n)), \quad (2.3)$$

for all $n \in \mathbb{N}$, that is, in accordance with property **(W1)**, it follows

$$\rho(u_n, u_{n+1}) < \rho(u_{n-1}, u_n) \text{ for all } n \in \mathbb{N}.$$

This further means that $\rho(u_n, u_{n+1}) \rightarrow \rho^* \geq 0$ as $n \rightarrow +\infty$, as well as $\rho(u_n, u_{n+1}) > \rho^*$ for all $n \in \mathbb{N} \cup \{0\}$. If we chose now that $\rho^* > 0$, then the relation

$$\theta + F(\rho(u_n, u_{n+1})) \leq F(\rho(u_{n-1}, u_n)), \quad (2.4)$$

implies

$$\theta + F(\rho^* + 0) \leq F(\rho^* + 0), \quad (2.5)$$

which is a contradiction. Thus, $\rho^* = 0$, i.e., $\lim_{n \rightarrow +\infty} \rho(u_n, u_{n+1}) = 0$.

Next, we show that $\{u_n\}, n \in \mathbb{N} \cup \{0\}$ is a Cauchy sequence by assuming the contrary. If we put $u = u_{n_k}$ and $v = u_{m_k}, k \in \mathbb{N}$ in F -contractive condition (2.1), we obtain

$$\theta + F\left(\rho(u_{n_k+1}, u_{m_k+1})\right) \leq F\left(\rho(u_{n_k}, u_{m_k})\right). \quad (2.6)$$

Since, according to Lemma 1.1, both $\rho(u_{n_k+1}, u_{m_k+1})$ and $\rho(u_{n_k}, u_{m_k})$ tend to ξ^+ as $k \rightarrow +\infty$, we have

$$\theta + F(\xi^+ + 0) \leq F(\xi^+ + 0), \quad (2.7)$$

which is wrong. We conclude that $\{u_n\}, n \in \mathbb{N} \cup \{0\}$ is a Cauchy sequence, hence it converges to some $u^* \in \Lambda$ as (Λ, ρ) is complete. Since S is continuous, it follows that $Su^* = u^*$, i.e., u^* is a unique fixed point of S and the Theorem is proved.

Now we state one consequence that can be obtained concerning the F -contractive self-mapping S which satisfies one of five conditions.

Corollary 2.1 Let (Λ, ρ) be a complete metric space and let $S: \Lambda \rightarrow \Lambda$ be a self-mapping such that there exists $\theta_i > 0, i = 1, 2, 3, 4, 5$ so for all $u, v \in \Lambda$ with $\rho(Su, Sv) > 0$, the following contractive conditions holds:

$$\begin{aligned} \theta_1 + \rho(Su, Sv) &\leq \rho(u, v), \\ \theta_2 - \frac{1}{\rho(Su, Sv)} &\leq -\frac{1}{\rho(u, v)}, \\ \theta_3 - \frac{1}{\rho(Su, Sv)} + \rho(Su, Sv) &\leq -\frac{1}{\rho(u, v)} + \rho(u, v), \\ \theta_4 + \frac{1}{1 - e^{\rho(Su, Sv)}} &\leq \frac{1}{1 - e^{\rho(u, v)}}, \\ \theta_5 + \frac{1}{e^{-\rho(Su, Sv)} - e^{\rho(Su, Sv)}} &\leq \frac{1}{e^{-\rho(u, v)} - e^{\rho(u, v)}}. \end{aligned}$$

Then, in each of these cases, S has a unique fixed point $u^* \in \Lambda$ and for all $u \in \Lambda$ the sequence $\{S^n u\}, n \in \mathbb{N}$ converges to u^* .

Proof. As each of the functions $F_1(r) = r, F_2(r) = -\frac{1}{r}, F_3(r) = -\frac{1}{r} + r,$
 $F_4(r) = \frac{1}{1 - \exp r}$ and $F_5(r) = \frac{1}{\exp(-r) - \exp r}$, where $r = \rho(u, v) > 0$, is

strictly increasing on $(0, +\infty)$, the proof immediately follows by Theorem 2.1.

The following result is related to the generalization and the improvement of Theorem 1.2.

Theorem 2.2 Let (Λ, ρ) be a complete metric space and let $S : \Lambda \rightarrow \Lambda$ is an F -Suzuki contraction mapping where F satisfies condition **(W1)**, that is, there exists $\theta > 0$ such that $\frac{1}{2}\rho(u, Su) < \rho(u, v)$ implies

$$\theta + F(\rho(Su, Sv)) \leq F(\rho(u, v)), \quad (2.8)$$

for all $u, v \in \Lambda$ with $\rho(Su, Sv) > 0$.

Then S has a unique fixed point $u^* \in \Lambda$ and for each $u \in \Lambda$ the sequence $\{S^n u\}$, $n \in \mathbb{N}$ converges to u^* .

Proof. It is easily seen that relation (2.8) implies the uniqueness of the fixed point if it exists. Indeed, if we suppose that there are two distinct fixed points u^* and v^* of S , then it is clear that from $\frac{1}{2}\rho(u^*, Su^*) < \rho(u^*, v^*)$ follows

$$\theta + F(\rho(Su^*, Sv^*)) \leq F(\rho(u^*, v^*)), \text{ i.e.}$$

$$\theta + F(\rho(u^*, v^*)) \leq F(\rho(u^*, v^*)),$$

which is a contradiction.

Now we show the existence of the fixed point. Let $u_0 \in \Lambda$ be an arbitrary point and $\{u_n\}$, $n \in \mathbb{N} \cup \{0\}$ is the corresponding Picard sequence, i.e. $u_n = S^n u_0$ with an initial value $u_0 = S^0 u_0$. If $u_k = u_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then u_k is a unique fixed point and the proof is complete.

Therefore, suppose $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. In this case $\frac{1}{2}\rho(u_n, u_{n+1}) < \rho(u_n, u_{n+1})$ holds true for all $n \in \mathbb{N} \cup \{0\}$, so from (2.8) we have

$$\theta + F(\rho(u_{n+1}, u_{n+2})) \leq F(\rho(u_n, u_{n+1})),$$

that is, according to condition **(W1)** we get

$$\rho(u_{n+1}, u_{n+2}) < \rho(u_n, u_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

As in the proof of Theorem 2.1, we obtain that $\rho(u_n, u_{n+1}) \rightarrow 0$ as $n \rightarrow +\infty$. Since $\rho(u_n, u_{n_k+1}) \rightarrow 0$ and $\rho(u_{n_k}, u_{m_k}) \rightarrow \xi^+$ as $k \rightarrow +\infty$ it follows that there is some $k_1 \in \mathbb{N}$ such that $\frac{1}{2}\rho(u_{n_k}, u_{m_k+1}) < \rho(u_{n_k}, u_{m_k})$, for all $k \in \mathbb{N}$ with $k \geq k_1$. Then, for $k \geq k_1$, we have

$$\theta + F(\rho(u_{n_k+1}, u_{m_k+1})) \leq F(\rho(u_{n_k}, u_{m_k})),$$

that is $\theta + F(\xi^+ + 0) \leq F(\xi^+ + 0)$, which is a contradiction because $\theta > 0$.

Hence we conclude that $\{u_n\}, n \in \mathbb{N} \cup \{0\}$ is a Cauchy sequence. The completeness of the metric space (Λ, ρ) guarantees the existence of some point $u^* \in \Lambda$ such that $\lim_{n \rightarrow +\infty} \rho(u_n, u^*) = 0$. The rest of the proof is analogous to the proof of Theorem 1.3 from (Piri & Kumam, 2014) on page 7. One can find that following inequalities hold

$$\frac{1}{2}\rho(u_n, Su_n) < \rho(u_n, u^*), \text{ or } \frac{1}{2}\rho(Su_n, S^2u_n) < \rho(Su_n, u^*),$$

for all $n \in \mathbb{N} \cup \{0\}$.

Further, from relation (2.8), it follows

$$\theta + F(\rho(Su_n, Su^*)) \leq F(\rho(u_n, u^*)), \text{ or } \theta + F(\rho(S^2u_n, Su^*)) \leq F(\rho(Su_n, u^*)),$$

i.e.,

$$\theta + F(\rho(u_{n+1}, Su^*)) \leq F(\rho(u_n, u^*)), \text{ or}$$

$$\theta + F(\rho(u_{n+2}, Su^*)) \leq F(\rho(u_{n+1}, u^*)). \quad (2.9)$$

Finally, using condition **(W1)**, we can write inequalities (2.9) in the form

$$\rho(u_{n+1}, Su^*) \leq \rho(u_n, u^*), \text{ or } \rho(u_{n+2}, Su^*) \leq \rho(u_{n+1}, u^*). \quad (2.10)$$

This proves that u^* is a unique fixed point of S , i.e., $Su^* = u^*$.

The immediate consequence of Theorem 2.2 is the following result.

Corollary 2.2 Let (Λ, ρ) be a complete metric space and let $S : \Lambda \rightarrow \Lambda$ be self-mapping such that there exists $\theta_i > 0, i=1,2,3,4,5,6$ so for all $u, v \in \Lambda$ with $\rho(Su, Sv) > 0$ the following implications hold true:

$\frac{1}{2}\rho(u, Su) < \rho(u, v)$ implies

$$\theta_1 + \rho(Su, Sv) \leq \rho(u, v), \text{ or}$$

$$\theta_2 - \frac{1}{\rho(Su, Sv)} \leq -\frac{1}{\rho(u, v)}, \text{ or}$$

$$\theta_3 - \frac{1}{\rho(Su, Sv)} + \rho(Su, Sv) \leq -\frac{1}{\rho(u, v)} + \rho(u, v), \text{ or}$$

$$\theta_4 + e^{\rho(Su, Sv)} \leq e^{\frac{1}{\rho(Su, Sv)}}, \text{ or}$$

$$\theta_5 + \frac{1}{1 - e^{\rho(Su, Sv)}} \leq \frac{1}{1 - e^{\rho(u, v)}}, \text{ or}$$

$$\theta_6 + \frac{1}{e^{-\rho(Su, Sv)} - e^{\rho(Su, Sv)}} \leq \frac{1}{e^{-\rho(u, v)} - e^{\rho(u, v)}}.$$

Then in each of these cases, S has a unique fixed point $u^* \in \Lambda$ and for all $u \in \Lambda$ the sequence $\{S^n u\}, n \in \mathbb{N}$ converges to u^* .

After all, we give the proof of Theorem 1.4 in an easier way: using only condition **(W1)** and Lemma 1.1.

Theorem 2.3 Let (Λ, ρ) be a complete metric space and $S : \Lambda \rightarrow \Lambda$ is a generalized F -contraction which satisfies condition **(W1)**, that is, there exists $\theta > 0$ such that for all $u, v \in \Lambda$ with $\rho(Su, Sv) > 0$ follows

$$\theta + F(\rho(Su, Sv)) \leq F(\Pi(u, v)), \tag{2.11}$$

where

$$\Pi(u, v) = \max \left\{ \rho(u, v), \rho(u, Su), \rho(v, Sv), \frac{\rho(u, Sv) + \rho(v, Su)}{2}, \frac{\rho(S^2u, u) + \rho(S^2u, Sv)}{2}, \rho(S^2u, Su), \rho(S^2u, v), \rho(S^2u, Sv) \right\}.$$

If S or F is continuous, then S has a unique fixed point $u^* \in \Lambda$ and for every $u \in \Lambda$ the sequence $\{S^n u\}$, $n \in \mathbb{N}$ converges to u^* .

Proof. Since $\rho(Su, Sv) > 0$ it follows that contractive condition (2.11) is well defined. Further, condition (2.11) implies the uniqueness of the fixed point if it exists. In order to show that S has a fixed point, suppose that u_0 is an arbitrary point in Λ and define a sequence $\{u_n\} \in \Lambda$, $n \in \mathbb{N} \cup \{0\}$ by $u_{n+1} = Su_n$. If we chose $u_k = u_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then u_k is a unique fixed point of S and the proof of the Theorem is completed.

Therefore, suppose that $\rho(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Using generalized F -contractive condition (2.11), of the mapping S , we obtain

$$F(\rho(u_n, u_{n+1})) < \theta + F(\rho(u_n, u_{n+1})) \leq F(\Pi(u_{n-1}, u_n)), \quad (2.12)$$

where

$$\begin{aligned} \Pi(u_{n-1}, u_n) &= \max \left\{ \rho(u_{n-1}, u_n), \rho(u_{n-1}, u_n), \rho(u_n, u_{n+1}), \right. \\ &\quad \left. \frac{\rho(u_{n-1}, u_{n+1}) + 0}{2}, \frac{\rho(u_{n+1}, u_{n-1}) + 0}{2}, \rho(u_{n+1}, u_n), \rho(u_{n+1}, u_n), 0 \right\} \\ &= \max \left\{ \rho(u_{n-1}, u_n), \rho(u_n, u_{n+1}), \frac{\rho(u_{n-1}, u_{n+1})}{2} \right\} \\ &\leq \max \left\{ \rho(u_{n-1}, u_n), \rho(u_n, u_{n+1}) \right\}. \end{aligned}$$

It is clear that $\max \left\{ \rho(u_{n-1}, u_n), \rho(u_n, u_{n+1}) \right\} = \rho(u_{n-1}, u_n)$, otherwise, we get a contradiction. This further means that $\lim_{n \rightarrow +\infty} \rho(u_n, u_{n+1}) = 0$.

Next, we prove that $\{u_n\}$, $n \in \mathbb{N} \cup \{0\}$ is a Cauchy sequence by supposing it is not. Putting $u = u_{n_k}$ and $v = u_{m_k}$ in generalized F -contractive condition (2.11) we have

$$\theta + F(\rho(u_{n_k+1}, u_{m_k+1})) \leq F(\rho(u_{n_k}, u_{m_k})), \quad (2.13)$$

where

$$\Pi(u_{n_k}, u_{m_k}) = \max \left\{ \rho(u_{n_k}, u_{m_k}), \rho(u_{n_k}, u_{n_k+1}), \rho(u_{m_k}, u_{m_k+1}), \right.$$

$$\frac{\rho(u_{n_k}, u_{m_k+1}) + \rho(u_{m_k}, u_{n_k+1})}{2}, \frac{\rho(u_{n_k+2}, u_{n_k}) + \rho(u_{n_k+2}, u_{m_k+1})}{2},$$

$$\rho(u_{n_k+2}, u_{n_k+1}), \rho(u_{n_k+2}, u_{m_k}), \rho(u_{n_k+2}, u_{m_k+1})\}.$$

According to Lemma 1.1, one can find that $\lim_{k \rightarrow +\infty} \Pi(u_{n_k}, u_{m_k}) = \xi^+$.

Therefore, from (2.13), it follows that $\theta + F(\xi^+ + 0) \leq F(\xi^+ + 0)$, which is a contradiction so we can conclude that the sequence $\{u_n\}, n \in \mathbb{N} \cup \{0\}$ is a Cauchy sequence. Since (Λ, ρ) is a complete metric space, it converges to some $u^* \in S$. Finally, if S is continuous, then $Su^* = u^*$ and the theorem is proved.

Suppose now that F is continuous and let $\rho(u^*, Su^*) > 0$. Putting $u = u_n$ and $v = u^*$ in condition (2.11) we get

$$\theta + F(\rho(u_{n+1}, u^*)) \leq F(\Pi(u_n, u^*)) \tag{2.14}$$

where

$$\Pi(u_n, u^*) = \max\{\rho(u_n, u^*), \rho(u_n, u_{n+1}), \rho(u^*, Su^*),$$

$$\frac{\rho(u_n, Su^*) + \rho(u^*, u_{n+1})}{2}, \frac{\rho(u_{n+2}, u_n) + \rho(u_{n+2}, Su^*)}{2},$$

$$\rho(u_{n+2}, u_{n+1}), \rho(u_{n+2}, u^*), \rho(u_{n+2}, Su^*)\}.$$

Since $F(\rho(u_{n+1}, Su^*)) \rightarrow F(\rho(u^*, Su^*))$ and $\Pi(u_n, u^*) \rightarrow \rho(u^*, Su^*)$ as $n \rightarrow +\infty$, taking the limit in (2.13), when n tends to infinity, we obtain

$$\theta + F(\rho(u^*, Su^*)) \leq F(\rho(u^*, Su^*)),$$

which is in contradiction with $\theta > 0$. Therefore, the assumption $\rho(u^*, Su^*) > 0$ is wrong. This means that $\rho(u^*, Su^*) = 0$, i.e., u^* is a unique fixed point of the generalized F -contraction S and the proof is finished.

From the previous theorem, one consequence can be obtained assuming some strictly increasing function instead of F .

Corollary 2.3 Let (Λ, ρ) be a complete metric space and let $S : \Lambda \rightarrow \Lambda$ is a generalized F -contraction such that there is $\theta_i > 0, i=1,2,3$ and that for all $u, v \in \Lambda$ with $\rho(Su, Sv) > 0$ the following implication holds true

$$\begin{aligned}\theta_1 + \rho(Su, Sv) &\leq \Pi(u, v), \\ \theta_2 - e^{\frac{1}{\rho(Su, Sv)}} + e^{\rho(Su, Sv)} &\leq -e^{\frac{1}{\Pi(u, v)}} + e^{\Pi(u, v)}, \\ \theta_3 - \frac{1}{\rho(Su, Sv)} + \rho(Su, Sv) &\leq -\frac{1}{\Pi(u, v)} + \Pi(u, v),\end{aligned}$$

where

$$\begin{aligned}\Pi(u, v) = \max \left\{ \rho(u, v), \rho(u, Su), \rho(v, Sv), \frac{\rho(u, Sv) + \rho(v, Su)}{2}, \right. \\ \left. \frac{\rho(S^2u, u) + \rho(S^2u, Sv)}{2}, \rho(S^2u, Su), \rho(S^2u, v), \rho(S^2u, Sv) \right\}.\end{aligned}$$

Then S has a unique fixed point $u^* \in \Lambda$ and for all $u \in \Lambda$ the sequence $\{S^n u\}, n \in \mathbb{N} \cup \{0\}$ converges to u^* .

Proof. Putting $F_1(r) = r$, $F_2(r) = -e^{\frac{1}{r}} + e^r$ and $F_3(r) = -\frac{1}{r} + r$ in relation to (2.12), the results follows.

Conclusion

In this article we get some new results concerning the F -contraction mappings of S in a complete metric space. To prove it, we used only property **(W1)**. We believe that this is a significant improvement of the known results in the existing literature.

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О НЕКОТОРЫХ F -СЖИМАЮЩИХ ОТОБРАЖЕНИЯХ ТИПА ПИРИ-КУМАН-ДУНГА В МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

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РУБРИКА ГРНТИ: 27.00.00 МАТЕМАТИКА;
27.39.27 Нелинейный функциональный анализ

ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: В данной статье представлены некоторые из новейших результатов отображений типа Пери-Кумама-Дунга в полном метрическом пространстве. Целью работы было улучшение опубликованных ранее результатов.

Методы: Используя свойство строго возрастающей функции, а также известную лемму, сформулированную в исследовании (Radenović et al, 2017) было доказано, что данная последовательность является последовательностью Пикара – Коши.

Результаты: В данной статье представлено несколько новых результатов, полученных путем наблюдения F – сжимающих отображений S в полном метрическом пространстве. В качестве доказательства использовалось исключительно свойство $(W1)$.

Выводы: Авторы считают, что полученные ими результаты представляют собой значительный вклад в данную область науки.

Ключевые слова: Принцип Банаха, F – сжимающее отображение, метрическое пространство, неподвижная точка.

О НЕКИМ F -КОНТРАКТИВНИМ ПРЕСЛИКАВАЊИМА ПИРИ-КУМАМ-ДУНГОВОГ ТИПА У МЕТРИЧКИМ ПРОСТОРИМА

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ОБЛАСТ: математика

ВРСТА ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: У чланку су успостављени нови резултати за пресликавања Пери–Кумам–Дунговог типа у комплетном метричком простору. Циљ рада јесте да се побољшају већ објављени резултати.

Метод: Користећи својство строго растуће функције, као и познату лему формулисану у (Radenović et al, 2017), доказано је да је Пикаров низ у ствари Кошијев.

Резултати: Добијено је неколико нових резултата посматрајући F -контрактивна пресликавања од S у комплетном метричком простору. У доказу је коришћена само особина $(W1)$.

Закључак: Аутори верују да добијени резултати представљају значајан допринос досадашњим познатим резултатима.

Кључне речи: Банахов принцип, F–контрактивно пресликавање, метрички простор, непокретна тачка.

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