REVISITING AND REVAMPING SOME NOVEL RESULTS IN $\mathcal{F}$-METRIC SPACES

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Abstract:

Introduction/purpose: This article establishes several new contractive conditions in the context of so-called $\mathcal{F}$-metric spaces. The main purpose was to generalize, extend, improve, complement, unify and enrich the already published results in the existing literature. We used only the property (F1) of Wardowski as well as one well–known lemma for the proof that Picard sequence is an $\mathcal{F}$-Cauchy in the framework of $\mathcal{F}$-metric space.

Methods: Fixed point metric theory methods were used.

Results: New results are enunciated concerning the F-contraction of two mappings S and T in the context of $\mathcal{F}-$complete $\mathcal{F}$-metric spaces.

Conclusions: The obtained results represent sharp and significant improvements of some recently published ones. At the end of the paper, an example is given, claiming that the results presented in this paper are proper generalizations of recent developments.

Key words: $\mathcal{F}$-metric space, F-contraction, fixed point.

Introduction and preliminaries

It is exactly one hundred years since S. Banach (Banach, 1922) proved the famous principle of contraction in his doctoral dissertation. Since then, many researchers have been trying to generalize that significant result in many directions. In one direction, new classes of metric spaces were created and the renowned results were extended to these spaces. Among them, b-metric and $\mathcal{F}$-metric spaces stand out. The former ones were introduced by Bakhtin (Bakhtin, 1989) and Czerwik (Czerwik, 1993) and the latter were recently introduced by Jleli and Samet (Jleli & Samet, 2018). Not that these two cases of spaces are intangible. Namely, there is a b-metric space that is not $\mathcal{F}$-metric, and vie versa, there is an $\mathcal{F}$-metric that is not *b*-metric. Note that convergence, Cauchyness and completeness of both types of spaces are defined for ordinary metric spaces. Also, it is worth mentioning that *b*-metric and $\mathcal{F}$-metric do not have to be continuous functions with two variables as is the case with ordinary metric. In both types of spaces, a convergent sequence is a Cauchy and it has a unique limit. This is what they have in common with ordinary metric spaces. The continuity of mapping in both classes of spaces is sequential, i.e., the same as in ordinary metric spaces. Let us now list the definitions of each of the mentioned types of spaces. For more new details on $\mathcal{F}$-metric spaces and new developments in the metric fixed point theory, one can see some noteworthy papers (Asif et al, 2019), (Aydi et al, 2019), (Derouiche & Ramoul, 2020), (Jahangir et al, 2021), (Kirk & Shazad, 2014), (Mitrović et al, 2019), (Salem et al, 2020), (Som et al, 2020), (Vujaković et al, 2020), (Vujaković & Radenović, 2020), (Younis et al, 2019a), (Younis et al, 2019b).

Definition 1. *((Bakhtin, 1989), (Czerwik, 1993)) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d_b : X \times X \to [0, +\infty)$ is said to be a b-metric with the coefficient s if for all $x, y, z \in X$ the following conditions are satisfied:*

(d_b1) $d_b(x, y) = 0$ if and only if $x = y$

(d_b2) $d_b(x, y) = d_b(y, x)$

(d_b3) $d_b(x, y) \leq s\left[d_b(x, z) + d_b(z, y)\right].$

Let $\mathcal{F}$ be the set of functions $f : (0, +\infty) \to (-\infty, +\infty)$ satisfying the following conditions:

$\mathcal{F}_1)$ f is non-decreasing,

$\mathcal{F}_2)$ For every sequence $\{t_n\} \subset (0, +\infty)$, we have

$$\lim_{n\to+\infty} t_n = 0 \text{ if and only if } \lim_{n\to+\infty} f(t_n) = -\infty.$$

Definition 2. *(Jleli & Samet, 2018) Let X be a (nonempty) set. A function $d_{\mathcal{F}} : X \times X \to [0, +\infty)$ is called a $\mathcal{F}$-metric on X if there exists $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that for all $x, y \in X$ the following conditions hold:*

$(d_{\mathcal{F}}1)$ $d_{\mathcal{F}}(x, y) = 0$ if and only if $x = y$.
$(d_{\mathcal{F}}2)$ $d_{\mathcal{F}}(x, y) = d_{\mathcal{F}}(y, x)$.
$(d_{\mathcal{F}}3)$ For every $N \in \mathbb{N}, N \geq 2$ and for every $\{u_i\}_{i=1}^{N} \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$d_{\mathcal{F}}(x, y) > 0 \text{ yields } f(d_{\mathcal{F}}(x, y)) \leq f\left(\sum_{i=1}^{N-1} d_{\mathcal{F}}(u_i, u_{i+1})\right) + \alpha.$$

In this case, the pair $(X, d_{\mathcal{F}})$ is called a $\mathcal{F}-$metric space.

Wardowski (Wardowski, 2012) considered a nonlinear function $F : (0, +\infty) \to (-\infty, +\infty)$ with the following characteristics:
(F1) F is strictly increasing.
(F2) $\mathcal{F}_2)$ above.
(F3) There exists $l \in (0, 1)$ such that $\lim_{t \to 0^+} t^l F(t) = 0$.

Wardowski (Wardowski, 2012) called the mapping $T : X \to X$, defined on a metric space (X, d), an $F-$contraction if there exist $\tau > 0$ and F satisfying (F1)-(F3) such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)) \text{ whenever } d(Tx, Ty) > 0.$$

The authors in (Asif et al, 2019) take $\mathcal{B} = \{F : (0, +\infty) \to (-\infty, +\infty) : \ F \text{ satisfies } \mathcal{F}_1) \text{ and } \mathcal{F}_2)\}$.

In 2019, A. Asif et al., (Asif et al, 2019) formulated and proved the fixed-point and common fixed-point results for single-valued Reich-type and Kannan-type F-contractions in the setting of $\mathcal{F}$-metric spaces:

Theorem 1. *Suppose $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ and $(X, d_{\mathcal{F}})$ is an $\mathcal{F}-$complete $\mathcal{F}$-metric space. Let $S, T : X \to X$ be self-mappings. Suppose there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\tau + F(d_{\mathcal{F}}(Sx, Ty)) \leq F(a \cdot d_{\mathcal{F}}(x, y) + b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty)) \quad (1)$$

for $a, b, c \in [0, . + \infty)$ such that $a + b + c < 1$ with

$$\min\{d_{\mathcal{F}}(Sx, Ty), d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, Sx), d_{\mathcal{F}}(y, Ty)\} > 0,$$

for all $(x, y) \in X \times X$. Then S and T have at most one common fixed point in X.

Corollary 1. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}$-*metric space Let* $S, T : X \to X$ *be self-mappings. Suppose that* $k \in [0, 1)$, *there exist* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\tau + F\left(d_{\mathcal{F}}\left(Sx, Ty\right)\right) \leq F\left(\frac{k}{2}\left(d_{\mathcal{F}}\left(x, Sx\right) + d_{\mathcal{F}}\left(y, Ty\right)\right)\right) \tag{2}$$

with $\min\left\{d_{\mathcal{F}}\left(Sx, Ty\right), d_{\mathcal{F}}\left(x, y\right), d_{\mathcal{F}}\left(x, Sx\right), d_{\mathcal{F}}\left(y, Ty\right)\right\} > 0$, *for all* $(x, y) \in X \times X$. *Then* S *and* T *have at most one common fixed point in* X.

By replacing S with T, the authors obtained the following result for single mapping.

Corollary 2. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}$-*metric space Let* $T : X \to X$ *be self-mapping. Suppose that for* $k \in [0, 1)$, *there exist* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\tau + F\left(d_{\mathcal{F}}\left(Tx, Ty\right)\right) \leq F\left(\frac{k}{2}\left(d_{\mathcal{F}}\left(x, Tx\right) + d_{\mathcal{F}}\left(y, Ty\right)\right)\right) \tag{3}$$

with $\min\left\{d_{\mathcal{F}}\left(Tx, Ty\right), d_{\mathcal{F}}\left(x, Tx\right), d_{\mathcal{F}}\left(y, Ty\right)\right\} > 0$, *for all* $(x, y) \in X \times X$. *Then* T *have at most one fixed point in* X.

Definition 3. *Let* $(X, d_{\mathcal{F}})$ *be an* $\mathcal{F}-$*complete* $\mathcal{F}$-*metric space and* $S, T : X \to X$ *be self-mappings. Suppose that* $a + b + c < 1$ *for* $a, b, c \in [0, +\infty)$. *Then the mapping* T *is called a Reich-type F-contraction on* $B\left(x_0, r\right) \subseteq X$ *if there exist* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that for all* $x, y \in B\left(x_0, r\right)$

$$\tau + F\left(d_{\mathcal{F}}\left(Sx, Ty\right)\right) \leq F\left(a \cdot d_{\mathcal{F}}\left(x, y\right) + b \cdot d_{\mathcal{F}}\left(x, Sx\right) + c \cdot d_{\mathcal{F}}\left(y, Ty\right)\right). \tag{4}$$

Theorem 2. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}$-*metric space. Let* T *be a Reich-type F-contraction on* $B\left(x_0, r\right) \subseteq X$. *Suppose that for* $x_0 \in X$ *and* $r > 0$, *the following conditions are satisfied:*

(a) $B\left(x_0, r\right)$ is $\mathcal{F}-$closed,
(b) $d_{\mathcal{F}}\left(x_0, x_1\right) \leq (1 - \lambda) r$, for $x_1 \in X$ and $\lambda = \frac{a+b}{1-c}$,
(c) There exist $0 < \varepsilon < r$ such that $f\left(\left(1 - \lambda^{k+1}\right) r\right) \leq f(\varepsilon) - \alpha$, where $k \in \mathbb{N}$.

Then S and T have at most one common fixed point in $B\left(x_0, r\right)$.

Taking $S = T$ in Theorem 2, the authors in ((Asif et al, 2019), Corollary 3.) obtained the following result for single mappings.

Corollary 3. *Suppose* $(f,\alpha) \in \mathcal{B} \times [0,+\infty), (F,\tau) \in \mathcal{B} \times (0,+\infty), (X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}$*-metric space and* $T : X \to X$ *is a self-mapping. Suppose that* $a+b+c<1$ *for* $a,b,c \in [0,+\infty)$. *Suppose that for* $x_0 \in X$ *and* $r>0$, *the following conditions are satisfied:*

(a) $B(x_0,r)$ is $\mathcal{F}-$closed,

(b) $\tau + F(d_{\mathcal{F}}(Sx,Ty)) \leq F(a \cdot d_{\mathcal{F}}(x,y) + b \cdot d_{\mathcal{F}}(x,Sx) + c \cdot d_{\mathcal{F}}(y,Ty))$for all $x,y \in B(x_0,r)$,

(c) $d_{\mathcal{F}}(x_0,x_1) \leq (1-\lambda)r$, for $x_1 \in X$ and $\lambda = \frac{a+b}{1-c}$,

(c) There exist $0<\varepsilon<r$ such that $f\left(\left(1-\lambda^{k+1}\right)r\right) \leq f(\varepsilon) - \alpha$, where $k \in \mathbb{N}$.

Then T has at most one fixed point in $B(x_0,r)$.

Corollary 4. *Suppose* $(f,\alpha) \in \mathcal{B} \times [0,+\infty), (F,\tau) \in \mathcal{B} \times (0,+\infty), (X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}$*-metric space. Let* $S,T : X \to X$ *be a self-mappings and* $k \in [0,1)$. *Suppose that for* $x_0 \in X$ *and* $r>0$, *the following conditions are satisfied:*

(a) $B(x_0,r)$ is $\mathcal{F}-$closed,

(b) $\tau + F(d_{\mathcal{F}}(Sx,Ty)) \leq F(a \cdot d_{\mathcal{F}}(x,y) + b \cdot d_{\mathcal{F}}(x,Sx) + c \cdot d_{\mathcal{F}}(y,Ty))$for all $x,y \in B(x_0,r)$,

(c) $d_{\mathcal{F}}(x_0,x_1) \leq (1-\lambda)r$, for $x_1 \in X$ and $\lambda = \frac{k}{2-k}$,

(c) There exist $0<\varepsilon<r$ such that $f\left(\left(1-\lambda^{k+1}\right)r\right) \leq f(\varepsilon) - \alpha$, where $k \in \mathbb{N}$.

Then S and T have at most one common fixed point in $B(x_0,r)$.

Further in the same paper ((Asif et al, 2019), Definitions 6, 8, Theorem 5, Corollary 5.), the authors gave the following:

Definition 4. *Let* $(X, d_{\mathcal{F}})$ *be a metric space. Let* $CB(X)$ *be the family of all non-empty closed and bounded subsets of* X. *Let* $H : CB(X) \times CB(X) \to [0,+\infty)$ *be a function defined by*

$$H(A,B) = \max\left\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\right\}, \tag{5}$$

where $D(x,B) = \inf\{d_{\mathcal{F}}(x,y) : y \in B\}$. *Then* H *defines a metric on* $CB(X)$ *called the Hausdorff-Pompeiu metric induced by* $d_{\mathcal{F}}$.

Definition 5. *Let* $(X, d_{\mathcal{F}})$ *be an* $\mathcal{F}$*-metric space. Suppose* $F \in \mathcal{B}$ *and* $H : CB(X) \times CB(X) \to [0, +\infty)$ *be the Hausdorff-Pompeiu metric function defined in Definition 2. A mapping* $T : X \to CB(X)$ *is known as a set-valued Reich-type contraction if there is some* $\tau > 0$ *such that*

$$2\tau + F(H(Tx, Ty)) \leq F(a \cdot d_{\mathcal{F}}(x, y) + b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty)) \quad (6)$$

for $(x, y) \in X \times X$ *and* $a, b, c \in [0, +\infty)$ *such that* $a + b + c < 1$.

Theorem 3. *Let* $(X, d_{\mathcal{F}})$ *be an* $\mathcal{F}-$*complete* $\mathcal{F}$*-metric space and* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$. *If the mapping* $T : X \to CB(X)$ *is a set-valued Reich-type F-contraction such that* F *is right continuous, then* T *has a fixed point in* X.

Corollary 5. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}$*-metric space. Let* $T : X \to CB(X)$ *be a Reich-type* $F-$*contraction such that* F *is right continuous. Suppose that for* $k \in [0, 1)$, *there exist* $F \in \mathcal{B}$ *and* $\tau > 0$ *such that*

$$\tau + F(H(Tx, Ty)) \leq F\left(\frac{k}{2}(d_{\mathcal{F}}(x, Tx) + d_{\mathcal{F}}(y, Ty))\right) \quad (7)$$

with $\min\{H(Tx, Ty), d_{\mathcal{F}}(x, Tx), d_{\mathcal{F}}(y, Ty)\} > 0$ *for all* $(x, y) \in X \times X$. *Then* T *has a fixed point in* X.

In the sequel, we will use the following two results:

Lemma 1. *((Mitrović et al, 2019), Lemma 1.) Let* (X, d_b) *(resp.* $(X, d_{\mathcal{F}})$ *be a b-metric (resp.* $\mathcal{F}-$*metric) space and* $\{x_n\}_{n=1}^{+\infty}$ *the sequence in it such that*

$$d_b(x_n, x_{n+1}) \leq \lambda \cdot d_b(x_{n-1}, x_n) \;\; (\text{resp. } d_{\mathcal{F}}(x_n, x_{n+1}) \leq \lambda \cdot d_{\mathcal{F}}(x_{n-1}, x_n)), \quad (8)$$

for all $n \in \mathbb{N}$, *where* $\lambda \in [0, 1)$. *Then* $\{x_n\}_{n=1}^{+\infty}$ *is a* d_b-*Cauchy sequence in* (X, d_b) *(resp.*$d_{\mathcal{F}}-$*Cauchy sequence in* $(X, d_{\mathcal{F}})$*).*

Lemma 2. *Let* $\{x_{n+1}\}_{n\in\mathbb{N}\cup\{0\}} = \{Tx_n\}_{n\in\mathbb{N}\cup\{0\}} = \{T^n x_0\}_{n\in\mathbb{N}\cup\{0\}}, T^0 x_0 = x_0$ *be a Picard sequence in* $\mathcal{F}$*-metric space inducing by mapping* $T : X \to X$ *and initial point* $x_0 \in X$. *If* $d_{\mathcal{F}}(x_n, x_{n+1}) < d_{\mathcal{F}}(x_{n-1}, x_n)$*) for all* $n \in \mathbb{N}$ *then* $x_n \neq x_m$ *whenever* $n \neq m$.

Proof. Let $x_n = x_m$ for some $n, m \in \mathbb{N}$ with $n < m$. Then $x_{n+1} = Tx_n = Tx_m = x_{m+1}$. Further, we get

$$d_{\mathcal{F}}(x_n, x_{n+1}) = d_{\mathcal{F}}(x_m, x_{m+1}) < d_{\mathcal{F}}(x_{m-1}, x_m) < ... < d_{\mathcal{F}}(x_n, x_{n+1}),$$

which is a contradiction. □

Some improved results

Firstly, since $F : (0, +\infty) \to (-\infty, +\infty)$ it follows that (1) is possible only if $d_{\mathcal{F}}(Sx, Ty) > 0$ where $x, y \in X$. Also, the condition (F1) yields that $a \cdot d_{\mathcal{F}}(x, y) + b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty) > 0$ for all $x, y \in X$ for which $d_{\mathcal{F}}(Sx, Ty) > 0$. This means that at least of $a, b, c \in [0, +\infty)$ must be distinct of 0. Now we can improve the formulation of Theorem 1 and all its corollaries from (Asif et al, 2019) and give new proofs as the following.

Theorem 4. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}-$*metric space. Let* $S, T : X \to X$ *be self mappings. Suppose there exist a strictly increasing function* $F : (0, +\infty) \to (-\infty, +\infty)$ *and* $\tau > 0$ *such that* $d_{\mathcal{F}}(Sx, Ty) > 0$ *yields*

$$\tau + F(d_{\mathcal{F}}(Sx, Ty)) \leq F(a \cdot d_{\mathcal{F}}(x, y) + b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty)), \quad (9)$$

for $a, b, c \in [0, +\infty)$ *such that* $a^2 + b^2 + c^2 > 0$ *and* $a + b + c < 1$.
Then S *and* T *have at most one common fixed point in* X, *if at least one of the mappings* S *or* T *is continuous.*

Proof. Already, we first eliminate the function F. Indeed, from (9) if $d_{\mathcal{F}}(Sx, Ty) > 0$ follows

$$d_{\mathcal{F}}(Sx, Ty) < a \cdot d_{\mathcal{F}}(x, y) + b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty), \quad (10)$$

where $a, b, c \in [0, +\infty)$, $a^2 + b^2 + c^2 > 0$ and $a + b + c < 1$. Further, we give the proof in several steps:

The step 1.

The point $\overline{x}$ is a fixed of S if and only if it is a fixed point of T. Let $S\overline{x} = \overline{x}$ and $T\overline{x} \neq \overline{x}$. Putting $x = y = \overline{x}$ in (10) we get

$$\begin{aligned} d_{\mathcal{F}}(\overline{x}, T\overline{x}) = d_{\mathcal{F}}(S\overline{x}, T\overline{x}) &< a \cdot d_{\mathcal{F}}(\overline{x}, \overline{x}) + b \cdot d_{\mathcal{F}}(\overline{x}, S\overline{x}) + c \cdot d_{\mathcal{F}}(\overline{x}, T\overline{x}) \\ &= a \cdot 0 + b \cdot 0 + c \cdot d_{\mathcal{F}}(\overline{x}, T\overline{x}), \end{aligned}$$

i.e., $(1 - c)\, d_{\mathcal{F}}(\overline{x}, T\overline{x}) < 0$. Since, $c \in [0, 1)$ we obtain the contradiction. Therefore, $T\overline{x} = \overline{x}$.

Conversely, let $\overline{x} = T\overline{x}$ and $S\overline{x} \neq \overline{x}$. In this case, we get

$$d_{\mathcal{F}}\left(\overline{Sx}, \overline{x}\right) = d_{\mathcal{F}}\left(\overline{Sx}, T\overline{x}\right) < a \cdot d_{\mathcal{F}}(\overline{x}, \overline{x}) + b \cdot d_{\mathcal{F}}(\overline{x}, S\overline{x}) + c \cdot d_{\mathcal{F}}(\overline{x}, \overline{x})$$

$$= a \cdot 0 + b \cdot d_{\mathcal{F}}(\overline{x}, S\overline{x}) + c \cdot 0,$$

i.e., $(1-b)\, d_{\mathcal{F}}\left(\overline{Sx}, \overline{x}\right) < 0$ which is a contradiction, because $b \in [0,1)$. Hence, it follows that $S\overline{x} = \overline{x}$.

The step 2.

The uniqueness of a possible common fixed point for S and T.

Let $\overline{x} \neq \overline{y}$ be two common fixed points for S and T. Then putting in (10) $x = \overline{x}$ and $y = \overline{y}$ we get:

$$d_{\mathcal{F}}(S\overline{x}, T\overline{y}) < a \cdot d_{\mathcal{F}}(\overline{x}, \overline{y}) + b \cdot d_{\mathcal{F}}(\overline{x}, S\overline{x}) + c \cdot d_{\mathcal{F}}(\overline{y}, T\overline{y}),$$

i.e., $(1-a) \cdot d_{\mathcal{F}}(\overline{x}, \overline{y}) < 0$. Which is a contradiction with $\overline{x} \neq \overline{y}$. This means that a possible common fixed point for S and T is unique.

The step 3.

In this step, we shall prove the existence of at least one common fixed point of S and T.

Therefore, suppose that x_0 is an arbitrary point and define a sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1},$$

for $n \in \mathbb{N} \cup \{0\}$. It is clear that $d_{\mathcal{F}}(x_{2n+1}, x_{2n+2}) > 0$ and $d_{\mathcal{F}}(x_{2n+3}, x_{2n+2}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Now, by (10) we further get ($x = x_{2n}, y = x_{2n+1}$)

$$d_{\mathcal{F}}(x_{2n+1}, x_{2n+2}) < a \cdot d_{\mathcal{F}}(x_{2n}, x_{2n+1}) + b \cdot d_{\mathcal{F}}(x_{2n}, x_{2n+1}) + c \cdot d_{\mathcal{F}}(x_{2n+1}, x_{2n+2}),$$

i.e., $d_{\mathcal{F}}(x_{2n+1}, x_{2n+2}) < k \cdot d_{\mathcal{F}}(x_{2n}, x_{2n+1})$ where $k = \frac{a+b}{1-c} \in [0,1)$. And similar $d_{\mathcal{F}}(x_{2n+3}, x_{2n+2}) < k \cdot d_{\mathcal{F}}(x_{2n+2}, x_{2n+1})$. Hence, for all $n \in \mathbb{N}$ we have

$$d_{\mathcal{F}}(x_n, x_{n+1}) < k \cdot d_{\mathcal{F}}(x_{n-1}, x_n) < d_{\mathcal{F}}(x_{n-1}, x_n).$$

According to Lemmas 9 and 10, we have that the sequence $\{x_n\}$ is a $d_{\mathcal{F}}-$Cauchy in an $\mathcal{F}-$complete $\mathcal{F}$-metric space $(X, d_{\mathcal{F}})$ and $x_n \neq x_m$ whenever $n \neq m$. This further means that there is (unique) $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$.

Firstly, let S be continuous. Then $x_{2n+1} = Sx_{2n} \to Sx^* = x^*$ since in each $\mathcal{F}-$metric space the subsequence of each convergent sequence converges to the unique limit. Now, we will prove that also $Tx^* = x^*$. Indeed, if $Tx^* \neq x^*$ then by using (10) with $x = y = x^*$ we get

$$d_{\mathcal{F}}(x^*, Tx^*) = d_{\mathcal{F}}(Sx^*, Tx^*) < a \cdot d_{\mathcal{F}}(x^*, x^*) + b \cdot d_{\mathcal{F}}(x^*, Sx^*) + c \cdot d_{\mathcal{F}}(x^*, Tx^*)$$

$$= a \cdot d_{\mathcal{F}}(x^*, x^*) + b \cdot d_{\mathcal{F}}(x^*, Sx^*) + c \cdot d_{\mathcal{F}}(x^*, Tx^*) = a \cdot 0 + b \cdot 0 + c \cdot d_{\mathcal{F}}(x^*, Tx^*).$$

Finally, we obtain that $(1-c) \cdot d_{\mathcal{F}}(x^*, Tx^*) < 0$ which is a contradiction because we suppose $d_{\mathcal{F}}(x^*, Tx^*) \neq 0$.

If the mapping T is continuous, the proof is similar.

The theorem is completely proved. □

Remark 1. Our Theorem 11 generalizes, improves, complements and unifies the corresponding Theorem 3 from (Asif et al, 2019) in several directions. First of all, it is worth to notice that some parts of the proof for Theorem 3 are doubtful. Namely, the authors in their proof use that $\mathcal{F}$-metric $d_{\mathcal{F}}$ is a continuous function with two variables ($d_{\mathcal{F}}(x_n, y_n) \to d_{\mathcal{F}}(x, y)$ if $d_{\mathcal{F}}(x_n, x) \to 0$ and $d_{\mathcal{F}}(y_n, y) \to 0$), which is not case. Also, it is clear that the function F in their Theorem 3 and in both Corollaries 1 and 2 is superfluous.

The next two corollaries follows from our Theorem 11.

Corollary 6. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}-$*metric space. Let* $S, T : X \to X$ *be self mappings. Suppose there exist strictly increasing function* $F : (0, +\infty) \to (-\infty, +\infty)$ *and* $\tau > 0$ *such that* $d_{\mathcal{F}}(Sx, Ty) > 0$ *yields*

$$\tau + F(d_{\mathcal{F}}(Sx, Ty)) \leq F\left(\frac{k}{2}(d_{\mathcal{F}}(x, Sx) + c d_{\mathcal{F}}(y, Ty))\right), \tag{11}$$

for $k \in [0, 1)$.
By replacing S *with* T, *we get the following result for single mapping:*

Corollary 7. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}-$*metric space. Let* $T : X \to X$ *be self mapping. Suppose there exist strictly increasing function* $F : (0, +\infty) \to (-\infty, +\infty)$ *and* $\tau > 0$ *such that*

$d_{\mathcal{F}}(Tx,Ty) > 0$ *yields*

$$\tau + F\left(d_{\mathcal{F}}(Tx,Ty)\right) \leq F\left(\frac{k}{2}\left(d_{\mathcal{F}}(x,Tx) + cd_{\mathcal{F}}(y,Ty)\right)\right), \tag{12}$$

for $k \in [0,1)$.
Then T *has at most one fixed point in* X, *if it is continuous.*

Remark 2. Now we give the following **Important Notice:**

It is useful to note that the other results from (Asif et al, 2019) can be repaired and supplemented in the same or similar way. It should also be said that the results on Hausdorff-Pompeiu metric given in (Asif et al, 2019) are dubious. This will be discussed in another of our papers.

The immediate consequences of our Theorem 11 are the following new contractive conditions that complement the ones given in (Collaco & Silva, 1997), (Rhoades, 1977) for usual metric spaces. For more contractive conditions in the framework of metric spaces see (Ćirić, 2003), (Consentino & Vetro, 2014), (Dey et al, 2019), (Karapınar, et al), (Piri & Kumam, 2014), (Salem et al, 2020), (Wardowski & Dung, 2014). In the sequel we will obtain several new contractive conditions in the framework of $\mathcal{F}-$metric spaces.

Corollary 8. *Suppose* $(f,\alpha) \in \mathcal{B} \times [0,+\infty)$ *and* $(X,d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}-$*metric space. Let* $S,T : X \to X$ *be self mappings. Suppose there exist* $\tau_1 > 0$ *such that* $d_{\mathcal{F}}(Sx,Ty) > 0$ *yields*

$$\tau_1 + d_{\mathcal{F}}(Sx,Ty) \leq a \cdot d_{\mathcal{F}}(x,y) + b \cdot d_{\mathcal{F}}(x,Sx) + c \cdot d_{\mathcal{F}}(y,Ty), \tag{13}$$

for $a,b,c \in [0,+\infty)$ *such that* $a^2+b^2+c^2 > 0$ *and* $a+b+c < 1$.
Then S *and* T *have at most one common fixed point in* X, *if one of the mappings* S *or* T *is continuous.*

Corollary 9. *Suppose* $(f,\alpha) \in \mathcal{B} \times [0,+\infty)$ *and* $(X,d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}-$*metric space. Let* $S,T : X \to X$ *be self mappings. Suppose there exist* $\tau_2 > 0$ *such that* $d_{\mathcal{F}}(Sx,Ty) > 0$ *yields*

$$\tau_2 + d_{\mathcal{F}}(Sx,Ty) \leq a \cdot d_{\mathcal{F}}(x,y), \tag{14}$$

for $a \in [0,1)$.
Then S *and* T *have at most one common fixed point in* X, *if one of the mappings* S *or* T *is continuous.*

Corollary 10. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}-$*metric space. Let* $S, T : X \to X$ *be self mappings. Suppose there exist* $\tau_3 > 0$ *such that* $d_{\mathcal{F}}(Sx, Ty) > 0$ *yields*

$$\tau_3 + d_{\mathcal{F}}(Sx, Ty) \leq b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty), \tag{15}$$

for $b, c \in [0, +\infty)$ *such that* $b^2 + c^2 > 0$ *and* $b + c < 1$.
Then S *and* T *have at most one common fixed point in* X, *if one of the mappings* S *or* T *is continuous.*

Corollary 11. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}-$*metric space. Let* $S, T : X \to X$ *be self mappings. Suppose there exist* $\tau_4 > 0$ *such that* $d_{\mathcal{F}}(Sx, Ty) > 0$ *yields*

$$\tau_4 - \frac{1}{d_{\mathcal{F}}(Sx, Ty)} \leq -\frac{1}{b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty)}, \tag{16}$$

for $b, c \in [0, +\infty)$ *such that* $b^2 + c^2 > 0$ *and* $b + c < 1$.
Then S *and* T *have at most one common fixed point in* X, *if one of the mappings* S *or* T *is continuous.*

Corollary 12. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}-$*metric space. Let* $S, T : X \to X$ *be self mappings. Suppose there exist* $\tau_5 > 0$ *such that* $d_{\mathcal{F}}(Sx, Ty) > 0$ *yields*

$$\begin{aligned}\tau_5 - \frac{1}{d_{\mathcal{F}}(Sx, Ty)} + d_{\mathcal{F}}(Sx, Ty) \leq &-\frac{1}{b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty)} \\ &+ b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty),\end{aligned} \tag{17}$$

for $b, c \in [0, +\infty)$ *such that* $b^2 + c^2 > 0$ *and* $b + c < 1$.
Then S *and* T *have at most one common fixed point in* X, *if at least one of the mappings* S *or* T *is continuous.*

Corollary 13. *Suppose* $(f, \alpha) \in \mathcal{B} \times [0, +\infty)$ *and* $(X, d_{\mathcal{F}})$ *is an* $\mathcal{F}-$*complete* $\mathcal{F}-$*metric space. Let* $S, T : X \to X$ *be self mappings. Suppose there exist* $\tau_6 > 0$ *such that* $d_{\mathcal{F}}(Sx, Ty) > 0$ *yields*

$$\tau_6 + \frac{1}{1 - \exp\left(d_{\mathcal{F}}(Sx, Ty)\right)} \leq -\frac{1}{1 - \exp\left(b \cdot d_{\mathcal{F}}(x, Sx) + c \cdot d_{\mathcal{F}}(y, Ty)\right)}, \tag{18}$$

for $b, c \in [0, +\infty)$ such that $b^2 + c^2 > 0$ and $b + c < 1$.
Then S and T have at most one common fixed point in X, if one of the mappings S or T is continuous.

Proof. As each of the functions $F_i(r) = r$, $i = \overline{1,3}$, $F_4(r) = -\frac{1}{r}$, $F_5(r) = -\frac{1}{r} + r$, $F_6(r) = \frac{1}{1-\exp(r)}$ is strictly increasing on $(0, +\infty)$, the proof immediately follows by our Theorem 11 and their corollaries. □

Example 1. Finally, we give the following simple example that support our Theorem 11 with $S = T$. Suppose that $X = \{2n + 1 : n \in \mathbb{N}\}$. Define the $d_{\mathcal{F}}-$metric given by the following

$$d_{\mathcal{F}}(x, y) = \begin{cases} 0 \text{ if } x = y \\ e^{|x-y|} \text{ if } x \neq y. \end{cases}$$

Let $F(r) = -e^{-r}$ and $T : X \to X$ is defined by

$$T(2n + 1) = \begin{cases} 3 \text{ if } n \in \{1, 2\} \\ 2n - 1 \text{ if } n \geq 3. \end{cases}.$$

It is clear that $d_{\mathcal{F}}$ is a $\mathcal{F}-$metric and F is strictly increasing on $(0, +\infty)$. All the conditions of Theorem 11 are satisfied. Indeed, putting in equation (9) $b = c = 0$, we get for $x \neq y$:

$$\tau - e^{-|Tx-Ty|} \leq -a \cdot e^{-|x-y|},$$

i.e., $e^{-|Tx-Ty|} > a \cdot e^{-|x-y|}$. Taking $x = 2n + 1, y = 2m + 1, n \neq m$ we further obtain $e^{-|2n-2m|} > a \cdot e^{-|2n-2m|}$. Since $n \neq m$ this means that there exists $a \in [0, 1)$ such that (9) holds true, i.e., T has a unique fixed point in $X = \{2n + 1 : n \in \mathbb{N}\}$, which is $x = 3$. Note that $\lim_{r \to +0} F(r) = -1$, then Theorem 3 from (Asif et al, 2019) is not applicable here. This shows that our results are proper generalizations of the ones from (Asif et al, 2019).

Conclusion

In this article, we obtained several new contractive conditions in the framework of $\mathcal{F}-$metric spaces. Our results improve, extend, complement, generalize, and unify various recent developments in the context of $\mathcal{F}-$metric spaces. An example shows that the main results of (Asif et al, 2019) are not applicable in our case. We think that this is a useful contribution in the framework of F-contraction introduced by D. Wardowski.

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ПЕРЕСМОТР И УЛУЧШЕНИЕ НЕКОТОРЫХ НОВЫХ РЕЗУЛЬТАТОВ В F-МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

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РУБРИКА ГРНТИ: 27.00.00 МАТЕМАТИКА:
27.25.17 Метрическая теория функций,
27.33.00 Интегральные уравнения,
27.39.29 Приближенные методы функционального анализа

ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: В данной статье устанавливается несколько новых сжимающих условий в контексте так называемых F-метрических пространств. Основная цель статьи заключается в обобщении, расширении, улучшении, дополнении, объединении ранее опубликованных результатов в существующей литературе. Мы использовали только свойство (F1) Вардовского, а также одну хорошо известную лемму для доказательства того, что последовательность Пикара тождественна F-Коши в рамках F-метрического пространства.

Методы: В статье применены методы метрической теории неподвижной точки.

Результаты: Сформулированы новые результаты о F-сжатии двух отображений S и T в контексте F-полных F-метрических пространств.

Выводы: Полученные результаты значительно улучшены по сравнению с некоторыми недавно опубликованными результатами. В заключении приводится пример, доказывающий, что результаты, представленные в данной статье, являются соответствующим обобщением недавних результатов.

Ключевые слова: F-метрическое пространство, F-сжатие, неподвижная точка.

РЕВИЗИЈА И ПОБОЉШАЊЕ НЕКИХ НОВИХ РЕЗУЛТАТА У $\mathcal{F}$-МЕТРИЧКИМ ПРОСТОРИМА

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ОБЛАСТ: математика

ВРСТА ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Овај рад успоставља неколико нових контрактивних услова у контексту такозваних $\mathcal{F}$-метричких простора. Главни циљ је генерализација, проширење, побољшање, допуна и обједињење већ добијених резултата у постојећој литератури. Коришћено је само својство (F_1) Вардовског, као и једна добро позната лема за доказ да је Пикаров низ $\mathcal{F}$-Кошијев у оквиру $\mathcal{F}$-метричког простора.

Методе: Коришћене су методе метричке теорије фиксне тачке.

Резултати: Објављени су нови резултати у вези са $\mathcal{F}$-контракцијама за два пресликавања у оквиру $\mathcal{F}$-комплетних $\mathcal{F}$-метричких простора.

Закључак: Добијени резултати представљају значајна побољшања, као и праву генерализацију неких недавно објављених резултата, што показује пример наведен на крају рада.

Кључне речи: $\mathcal{F}$-метрички простор, F-контракција, фиксна тачка.

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