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ПРЕГЛЕДНИ РАДОВИ

ОБЗОРНЫЕ СТАТЬИ

REVIEW PAPERS

QUANTUM ELECTRODYNAMICS DIVERGENCIES

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Abstract:

Introduction/purpose: The problem of divergencies in Quantum Electrodynamics (QED) is discussed.

Methods: The renormalisation group method is employed for dealing with infinities in QED.

Results: The integrals in QED giving physical observables are finite.

Conclusions: The divergencies in QED can be treated in a consistent way providing mathematical rigorous results.

Key words: Quantum Electrodynamics, Quantum Field Theory, Renormalisation Group.

Renormalisation group

The quantum field brings together two fundamental notions of classical and non-relativistic quantum physics: particles and fields. For instance, the quantum electromagnetic field can be reduced to particles called photons, or to a wave process described by a classical field. The same is true for the quantum Dirac field, for electrons.

Quantum field theory, QFT for short, as the theory of interacting quantum fields, includes the remarkable phenomenon of virtual particles related to virtual transitions in quantum mechanics. For example, a photon propagating through empty space, the classical vacuum, undergoes a virtual transition into an electron–positron pair. Usually, this pair undergoes the

reverse transformation: annihilation back into a photon. This sequence of two transitions is known as the process of vacuum polarisation. Hence the vacuum in QFT is not an empty space, rather it is filled by virtual particle–antiparticle pairs.

Another example is the electromagnetic interaction between two electric charges, e.g. between two electrons. In QFT, rather than a Coulomb force described by a potential, the interaction corresponds to an exchange of virtual photons, which, in turn, propagate in spacetime accompanied by virtual electron–positron pairs.

QFT calculation usually results in a series of terms, each of which represents the contribution of different vacuum polarisation mechanisms. Unfortunately, most of these terms turn out to be infinite.

The puzzle was resolved in the late 1940s, mainly by Bethe (Bethe, 1947), Feynman (Feynman, 1949a), Schwinger (Schwinger, 1948), Tomonaga (Tomonaga, 1946) and Dyson (Dyson, 1949). These famous theoreticians were able to show that all infinite contributions can be grouped into a few mathematical combinations that correspond to a change of normalisation of quantum fields, ultimately resulting in a redefinition, i.e. "renormalisation" of masses and coupling constants. Physically, this effect is a close analogue of a classical "dressing process" for a particle interacting with a surrounding medium.

The most important feature of renormalisation is that the calculation of physical quantities gives finite functions of new "renormalised" couplings, such as electron charge and masses, all infinities being swallowed by the factors of the renormalisation redefinition. The "bare" values of mass and electric charge do not appear in the physical expression. At the same time, the renormalised parameters should be related to the physical ones, measured experimentally.

Dealing with infinities

Infinities are disturbing. In Nature infinities seldom happen. There are "only" about 10^{80} atoms in the Universe. The Universe itself had a beginning, but (as of 2021) it will expand forever to a state of thermodynamically no free energy. Mathematics itself started using infinite numbers only in 17^{th} century, and division by zero is invalid even in hyperreal numbers.

Yet in physics we have to deal with infinities even from classical electromagnetism. Consider a static electrical field generated by a single particle,

say an electron e. Then we have

$$\operatorname{div} \vec{\mathcal{E}} = \delta(\mathbf{r}) \, . \tag{1}$$

By symmetry arguments, we immediately obtain the electrical field $\vec{\mathcal{E}}$, that is

$$\vec{\mathcal{E}} = \frac{e}{4\pi \mathbf{r}^2} \tag{2}$$

and computing the total energy of the field, proportional to its electromagnetic mass m_{em},

$$E = \frac{1}{2} \int \mathrm{d}V \vec{\mathcal{E}}^2 = \frac{1}{2} \int_{r_e}^{\infty} \mathrm{d}r \left(\frac{e}{4\pi r^2} \right)^2 4\pi r^2 = \frac{e^2}{8\pi r_e} \tag{3}$$

one sees that becomes infinite as the electron radius r_e goes to 0. In late 19^{th} century, that meant that an electron needed infinite energy to be accelerated.

When quantising the electromagnetic field, we are faced with another divergence problem. Writing down the Hamiltonian of harmonic oscillators for each radiation mode identified by $(\mathbf{k}, r)$

$$H = \sum_{\mathbf{k}} \sum_{r} \hbar \omega_{\mathbf{k}} \left(a_r^{\dagger}(\mathbf{k}) a_r(\mathbf{k}) + \frac{1}{2} \right) \tag{4}$$

we see that the energy of vacuum state $|0\rangle$ equals to $\frac{1}{2} \sum_{\mathbf{k}} \sum_r \hbar \omega_{\mathbf{k}}$, which is an infinite constant $(\omega_{\mathbf{k}} = c|\mathrm{k}|)$. This problem is usually dealt with by shifting the Hamiltonian by this infinite value, as one has the freedom to redefine the zero of energy scale.

During the development of quantum electrodynamics (QED) it was discovered that many integrals were divergent (Dirac, 1927), (Dirac, 1934), (Heisenberg, 1934). Those were present in perturbative calculations involving Feynman diagrams (Feynman, 1949a), (Feynman, 1949b) containing closed loops. Particles circulating in closed loops are called *virtual particles* because they are off–shell, that is $p^2 \neq m^2$. In a loop, their momentum and energy are not determined and thus they have to be integrated over all values allowed by the whole 4–momentum conservation. Such integrals are divergent, i.e. give infinite results. Divergencies that are troublesome in quantum field theory are almost always due to large energy and momenta i.e. *ultraviolet divergencies* – (UV), or conversely short distance behaviour.

It is possible to give perfectly meaningful and rigorous results to this divergent integrals in an ample class of theories, as we will see.

QED divergencies

We will now treat the divergencies given by loops present in quantum electrodynamics. Recall that the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}\left[i\gamma_\mu(\partial^\mu - ieA^\mu) - m\right]\psi \tag{5}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, ψ is a 4 component Dirac spinor and e the electric charge. Recall that under a gauge transformation $\mathcal{L}$ is invariant for $\psi \to e^{i\alpha}\psi$ and $A_\mu \to A_\mu + \partial_\mu\alpha$.

In order to perform the calculation, we will make use of the so–called *dimensional regularisation* (Bollini, 1972), (Hooft, 1972). This technique was introduced during investigation of the renormalisation of non Abelian gauge fields. It consists of writing Euclidean Feynman integrals obtained by means of a Wick rotation (Wick, 1954) $x_0 \to ix_0$ (i.e. with positive defined metric $x_0^2 + x_1^2 + x_3^2 + x_3^2$) in a space with generic dimension D, and making an analytic continuation in D itself which assumes non–integer values. Recalling that the volume of a D dimensional sphere S_D is

$$\int_{S_D} \mathrm{d}^D x = \int_0^R \mathrm{d}r\,\mathrm{d}\Omega_D\, r^{D-1} = R^D\,\frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}+1\right)} \tag{6}$$

i.e. $\Omega_D = \pi^{D/2}/\Gamma(D/2+1)$, one could infer that all possible divergencies stem only from the Gamma function term. Eventually, we will get rid of those infinities obtaining meaningful results.

As we are going to do our calculations in D dimensions, a few observations on the Lagrangian (5) are in order. One half spin spinors ψ have the dimension $(1-D)/2$, while vector fields A_μ have the dimension $(2-D)/2$. Clifford algebra of γ matrices retains the usual form

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{7}$$

while $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, \ldots -1)$. The coupling constant e is replaced by

$$e \to e\mu^{(4-D)/2} \tag{8}$$

μ being a mass parameter of dimensional regularisation. In Euclidean space photon propagator becomes simply $\delta_{\mu\nu}/p^2$, while the fermion pro-

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pagator is $-i/(\not{p}+m)$. The vertex of two fermions and a vector becomes $-ie\mu^{(4-D)/2}\gamma_\rho$.

Vacuum polarisation

The first loop diagram of QED we consider is the so–called vacuum polarisation diagram, i.e. the correction to the photon line, shown in Fig. 1.

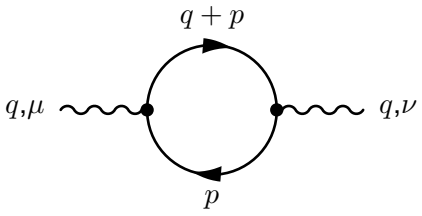

Figure 1 – Vacuum polarisation diagram
Рис. 1 − Диаграмма поляризации вакуума
Слика 1 – Дијаграм поларизације вакуума

The expression for the loop is given by

$$\Pi_{\mu\nu}(q) = -(e\mu^{(4-D)/2})^2 \int \frac{\mathrm{d}^D p}{(2\pi)^D}\ \mathrm{Tr}\left[\gamma_\mu \frac{1}{\not{p}+\not{q}+m}\gamma_\nu \frac{1}{\not{p}+m}\right] \tag{9}$$

where the minus sign comes because of the fermion loop, the trace is calculated over γ matrices. This will be rewritten in a more suitable form for calculations

$$\Pi_{\mu\nu}(q) = -(e\mu^{(4-D)/2})^2 \int \frac{\mathrm{d}^D p}{(2\pi)^D}\ \frac{\mathrm{Tr}[\gamma_\mu(\not{p}+\not{q}-m)\gamma_\nu(\not{p}-m)]}{[(p+q)^2+m^2][p^2+m^2]}\ . \tag{10}$$

From eq. (9), it is readily apparent that in $D=4$ the integral is quadratically divergent, with a sub-leading logarithmic divergent term. In fact, for large p we have

$$\int^\Lambda \frac{\mathrm{d}^D p}{(2\pi)^D}\ \frac{1}{\not{p}+\not{q}+m}\frac{1}{\not{p}+m} \sim \int^\Lambda \mathrm{d}p\ p^{D-1}\frac{1}{p^2} \sim \Lambda^{D-2}\ . \tag{11}$$

Let us introduce a trick due to Feynman for writing denominators, using

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{[(1-x)A+xB]^2}\ . \tag{12}$$

We will now shift the integration variable with the Feynman parameter x

$$p' = p + qx. \tag{13}$$

Plugging it back into eq. (10), we obtain

$$\Pi_{\mu\nu}(q) = -(e\mu^{(4-D)/2})^2 \times$$
$$\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D p'}{(2\pi)^D} \frac{\mathrm{Tr}[\gamma_\mu(\not{p}' + \not{q}(1-x) - m)\gamma_\nu(\not{p}' - \not{q}x - m)]}{[p'^2 + m^2 + q^2 x(1-x)]^2} \,. \tag{14}$$

By symmetry considerations, that is $p' \to -p'$, terms odd in p' do not contribute to the integral. For the trace in D dimensions of γ matrices, we have the following formulæ:

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu) = -2^{D/2}\delta_{\mu\nu} \tag{15}$$

and

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 2^{D/2}[\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}] \,. \tag{16}$$

Using these expressions, we will rewrite the trace of eq. (14) as follows

$$[p'_\rho p'_\sigma - q_\rho q_\sigma x(1-x)]\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) + m^2\mathrm{Tr}(\gamma_\mu\gamma_\nu) \,, \tag{17}$$

remembering that the trace of an odd number of gamma matrices is zero.

Using eqs. (15) and (16), we arrive at the following expression

$$2^{D/2}\{2p'_\mu p'_\nu - 2x(1-x)(q_\mu q_\nu - \delta_{\mu\nu}q^2) - \delta_{\mu\nu}[(p'^2 + m^2 + q^2x(1-x)]\} \,, \tag{18}$$

after having added and subtracted $\delta_{\mu\nu}q^2x(1-x)$. We end up with the following integral, after dropping the prime for simplicity

$$\Pi_{\mu\nu}(q) =$$
$$-(e\mu^{(4-D)/2})^2 2^{D/2} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D p}{(2\pi)^D} \left\{ \frac{2p_\mu p_\nu}{[p^2 + m^2 + q^2x(1-x)]^2} - \right.$$
$$\left. \frac{2\delta_{\mu\nu}}{[p^2 + m^2 + q^2x(1-x)]} - \frac{2x(1-x)[q_\mu q_\nu - \delta_{\mu\nu}q^2]}{[p^2 + m^2 + q^2x(1-x)]^2} \right\} \,. \tag{19}$$

Integrating over the loop with the aid of eqs. (67) and (69) shows that the first two terms cancel out. Defining $\varepsilon = (4-D)/2$, the integral reads

$$\Pi_{\mu\nu}(q) = \frac{e^2}{2\pi^2}\Gamma(\varepsilon)(q_\mu q_\nu - \delta_{\mu\nu}q^2)\int_0^1 \mathrm{d}x\; x(1-x)\left[\frac{m^2+q^2x(1-x)}{2\pi\mu^2}\right]^{-\varepsilon} . \tag{20}$$

As D approaches 4, we can expand eq. (20) in powers of ε. Noticing that a small power is indistinguishable from a logarithm, namely,

$$x^{-\varepsilon} = 1 - \varepsilon\log(x) + \mathcal{O}(\varepsilon^2) \tag{21}$$

and using the properties of the Gamma function illustrated in the appendix , we can see that eq. (20) becomes

$$\Pi_{\mu\nu}(q) = \frac{e^2}{2\pi^2}(q_\mu q_\nu - \delta_{\mu\nu}q^2)\left\{\frac{1}{6\varepsilon} - \frac{1}{6}\gamma - \right.$$
$$\left.\int_0^1 \mathrm{d}x\; x(1-x)\log\left[\frac{m^2+q^2x(1-x)}{2\pi\mu^2}\right]\right\} + \mathcal{O}(\varepsilon) , \tag{22}$$

where we have used the result of

$$\int_0^1 \mathrm{d}x\; x(1-x) = \frac{1}{6} . \tag{23}$$

Properties of vacuum polarisation

A few remarks on formula (22) are in order. First of all, we have actually checked out that thanks to dimensional regularisation and because of the Gamma function properties of its singularities, the loop integral has only a divergence given by a simple pole for $D \to 4$, all other terms being finite. The technique used when D approaches 4 is known by the name *"epsilon expansion"* (Wilson and Kogut, 1974), very often employed in Statistical Mechanics as well.

If just the finite part is considered, eq. (22) could be rewritten as

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - \delta_{\mu\nu}q^2)\pi(q^2) \tag{24}$$

where at one-loop order $\pi(q^2)$ is given by

$$\pi(q^2) = -\frac{e^2}{2\pi^2}\int_0^1 \mathrm{d}x\; x(1-x)\log\left[\frac{m^2+q^2x(1-x)}{2\pi\mu^2}\right] . \tag{25}$$

The form of eq. (24) shows that vacuum polarisation $\Pi_{\mu\nu}(q^2)$ obtained is also a Lorentz invariant as it should be, since all its parts are Lorentz

invariant. It can be shown that this statement holds true to any order of perturbation theory. It also obeys to the equation

$$q^{\mu}\Pi_{\mu\nu}(q^2) = 0 \tag{26}$$

known as Ward–Takahashi identity (Ward, 1950), (Takahashi, 1957). In QED, this means that non–transverse photon polarisation can be consistently ignored: a photon cannot acquire mass.

Observe also that vacuum polarisation depends upon a mass parameter μ, that is an energy scale.

Fermion propagator

Continue now to the computation to the correction of the fermion line shown in Fig. 2

Figure 2 – Fermion propagator diagram
Рис. 2 − Диаграмма пропагатора фермионов
Слика 2 – Дијаграм пропагатора фермиона

This diagram is usually denoted by $\Sigma(p)$. As for the vacuum polarisation graph, we will proceed with dimensional regularisation in Euclidean space, for which

$$\Sigma(p) = -(e\mu^{(4-D)/2})^2 \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \gamma_\mu \frac{(-i)}{\not p - \not k + m} \gamma_\nu \frac{\delta_{\mu\nu}}{k^2}\ . \tag{27}$$

For Euclidean space, the Clifford algebra of gamma matrices has the form

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}\ , \tag{28}$$

and rewrite the fermion propagator as

$$\frac{(-i)}{\not p + m} = i\frac{\not p - m}{p^2 + m^2}\ . \tag{29}$$

Introducing the Feynman parameter integration as seen before , we have

$$\Sigma(p) = -i(e\mu^{(4-D)/2})^2 \times$$
$$\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\gamma_\mu(\not{p} - \not{k} - m)\gamma_\mu}{[k^2(1-x) + (p-k)^2 x + m^2 x]^2} . \tag{30}$$

Shift the integration variable as

$$k' = k - px \tag{31}$$

and plug it back into (30) obtaining

$$\Sigma(p) = -ie^2\mu^{4-D} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k'}{(2\pi)^D} \frac{\gamma_\mu(\not{p}(1-x) - \not{k}' - m)\gamma_\mu}{[k'^2 + m^2 x + p^2 x(1-x)]^2} . \tag{32}$$

Because of symmetry, the terms linear in k' vanish. The remaining terms give

$$\Sigma(p) = -ie^2\mu^{4-D} \times$$
$$\int_0^1 \mathrm{d}x \, \gamma_\mu[\not{p}(1-x) - m]\gamma_\mu \frac{\Gamma\left(\frac{4-D}{2}\right)}{(4\pi)^{D/2}} [p^2 x(1-x) + m^2 x]^{(D-4)/2} \tag{33}$$

because of eq. (67).

Before expanding in $\varepsilon = (4-D)/2$, we have to make use of gamma matrices relations

$$\gamma_\mu\gamma_\mu = -D \tag{34}$$

and

$$\gamma_\mu\gamma_\rho\gamma_\mu = (2 - (4-D))\gamma_\rho . \tag{35}$$

In terms of ε, eq. (33) becomes

$$\Sigma(p) = -2i\frac{e^2}{16\pi^2}\Gamma(\varepsilon) \int_0^1 \mathrm{d}x \left[\frac{p^2 x(1-x) + m^2 x}{4\pi\mu^2}\right]^{-\varepsilon} \times$$
$$[\not{p}(1-x) + 2m - \varepsilon(\not{p}(1-x) + m)] . \tag{36}$$

Expanding around $\varepsilon = 0$ furnishes us with the result

$$\Sigma(p) = \frac{-i}{\varepsilon}\frac{e^2}{16\pi^2}(\not{p} + 4m) + i\frac{e^2}{8\pi^2}\left[\frac{1}{2}\not{p}(1+\gamma) + m(1+2\gamma) + \right.$$
$$\left. \int_0^1 \mathrm{d}x \, [\not{p}(1-x) + 2m] \log\left(\frac{p^2 x(1-x) + m^2 x}{4\pi\mu^2}\right)\right] + \mathcal{O}(\varepsilon) . \tag{37}$$

Vertex correction

Last one–loop QED correction is the vertex correction shown in Fig. 3.

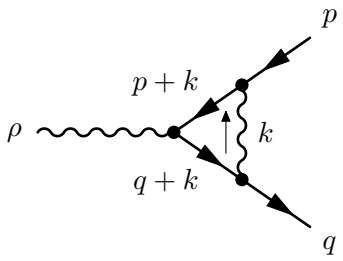

Figure 3 – Vertex correction diagram
Рис. 3 − Диаграмма коррекции вершин
Слика 3 – Дијаграм корекције вертекса

$$\Gamma_\rho(p,q) = -i(e\mu^{(4-D)/2})^3 \times$$
$$\int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \gamma_\mu \frac{1}{\not{p}+\not{k}+m}\gamma_\rho \frac{1}{\not{q}+\not{k}+m}\gamma_\nu \frac{\delta_{\mu\nu}}{k^2}\,. \tag{38}$$

This time, unlike in the previous cases, we have to deal with three propagators inside the loop, so this case is more complicated. It is necessary to introduce two Feynman parameters, x and y, make use of the relation

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{1}{[Ay + B(x-y) + C(1-x)]^3} \tag{39}$$

and rewrite eq. (38) as

$$\Gamma_\rho(p,q) = -2i(e\mu^{(4-D)/2})^3 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^D k}{(2\pi)^D}$$
$$\frac{\gamma_\nu(\not{p}+\not{k}-m)\gamma_\rho(\not{q}+\not{k}-m)\gamma_\nu}{[k^2 + m^2(x+y) + 2k\cdot(px+qy) + p^2 x + q^2 y]^3}\,. \tag{40}$$

Shifting the integration variable as follows

$$k' = k + px + qy \tag{41}$$

and plugging it back in the integral, eq. (40) becomes

$$\Gamma_\rho(p,q) = -2i(e\mu^{(4-D)/2})^3 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^D k}{(2\pi)^D}$$

$$\frac{\gamma_\nu(\not{k}-\not{q}y+\not{p}(1-x)-m)\gamma_\rho(\not{k}-\not{p}x+\not{q}(1-y)-m)\gamma_\nu}{[k^2+m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p\cdot qxy]^3}\,. \tag{42}$$

By inspection of eq. (42), the denominator goes like $1/k^6$ for large k and the numerator as k^2, thus (42) behaves like

$$\int^\Lambda \frac{\mathrm{d}^D k}{(2\pi)^D}\,\frac{k^2}{k^6}\sim\Lambda^{D-4}\,, \tag{43}$$

and in $D\to 4$ only the piece quadratic in k is problematic. Writing

$$\Gamma_\rho(p,q)=\Gamma^{(1)}_\rho(p,q)+\Gamma^{(2)}_\rho(p,q) \tag{44}$$

where $\Gamma^{(1)}_\rho(p,q)$ contains only the quadratic part in k, we find using eq. (67)

$$\Gamma_\rho(p,q)=-i\frac{(e\mu^{(4-D)/2})^3}{2(4\pi)^{D/2}}\Gamma\left(\frac{4-D}{2}\right)\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y \\ \frac{\gamma_\nu\gamma_\sigma\gamma_\rho\gamma_\sigma\gamma_\nu}{[m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p\cdot qxy]^{(4-D)/2}} \tag{45}$$

for the divergent part and for the convergent part using eq. (69)

$$\Gamma_\rho(p,q)=-i\frac{(e\mu^{(6-D)/2})^3}{2(4\pi)^{D/2}}\Gamma\left(\frac{6-D}{2}\right)\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y \\ \frac{\gamma_\nu[\not{p}(1-x)-\not{q}y-m]\gamma_\rho[\not{q}(1-y)-\not{p}x-m]\gamma_\nu}{[m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p\cdot qxy]^{(6-D)/2}}\,. \tag{46}$$

As eq. (46) is convergent, we are allowed to put directly $D=4$ obtaining

$$\Gamma_\rho(p,q)=-i((e\mu^{(6-D)/2})^3)\frac{e^2}{16\pi^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y \\ \frac{\gamma_\nu[\not{p}(1-x)-\not{q}y-m]\gamma_\rho[\not{q}(1-y)-\not{p}x-m]\gamma_\nu}{[m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p\cdot qxy]}\,. \tag{47}$$

For the divergent part, using the identity

$$\gamma_\sigma\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma=2\gamma_\nu\gamma_\rho\gamma_\mu-(D-4)\gamma_\mu\gamma_\rho\gamma_\nu \tag{48}$$

together with eq. (35) allows us to rewrite eq. (45) in the form

$$\Gamma_\rho(p,q)=-ie\mu^\varepsilon\gamma_\rho\frac{e^2}{8\pi^2}\Gamma(\varepsilon)(1-\varepsilon)^2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y$$

$$\left[\frac{m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p\cdot qxy}{4\pi\mu^2}\right]^{-\varepsilon}\ , \tag{49}$$

that is

$$\Gamma_\rho(p,q) = -ie\mu^\varepsilon\gamma_\rho\frac{e^2}{16\pi^2}\left\{\frac{1}{\varepsilon}-\gamma-1-2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\right.$$
$$\left.\log\left[\frac{m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p\cdot qxy}{4\pi\mu^2}\right]\right\}\ . \tag{50}$$

Other one–loop diagrams

At the one-loop level for QED, we have to consider also diagrams with an internal fermionic loop and an odd number of external photons, as shown in Figs. 4 and 5, and one with four external photons as in Fig. 6.

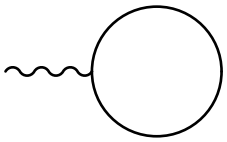

Figure 4 – One external photon
Рис. 4 − Один внешний фотон
Слика 4 – Један спољашњи фотон

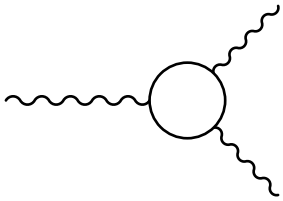

Figure 5 – Three external photons
Рис. 5 − Три внешних фотона
Слика 5 – Три спољашња фотона

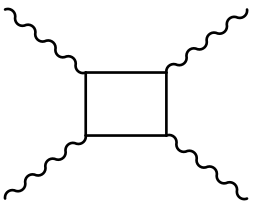

Figure 6 – Feynman diagram for photon scattering
Рис. 6 − Диаграмма Фейнмана для рассеяния фотонов
Слика 6 – Фејнманов дијаграм расејања фотона

Fermion loops with an odd number of external photon lines vanish because of symmetry reasons. Consider such a diagram G_n with n points

that can be written as

$$G_n = \mathrm{Tr}[\gamma_{\mu(1)} S_F(x_1, x_n)\gamma_{\mu(n)} S_F(x_n, x_{n-1})\gamma_{\mu(n-1)} \dots \gamma_{\mu(2)} S_F(x_2, x_1)] \quad (51)$$

where $S_F(x_i, x_j)$ is the fermion propagator that connects points x_i and x_j, where photon lines insert. From gamma matrices algebra, we recall the existence of a matrix $C = i\gamma^2\gamma^0$ such that transposes each gamma matrix:

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T \, . \quad (52)$$

Therefore, this relation holds true for the fermionic propagator as well

$$CS_F(x, y)C^{-1} = S_F(y, x)^T \, , \quad (53)$$

where the inversion of coordinates in the propagator should be noticed. Insert now the term CC^{-1} between the propagators in eq. (51) and rewrite

$$\begin{gathered} G_n = (-1)^n \mathrm{Tr}[\gamma_{\mu(1)}^T S_F^T(x_n, x_1)\gamma_{\mu(n)}^T S_F^T(x_{n-1}, x_n)\gamma_{\mu(n-1)}^T \dots \\ \gamma_{\mu(2)}^T S_F^T(x_1, x_2)] = \\ (-1)^n \mathrm{Tr}[\gamma_{\mu(1)} S_F(x_1, x_n)\gamma_{\mu(n)} S_F(x_n, x_{n-1})\gamma_{\mu(n-1)} \dots \\ \gamma_{\mu(2)} S_F(x_2, x_1)] \, . \end{gathered} \quad (54)$$

For odd n, therefore, $G_n = -G_n$ so it implies $G_n = 0$. This proves the statement, known as Furry's theorem (Furry, 1937).

About the box diagram with four external photons depicted in Fig. 6, often referred to as a light–light scattering diagram, the internal loop is made out of four fermions; therefore, for $D = 4$, it is expected to diverge like

$$\int^\Lambda \mathrm{d}^4 p \, \frac{1}{\not{p}^4} \sim \log \Lambda \, . \quad (55)$$

Luckily, this diagram is actually convergent after an explicit calculation.

Suppose that the box diagram was actually divergent. This would mean that we are faced with new interactions among four photons, $A_\mu A_\nu A_\rho A_\sigma$, not present in the original Lagrangian (5). Because of gauge invariance, this new term should be proportional to $(F_{\mu\nu})^4$, implying a kinetic term with more than two derivatives, possibly spoiling causality.

Appendix

Gaussian and Feynman integrals

$$\int_{-\infty}^{+\infty} \mathrm{d}x\ e^{-ax^2} = \sqrt{\frac{\pi}{a}} \tag{56}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\ e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}} \tag{57}$$

Momenta:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\ x^{2n} e^{-ax^2} =$$

$$\left(\frac{d}{db}\right)^{2n} \int_{-\infty}^{+\infty} \mathrm{d}x\ e^{-ax^2+bx}\bigg|_{b=0} = \sqrt{\frac{\pi}{a}}\,\frac{1}{a^n}\,(2n-1)!! \tag{58}$$

odd momenta are zero by symmetry.

In Euclidean D dimensions, we have:

$$\int \mathrm{d}^D x\ \exp(-\alpha x^2 + \beta\cdot x) = \left(\frac{\pi}{\alpha}\right)^{D/2} \exp\left(\frac{\beta^2}{4\alpha}\right)\ . \tag{59}$$

For the operators A and J, where $\mathbf{x}\cdot A\cdot\mathbf{x} = x_i A_{ij} x_j$ and $J\cdot\mathbf{x} = J_{ij}x_j$, with repeated indices summed over:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^{N}\mathrm{d}x_i\ e^{-\frac{1}{2}\mathbf{x}\cdot A\cdot\mathbf{x}+J\cdot\mathbf{x}} = \left[\frac{(2\pi)^N}{\det(A)}\right]^{\frac{1}{2}} e^{\frac{1}{2}J\cdot A^{-1}\cdot J}\ . \tag{60}$$

An almost trivial yet very useful expression is the following:

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)}\int_0^{+\infty}\mathrm{d}\alpha\ \alpha^{n-1}\exp(-\alpha A) \tag{61}$$

that allows us to combine propagators and Gaussian integrals.

Applying eqs. (59) and (61) to the propagators written in Euclidean space (i.e. $k^2 = k_0^2 + \mathbf{k}^2$), one obtains:

$$\frac{1}{k^2+m^2} = \int_0^{+\infty}\mathrm{d}\alpha\ \exp[-\alpha(k^2+m^2)] \tag{62}$$

$$\int \mathrm{d}^D k\ \frac{1}{(k^2+m^2)} = \int_0^{+\infty}\mathrm{d}\alpha\ \int \mathrm{d}^D k\ \exp[-\alpha(k^2+m^2)] =$$

$$\pi^{D/2}\int_0^{+\infty}\mathrm{d}\alpha\;\alpha^{-D/2}\exp(-\alpha m^2)=\pi^{D/2}\Gamma\left(1-\frac{D}{2}\right)m^{(D-2)} \tag{63}$$

$$\int\mathrm{d}^Dk\;\frac{1}{(k^2+m^2)^2}=$$
$$\pi^{D/2}\int_0^{+\infty}\mathrm{d}\alpha\;\alpha^{-D/2-1}\exp(-\alpha m^2)=\pi^{D/2}\Gamma\left(2-\frac{D}{2}\right)m^{(D-4)} \tag{64}$$

$$\int\mathrm{d}^Dk\;\frac{1}{(k^2+m^2)^3}=\frac{1}{2}\pi^{D/2}\Gamma\left(3-\frac{D}{2}\right)m^{(D-6)}\;. \tag{65}$$

By induction, one obtains the formula for a generic power of a propagator:

$$\int\mathrm{d}^Dk\;\frac{1}{(k^2+m^2)^n}=\pi^{D/2}\frac{\Gamma\left(n-\frac{D}{2}\right)}{\Gamma(n)}m^{(D-2n)}\;. \tag{66}$$

Let us now shift the integration variable $k=k'+p$ and insert it back into eq. (66), obtaining

$$\int\mathrm{d}^Dk\;\frac{1}{(k^2+2p\cdot k+(m^2+p^2))^n}=\pi^{D/2}\frac{\Gamma\left(n-\frac{D}{2}\right)}{\Gamma(n)}m^{(D-2n)}\;. \tag{67}$$

By repeated differentation of eq. (67) $\partial/\partial p_\mu$ we obtain the expressions

$$\int\mathrm{d}^Dk\;\frac{k_\mu}{(k^2+2p\cdot k+(m^2+p^2))^n}=\pi^{D/2}\frac{\Gamma\left(n-\frac{D}{2}\right)}{\Gamma(n)}m^{(D-2n)}(-p_\mu)\;. \tag{68}$$

and

$$\int\mathrm{d}^Dk\;\frac{k_\mu k_\nu}{(k^2+2p\cdot k+(m^2+p^2))^n}=$$
$$\frac{\pi^{D/2}}{\Gamma(n)}m^{(D-2n)}\left[\Gamma\left(n-\frac{D}{2}\right)p_\mu p_\nu+\frac{1}{2}\delta_{\mu\nu}\Gamma\left(n-1-\frac{D}{2}\right)m^2\right] \tag{69}$$

Riemann's Gamma function

Riemann's Gamma function is defined by the integral

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\; t^{z-1} e^{-t}\;, \tag{70}$$

for $\Re(z) > 0$, which satisfies the recursive relation $\Gamma(z+1) = z\Gamma(z)$, thus having the property that for integer values $\Gamma(n) = (n-1)!$ since $\Gamma(1) = 1$.

Originally, the function was defined by Weierstrass as an infinite product by the relation

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right] \tag{71}$$

γ being the Euler–Mascheroni's constant, $\gamma \approx 0.57721$. From (71), it is readily apparent that $\Gamma(z)$ is analytic except for negative integers $z = 0, -1, -2, \ldots$ where it has simple poles. It is also possible to obtain the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{72}$$

that allows us to obtain the value $\left\{\Gamma(\frac{1}{2})\right\}^2 = \pi$. Defining the function $\psi(z)$ (or digamma) as the logarithmic derivative of $\Gamma(z)$, i.e.

$$\psi(z) = \frac{\Gamma(z)'}{\Gamma(z)} \tag{73}$$

from the reflection formula it has the property that $\psi(1-z) - \psi(z) = \pi \cot \pi z$.

Subsequent derivatives are defined by the functions

$$\psi^{(n)}(z) = \left(\frac{d}{dz}\right)^{(n+1)} \log(\Gamma(z)) \tag{74}$$

that allows us to express the Gamma function near the simple poles $z = -n$:

$$\Gamma(z) = \frac{(-1)^n}{n!(z+n)} + \frac{(-1)^n \psi(n+1)}{n!} + \mathcal{O}(z+n) \tag{75}$$

For integers n

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k} = -\gamma + H_n \tag{76}$$

where H_n is a harmonic number, defined as seen above from the sum of the reciprocal of integers, having the property that $H_n \sim \log n + \gamma + 1/(2n) + \mathcal{O}(1/n^2)$ for large n.

Another relevant formula related to the Gamma function is the asymptotic expansion, known as Stirling's series:

$$\Gamma(z+1) = \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \mathcal{O}\left(\frac{1}{z^5}\right)\right] \tag{77}$$

that can be computed from a saddle approximation of the Gamma function.

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КВАНТОВЫЕ ЭЛЕКТРОДИНАМИЧЕСКИЕ РАСХОДИМОСТИ

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РУБРИКА ГРНТИ: 27.35.00 Математические модели естественных наук и технических наук. Уравнения математической физики
27.35.57 Математические модели квантовой физики
27.35.59 Методы теории возмущений
ВИД СТАТЬИ: обзорная статья

Резюме:

Введение/цель: В данной статье обсуждается проблема расходимостей в квантовой электродинамике (QED).

Методы: В статье применялся метод ренормализационной группы в работе с бесконечностями в QED.

Результаты: Интегралы в QED, дающие физические наблюдаемые величины, конечны.

Выводы: Расхождения в QED можно рассматривать последовательным образом, обеспечивая строгие математические результаты.

Ключевые слова: квантовая электродинамика, квантовая теория поля, ренормализационная группа.

КВАНТНЕ ЕЛЕКТРОДИНАМИЧКЕ ДИВЕРГЕНЦИЈЕ

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ОБЛАСТ: математика, физика
ВРСТА ЧЛАНКА: прегледни рад

Сажетак:

Увод/циљ: У раду се разматра проблем дивергенција у квантној електродинамици (QED).

Методе: Метода ренормализационе групе користи се за решавање бесконачности у QED.

Резултати: Интегрални у QED који дају физичке опсервабилности јесу коначнии.

Закључак: Разлике у QED могу се третирати на доследан начин пружајући строге математичке резултате.

Кључне речи: квантна електродинамика, квантна теорија поља, ренормализациона група.

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