

ПРЕГЛЕДНИ РАДОВИ

ОБЗОРНЫЕ СТАТЬИ

REVIEW PAPERS

## CORRECTIONS TO PROPAGATORS OF QUANTUM ELECTRODYNAMICS

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*Abstract:*

*Introduction/purpose:* The problem of quantum corrections to propagators in Quantum Electrodynamics (QED) is discussed.

*Methods:* The Dyson–Schwinger equation is employed for correcting propagators in QED.

*Results:* The observable quantities in QED are finite.

*Conclusions:* QED divergencies can be avoided by redefining physical quantities in a suitable manner.

*Key words:* Quantum Electrodynamics, Quantum Field Theory, Renormalization Group

### QED loops

#### Corrected photon propagator

In (Fabiano, 2021) we have computed the correction to the photon line at one-loop level in QED. Remembering that the bare photon propagator is given by the expression

$$iD_{\mu\nu}(q) = -i \frac{g_{\mu\nu}}{q^2 + i\varepsilon} \quad (1)$$

obtained, roughly speaking, by inverting the term  $F^2$  in the Lagrangian (5) of (Fabiano, 2021). In Minkowskian metric the vacuum polarisation is given

by

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu} q^2) \pi(q^2) . \quad (2)$$

The *physical* or *renormalised* photon propagator is obtained by considering all possible corrections to the photon line, as illustrated in eq. (3).

$$iD_{\mu\nu}^P = \text{diagram with black circle} = \text{diagram with wavy line} + \text{diagram with one loop} + \text{diagram with two loops} + \dots \quad (3)$$

As we can see, the physical photon propagator  $iD_{\mu\nu}^P$  is obtained by repeated insertions of vacuum polarisation diagrams at one-loop level, in the following manner:

$$\begin{aligned} iD_{\mu\nu}^P(q) &= iD_{\mu\nu}(q) + iD_{\mu\lambda}(q) i\Pi_{\lambda\rho}(q) iD_{\rho\nu}(q) + \\ & iD_{\mu\lambda}(q) i\Pi_{\lambda\rho}(q) iD_{\rho\kappa}(q) i\Pi_{\kappa\sigma}(q) iD_{\sigma\nu}(q) + \dots \end{aligned} \quad (4)$$

Recalling the geometric series for which this expression holds true

$$\sum_{n=1}^{+\infty} (-x)^n = \frac{1}{1+x} , \quad (5)$$

one could immediately recognise the same pattern in eq. (4) and rewrite it as (Dyson, 1949), (Schwinger, 1951)

$$\begin{aligned} iD_{\mu\nu}^P(q) &= \frac{-i}{q^2} g_{\mu\nu} \{ 1 - \pi(q^2) + [\pi(q^2)]^2 + \dots \} + \mathcal{O}(q_\mu q_\nu) = \\ & \frac{-i}{q^2} g_{\mu\nu} \frac{1}{1 + \pi(q^2)} + \mathcal{O}(q_\mu q_\nu) . \end{aligned} \quad (6)$$

### Corrected electron propagator

Proceeding in a manner completely analogous to previous section we could calculate the physical electron propagator. The bare electron propa-



gator is given by

$$S(p) = i \frac{1}{\not{p} - m + i\varepsilon} = i \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}, \quad (7)$$

while the physical propagator  $S^P(p)$  is obtained by repeated insertions of  $\Sigma(p)$  calculated in (Fabiano, 2021), formula (27):

$$S^P = \text{---} \blacktriangle \text{---} = \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} + \dots \quad (8)$$

The expression for  $S^P$  is pictorially represented in eq. (8), this translates to:

$$S^P(p) = S(p) + S(p)\Sigma(p)S(p) + S(p)\Sigma(p)S(p)\Sigma(p)S(p) + \dots \quad (9)$$

and using eq. (5) we end with the expression

$$S^P(p) = i \frac{1}{\not{p} - m - \Sigma(p) + i\varepsilon}. \quad (10)$$

### Counterterms

Up to now, we have computed all possible fundamental divergencies in QED. Those are necessary to build the necessary *counterterms* in order to renormalise QED. Those counterterms are suitably constructed terms in the Lagrangian in order to cancel out divergencies and make results finite. To recap, we started with this classical Lagrangian in  $D$  dimensions

$$\mathcal{L}_{cl} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\not{\partial}\psi + e\mu^{(4-D)/2}\bar{\psi}\not{A}\psi - m\bar{\psi}\psi, \quad (11)$$

and we add a counterterm Lagrangian with the *same form* of the present Lagrangian of (11)

$$\mathcal{L}_{ct} = -\frac{1}{4}K_3F^{\mu\nu}F_{\mu\nu} + iK_2\bar{\psi}\not{\partial}\psi + e\mu^{(4-D)/2}K_1\bar{\psi}\not{A}\psi - mK_m\bar{\psi}\psi. \quad (12)$$

The obtained renormalised Lagrangian

$$\mathcal{L}_{ren} = \mathcal{L}_{cl} + \mathcal{L}_{ct} \quad (13)$$

could be expressed in terms of the bare quantities defined in the following way:

$$\psi_0 = \sqrt{1 + K_2} \psi = Z_2^{1/2} \psi \quad (14)$$

$$A_0^\mu = \sqrt{1 + K_3} A^\mu = Z_3^{1/2} A^\mu \quad (15)$$

$$e_0 = e \mu^{(4-D)/2} \frac{1 + K_1}{(1 + K_2) \sqrt{1 + K_3}} = \frac{Z_1}{Z_2 Z_3^{1/2}} e \mu^{(4-D)/2} \quad (16)$$

$$m_0 = m \frac{1 + K_m}{1 + K_2} = \frac{Z_m}{Z_2} m \quad (17)$$

where we have introduced Dyson's  $Z$  notation (Dyson, 1952), and bare quantities, which do not depend on the scale  $\mu$ , are denoted by a 0 subscript. Often, eq. (14) is called *wave function renormalisation*. The renormalised Lagrangian is

$$\mathcal{L}_{ren} = -\frac{1}{4} F_0^{\mu\nu} F_{0\mu\nu} + i \bar{\psi}_0 \not{\partial} \psi_0 + e_0 \bar{\psi}_0 \not{A}_0 \psi_0 - m_0 \bar{\psi}_0 \psi_0, \quad (18)$$

or in Dyson's notation

$$\mathcal{L}_{ren} = -\frac{Z_3}{4} F^{\mu\nu} F_{\mu\nu} + i Z_2 \bar{\psi} \not{\partial} \psi + e Z_1 \bar{\psi} \not{A} \psi - m Z_m \bar{\psi} \psi. \quad (19)$$

The covariant derivative in  $\mathcal{L}_{ren}$  transforms as

$$\mathcal{D}_{ren}^\mu = \partial^\mu - i e \frac{Z_1}{Z_2} A^\mu \quad (20)$$

and, in order not to spoil gauge invariance of the Lagrangian it needs to be  $Z_1 = Z_2$ . It is possible to show that this is actually the case to all orders of perturbation theory.

The counterterms can be read off the one-loop calculations encountered in (Fabiano, 2021). Starting with fermion line correction, from eq. (37) of (Fabiano, 2021) we extract the term

$$\Sigma(p) = -i \frac{e^2}{16\pi^2} (\not{p} + 4m) \frac{1}{\varepsilon} + \text{finite terms} \quad (21)$$

and comparing to the inverse of the bare electron propagator, eq. (7)

$$S^{-1}(p) = -i(\not{p} - m + i\varepsilon) \quad (22)$$



one could infer that the term in  $\not{p}$  is related to  $Z_2$ , while the term proportional to  $m$  is related to  $Z_m$ . Therefore

$$K_2 = -\frac{e^2}{16\pi^2} \left[ \frac{1}{\varepsilon} + F_2 \left( \varepsilon, \frac{m}{\mu} \right) \right] \quad (23)$$

and

$$K_m = -\frac{e^2}{4\pi^2} \left[ \frac{1}{\varepsilon} + F_m \left( \varepsilon, \frac{m}{\mu} \right) \right], \quad (24)$$

where functions  $F_2$  and  $F_m$  are arbitrary finite parts depending upon  $\varepsilon$  and  $m/\mu$ , and are analytical as  $\varepsilon \rightarrow 0$ . It means that the counterterms contain just the part proportional to  $\frac{1}{\varepsilon}$  necessary to cancel the overall divergencies.

The second correction we tackle is the one for the photon line encountered in (Fabiano, 2021). From eq. (22) of (Fabiano, 2021) we have

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - \delta_{\mu\nu} q^2) \left[ \frac{e^2}{12\pi^2} \frac{1}{\varepsilon} + \text{finite terms} \right], \quad (25)$$

and using the relation of eq. (4) we have for the one-loop propagator

$$D_{\mu\nu}(p) = \frac{\delta_{\mu\nu}}{p^2} + \frac{\delta_{\mu\nu}}{p^2} \Pi \frac{\delta_{\mu\nu}}{p^2} + \dots = \frac{\delta_{\mu\nu}}{p^2} \left[ 1 - \frac{e^2}{12\pi^2} \frac{1}{\varepsilon} \right] + \frac{p_\mu p_\nu}{p^4} \frac{e^2}{12\pi^2} \frac{1}{\varepsilon} + \dots \quad (26)$$

so that

$$K_3 = -\frac{e^2}{12\pi^2} \left[ \frac{1}{\varepsilon} + F_3 \right] \quad (27)$$

where  $F_3$  is an arbitrary dimensionless finite function.

Last comes the vertex correction, from (Fabiano, 2021) eq. (50) we have

$$\Gamma_\rho(p, q) = -ie\mu^\varepsilon \gamma_\rho \left[ \frac{e^2}{16\pi^2} \frac{1}{\varepsilon} + \text{finite terms} \right] \quad (28)$$

that gives

$$K_1 = -\frac{e^2}{12\pi^2} \left[ \frac{1}{\varepsilon} + F_1 \right] \quad (29)$$

where, once more,  $F_1$  is a finite function. In terms of the  $Z$  notation, we summarise our results as

$$Z_1 = 1 - \frac{e^2}{16\pi^2} \left[ \frac{1}{\varepsilon} + F_1 \right] + \mathcal{O}(e^4) \quad (30)$$

$$Z_2 = 1 - \frac{e^2}{16\pi^2} \left[ \frac{1}{\varepsilon} + F_2 \right] + \dots \tag{31}$$

$$Z_3 = 1 - \frac{e^2}{12\pi^2} \left[ \frac{1}{\varepsilon} + F_3 \right] + \dots \tag{32}$$

$$Z_m = 1 - \frac{e^2}{4\pi^2} \left[ \frac{1}{\varepsilon} + F_m \right] + \dots \tag{33}$$

We remark once more that  $Z_1 = Z_2$  is satisfied to this order in perturbation theory. So using the relation of eq. (16) and remembering that  $\varepsilon = (4 - D)/2$ , for  $D \rightarrow 4$  we have

$$e_0 = e\mu^{2\varepsilon} \left[ 1 + \frac{e^2}{24\pi^2} \frac{1}{\varepsilon} + \text{finite terms} + \mathcal{O}(e^3) \right]. \tag{34}$$

If we ignore the finite part of the counterterms by adopting a mass-independent prescription, also known as the *minimal subtraction scheme*, or MS scheme ('t Hooft, 1973), (Weinberg, 1973), for which the finite part is zero, we can compute the so-called *beta function* due to Gell–Mann and Low (Gell–Mann and Low, 1954) defined in the following way:

$$\beta(e) = \lim_{\varepsilon \rightarrow 0} \mu \frac{\partial e}{\partial \mu}, \tag{35}$$

which is an analytic function in  $\varepsilon$ . Compute the beta function from eq. (34) by differentiating with respect to  $\mu$ , remembering that  $\mu_0$  is constant taking the prescribed limit  $\varepsilon \rightarrow 0$ , and obtain

$$\beta(e) = \mu \frac{\partial e}{\partial \mu} = \frac{e^3}{12\pi^2} \tag{36}$$

which is actually a differential equation for electric charge  $e$  as a function of a mass scale  $\mu$ :

$$12\pi^2 \int_{e_0}^e de \frac{1}{e^3} = \int_{\mu_0}^{\mu} d\mu \frac{1}{\mu}, \tag{37}$$

where  $\mu_0$  is an arbitrary scale. The explicit solution to this equation is

$$\frac{1}{e^2(\mu)} - \frac{1}{e^2(\mu_0)} = -\frac{1}{6\pi^2} \log \left( \frac{\mu}{\mu_0} \right) \tag{38}$$

which can be written in an explicit form for  $e^2(\mu)$ :

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 - \frac{e^2(\mu_0)}{6\pi^2} \log \left( \frac{\mu}{\mu_0} \right)}. \tag{39}$$

A few comments on eq. (39). It has a singularity at the point

$$\mu = \mu_0 \exp[6\pi^2 e^{-2}(\mu_0)] , \quad (40)$$

better known as the *Landau pole* (Landau et al, 1954), (Landau and Pomeranchuk, 1955). A careful evaluation in QED shows that the Landau pole is of order of  $10^{284}$  eV, a huge scale much larger than anything envisaged so far – for instance the Large Hadron Collider (LHC) works at about  $10^{13}$  eV, while the *Planck scale*, that is a scale at which quantum gravity effects should become relevant,  $\sqrt{\hbar c^5/G}$ , is at “only”  $10^{28}$  eV.

As the energy scale increases, or conversely, the distance decreases, the electron charge increases.

### Running coupling constant

The formalism of the beta function and the existence of a so-called *running coupling constant*<sup>1</sup> is not a peculiarity of QED but it is standard behaviour in any quantum field theory. We have seen that in the minimal subtraction scheme the counterterms in the Lagrangian have no finite parts, therefore can be expanded in a Laurent series in  $\varepsilon$  containing only divergent parts. Call the generic renormalised coupling constant  $g$  and its bare version  $g_0$ , then the above statement could be written as (hereafter,  $\varepsilon = 4 - D$ )

$$g_0 = \mu^\varepsilon \left[ g + \sum_{k=1}^{+\infty} \frac{g_k(g)}{\varepsilon^k} \right] , \quad (41)$$

where  $g_k$  are regular functions in  $g$ . Analogous expansions exist for bare mass  $m_0$  and bare fields  $\psi_0, A_0^\mu$ .

Now, a crucial observation is that all bare quantities are independent of the scale by definition. As the bare coupling constant is not dependent upon  $\mu$ ,  $dg_0/d\mu = 0$ . Applying the derivative to eq. (41), one obtains

$$\varepsilon g + \left( g_1 + \mu \frac{\partial g}{\partial \mu} \right) + \sum_{k=1}^{+\infty} \frac{1}{\varepsilon^k} \left[ \frac{dg_k}{dg} \mu \frac{\partial g}{\partial \mu} + g_{k+1} \right] = 0 . \quad (42)$$

We have already discussed that  $\mu \partial g / \partial \mu$  is an analytical function in  $\varepsilon$ , so we can write it as follows:

$$\mu \frac{\partial g}{\partial \mu} = \sum_{n=0}^{+\infty} d_n \varepsilon^n , \quad (43)$$

<sup>1</sup>An oxymoron!

and insert this form into eq. (42).

We obtain the equation for coefficients  $d$  of the beta function:

$$\varepsilon(g + d_1) + \left( g_1 + d_0 + d_1 \frac{dg_1}{dg} \right) + \sum_{k=1}^{+\infty} \frac{1}{\varepsilon^k} \left[ g_{k+1} + d_0 \frac{dg_k}{dg} + d_1 \frac{dg_{k+1}}{dg} \right] = 0, \quad (44)$$

and observe that only the first two  $d$  terms survive,  $d_0$  and  $d_1$ , so that eq. (43) is only linear in  $\varepsilon$ . We group different powers of  $\varepsilon$ , and each one of them has to vanish separately, so we have

$$\begin{aligned} (g + d_1) &= 0 \\ g_1 + d_1 \frac{dg_1}{dg} &= -d_0 \\ \left( 1 + d_1 \frac{d}{dg} \right) g_{k+1} &= -d_0 \frac{dg_k}{dg}. \end{aligned} \quad (45)$$

Solving eqs. (45) and plugging it back in eq. (43) we end up with

$$\mu \frac{\partial g}{\partial \mu} = -g_1 + g \frac{dg_1}{dg} - g\varepsilon, \quad (46)$$

and taking the limit  $\varepsilon \rightarrow 0$ :

$$\beta(g) = -g_1 + g \frac{dg_1}{dg}. \quad (47)$$

We also found the recurrence relation for the coefficients of the counterterms:

$$\left( 1 - g \frac{d}{dg} \right) [g_{k+1}(g) - g_1(g)] = \frac{d}{dg} g_k(g). \quad (48)$$

This recursion relation is very important because it shows that the coefficients of higher order poles can, at least in principle, be computed from just the knowledge of the simple pole term. So, in the minimal subtraction scheme we have seen that the beta function depends only on the coupling constant  $g$ , and the latter depends only on the scale  $\mu$ ; therefore, we can write

$$\mu \frac{dg(\mu)}{d\mu} = \beta(g(\mu)). \quad (49)$$

This equation is known as the Callan–Symanzik equation (Callan, 1970), (Symanzik, 1970).



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## ПОПРАВКИ ПРОПАГАТОРОВ КВАНТОВОЙ ЭЛЕКТРОДИНАМИКИ

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РУБРИКА ГРНТИ: 27.35.00 Математические модели  
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27.35.57 Математические модели квантовой  
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27.35.59 Методы теории возмущений

ВИД СТАТЬИ: обзорная статья

*Резюме:*

*Введение/цель:* В данной статье обсуждается проблема квантовых поправок к пропагаторам в квантовой электродинамике (КЭД).

*Методы:* Для поправок к пропагаторам в КЭД использовалось уравнение Дайсона – Швингера.

*Результаты:* Наблюдаемые величины в QED конечны.

*Выводы:* Расхождения в КЭД можно избежать, соответствующим образом переопределив физические величины.

*Ключевые слова:* квантовая электродинамика, квантовая теория поля, ренормализационная группа.

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## КОРЕКЦИЈЕ ПРОПАГАТОРА КВАНТНЕ ЕЛЕКТРОДИНАМИКЕ

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ОБЛАСТ: математика

ВРСТА ЧЛАНКА: прегледни рад

**Сажетак:**

**Увод/циљ:** У раду се разматра проблем квантних корекција пропагатора у квантној електродинамици (QED).

**Метод:** Коришћена је Дајсон-Швингерова једначина за корекцију пропагатора у QED-у.

**Резултати:** Посматране количине у QED-у су коначне.

**Закључак:** Дивергенције у квантној електродинамици могу се избећи редефинисањем физичких величина на одговарајући начин.

**Кључне речи:** квантна електродинамика, квантна теорија поља, ренормализациона група.

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