# A NEW VERSION OF THE RESULTS OF $U_n$–HYPERMETRIC SPACES

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*Abstract*:

*Introduction/purpose: The aim of this paper is to present the concept of a universal hypermetric space. An $n$-dimensional $(n \geq 2)$ hypermetric distance over an arbitrary non-empty set $X$ is generalized. This hypermetric distance measures how separated all $n$ points of the space are. The paper discusses the concept of completeness, with respect to this hypermetric as well as the fixed point theorem which play an important role in applied mathematics in a variety of fields.*

*Methods: Standard proof based theoretical methods of the functional analysis are employed.*

*Results: The concept of a universal hypermetric space is presented. The universal properties of hypermetric spaces are described.*

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*Conclusion: This new version of the results for $U_n$-hypermetric spaces may have applications in various disciplines where the degree of clustering is sought for.*

*Key words: $U_n$-hypermetric spaces, OG-metric, G-metric.*

## Introduction

The role of distance in understanding the world is undeniable. Our intuitive understanding of the concept of distance in the real world, however, is different from the one proposed in mathematics. Some of the properties that belong to our understanding of distance from the real world, such as symmetry and single-valuedness, are not necessarily established within certain abstract distances.

This will, in fact, be our main motivation for presenting a generalized concept of distance as a set-valued function in this paper.
The notion of $2$-metric spaces, as a possible generalization of metric spaces, was introduced by Gähler (Gähler, 1963), (Gähler, 1964), (Gähler, 1966). See also (Diminnie et al, 2017), (Ha et al, 1990) for further developments. The $2$-metric $d(x, y, z)$ is a function of $3$ variables, and Gähler geometrically interpreted it as an area of a triangle with the vertices at $x$ , $y$ and $z$, respectively.

This led B. C. Dhage, in his PhD thesis in 1992, to introduce the notion of $D$-metric (Dhage et al, 2000) that does, in fact, generalize metric spaces. Subsequently, Dhage published a series of papers attempting to develop topological structures in such spaces and prove several fix point results.

Most of the claims, however, concerning the fundamental topological properties of $D$-metric spaces, are incorrect. In 2003, Mustafa and Sims demonstrated that in a strong remark (Mustafa & Sims, 2003). This led them to introducing the notion of a $G$-metric space (Mustafa & Sims, 2006), as a generalization of metric spaces. In this type of spaces, a non-negative real number is assigned to every triplet of elements.

In an attempt to generalize the notion of a $G$-metric space to more than three variables, Khan first introduced the notion of a $K$-metric, and later the notion of a generalized $n$-metric space (for any $n \geq 2$) (Khan, 2012), (Khan, 2014). He also proved a common fixed point theorem for such spaces.

$G$-metric spaces were generalized to universal metrics in (Dehghan Nezhad & Mazaheri, 2010), (Dehghan Nezhad & Aral, 2011), (Dehghan

Nezhad & Khajuee, 2013), (Dehghan Nezhad et al, 2017). The interpretation of the perimeter of a triangle is applied, but this time on $G$-metric spaces. Since then, many authors have obtained fixed point results for $G$-metric spaces.

The main purpose of this paper is a generalization of universal metric spaces into universal hypermetric spaces of the $n$-dimension (see (Kelly, 1975) for a discussion on hypermetric spaces). In the first part, we generalize an $n$-dimensional $(n \geq 2)$ hypermetric distance over an arbitrary non-empty set $X$. This hypermetric measures how separated all $n$ points of the space are. The hyperdistance function is defined in any way we like, the only constraint being the simultaneous satisfaction of the three properties, viz non-negativity and positive-definiteness, symmetry and triangle inequality. In the second part, we discuss the concept of completness, with respect to this hypermetric, and the fixed point theorem, which play an important role in applied mathematics in a variety of fields. Examples show a fundamental difference between our results and the well-known ones. This concept is the first view of novel methods for selecting the clusters by hypermetric. The purpose definition is applicable for engineering science (for example, the theory of clustering).

By a strict order relation of a set X, we mean a binary relation " $<$ ", which is transitive ($\alpha < \beta$ and $\beta < \gamma$ implies $\alpha < \gamma$), such that $\alpha < \beta$ and $\beta < \alpha$ cannot both hold. It is a strict total order relation, if for every $\alpha$,$\beta$ belonging to $X$, exactly one and only one of $\alpha < \beta$, $\beta < \alpha$ or $\alpha = \beta$ holds. A group $G$ is called left-ordered, if endowed with a strict total relation " $<$ " which is left invariant, meaning that $\alpha < \beta$ implies $\gamma + \alpha < \gamma + \beta$, for all $\alpha$,$\beta$,$\gamma \in G$. We will say that $G$ is bi-ordered, if it admits the left and right invariant properties simultaneously (historically, this has been called simple-ordered). We refer to the ordered pair $(G, <)$ as an ordered group (Cohen & Goffman, 1949). From now on, we assume that $1$ denotes the identity element of $G$. It should be noted that, for abelian additive groups, the identity element may be denoted by $0$. This is common to an ordered group with the symbol " $\leq$ " that has the obvious meaning : $\alpha \leq \beta$ means $\alpha < \beta$ or $\alpha = \beta$. We denote $G^+$ a set of non-negative elements of $G$, namely $G^+ := \{g \in G \| e \leq g\}$. Two positive elements, $x, y$, of an ordered group are relatively Archimedean if there are positive integers $m, n$ such that $mx \geq y$ and $ny \geq x$. If every two positive elements of an ordered group are relatively Archimedean, then the ordered group is Archimedean.

Every Archimedean ordered group is isomorphic to an ordered subgroup of the additive group of the real numbers. An ordered group $G$ is order complete if every non-empty subset of $G$ that has an upper bound has a least upper bound.

## Universal hypermetric spaces of the dimension $n$

The goal of this section is to describe a few properties of the universal hypermetric spaces.

Definition 1. *Let $G$ be an ordered group. An ordered group metric (or OG-metric ) on a non-empty set $X$ is a symmetric non-negative function $d_G$ from $X \times X$ into $G$ such that $d_G(x, y) = 0$ if and only if $x = y$ and such that the triangle inequality is satisfied; the pair $(X, d_G)$ is an ordered group metric space (or OG-metric space).*

Now we first recall and introduce some notation. For $n \geq 2$, let $X^n$ denote the n-times Cartesian product $\underbrace{X \times \ldots \times X}_{n-times}$ and $G$ be an ordered group. Let $P^*(G)$ denote the family of all non-empty subsets of $G$. We begin with the following definition.

Definition 2. *Let $X$ be a non-empty set. Let $\mathbb{U}_n : X^n \to P^*(G^+)$ be a function that satisfies the following conditions:*

*(U1)* $\mathbb{U}_n(x_1, \ldots, x_n) = \{0\}$, *if* $x_1 = \ldots = x_n$,

*(U2)* $\mathbb{U}_n(x_1, \ldots, x_n) \supset \{0\}$, *for all* $x_1, ..., x_n$ *with* $x_i \neq x_j$, *for some* $i, j \in \{1, ..., n\}$,

*(U3)* $\mathbb{U}_n(x_1, \ldots, x_n) = \mathbb{U}_n(x_{\pi_1}, \ldots, x_{\pi_n})$, *for every permutation* $(\pi_{(1)}, ..., \pi_{(n)})$ *of* $(1, 2, ..., n)$,

*(U4)* $\mathbb{U}_n(x_1, x_2, \ldots, x_{n-1}, x_{n-1}) \subseteq \mathbb{U}_n(x_1, x_2, \ldots, x_{n-1}, x_n)$, *for all* $x_1, \ldots, x_n \in X$,

*(U5)* $\mathbb{U}_n(x_1, x_2, \ldots, x_n) \subseteq \mathbb{U}_n(x_1, a, ..., a) + \mathbb{U}_n(a, x_2, ..., x_n)$, *for all* $x_1, \ldots, x_n, a \in X$.

Let $A_i$ be subsets of $X$, $i = 1, \ldots, n$. We define

$$\mathbb{U}_n(A_1, \ldots, A_n) = \bigcup \Big\{ \mathbb{U}_n(x_1, \ldots, x_n) \mid x_i \in A_i, \quad i = 1, \ldots, n \Big\}, \ and$$

$$A_i + A_j = \{x_i + x_j \,|\, x_i \in A_i, x_j \in A_j; 1 \leq i, j \leq n\},$$

We will use the following abbreviated notation: The function $\mathbb{U}_n$ is called a *universal ordered hypermetric group* of the dimension $n$, or more specifically an $UO_n$-hypermetric (or $U_n$-hypermetric ) on $X$, and the pair $(X, \mathbb{U}_n)$ is called an $U_n$-hypermetric space. For example, we can set $G^+ = \mathbb{Z}_+^0$ or $\mathbb{R}_+^0$, where $\mathbb{Z}_+^0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$ and $\mathbb{R}_+^0 := [0, +\infty)$.

In the sequel, for simplicity we assume that $G^+ = \mathbb{R}_+^0$. The following useful properties of a $U_n$-hypermetric are easily derived from the axioms.

Proposition 1. *(example) Let $X = \{a_1, \dots a_\ell\}$ be an $\ell$-element set and $\mathbb{N}_\ell = \{1, \dots, \ell\}$. Define*

$$\mathbb{F}_2 : X \times X \to P^*(\mathbb{R}_+^0)$$

*with,*

$$\mathbb{F}_2(a_i, a_j) = \begin{cases} \{\{0\}, \dots, \{j\}\} & ; \quad i < j \\ \{\{0\}, \dots, \{i-1\}\} & ; \quad i = j \\ \{\{0\}, \dots, \{i\}\} & ; \quad i > j \end{cases} \qquad ; \qquad (for\ all \quad i, j \in \mathbb{N}_\ell)$$

*and also assume $A + B = A \cup B$, for all $A, B \subseteq P(\mathbb{R}_+^*)$. Then $\mathbb{F}_2$ is a $U_2$-hypermetric space.*

*Proof.* It is sufficient to show that $\mathbb{F}_2$ satisfies all the properties $[(U1)], [(U2)], \dots, [(U5)]$ . The proofs of $[(U1)], \dots, [(U4)]$, follow immediately from the definition of $\mathbb{F}_2$. We only need to show that $\mathbb{F}_2$ satisfies the following relation

$$\mathbb{F}_2(a_i, a_j) = \mathbb{F}_2(a_i, a_k) + \mathbb{F}_2(a_k, a_j) \qquad ; \qquad (for\ all \quad i, j, k \in \mathbb{N}_\ell),$$

so we prove that in the following cases.

(i) $(i = j)$

We have $\{\{0\}, \dots, \{j-1\}\} \subseteq \{0, \dots, k\}$, if $j < k$ and also, $\{\{0\}, \dots, \{j-1\}\} = \{\{0\}, \dots, \{k-1\}\}$, if $j = k$. Finally we have $\{\{0\}, \dots, \{j-1\}\} \subseteq \{\{0\}, \dots, \{j\}\}$, if $j > k$

(ii) $(i < j)$

We have $\{\{0\}, \dots, \{j\}\} \subseteq \{\{0\}, \dots, \{j\}\}$, if $j > k$, and $\{\{0\}, \dots, \{j\}\} \subseteq \{\{0\}, \dots, \{k\}\}$, if $k > j$, and at last the equality holds if $i = j$.

(iii) $(i > j)$

The same reasoning applies to this case, with $j$ replaced by $i$ in (ii) and the proof is completed.

□

Proposition 2. *Let* $(X, \mathbb{U}_n)$ *be a* $U_n$*-hypermetric space, then for any* $x_1, ..., x_n, a \in X$ *it follows that:*
*(1) If* $\mathbb{U}_n(x_1, ..., x_n) = \{0\}$*, then* $x_1 = ... = x_n$*,*
*(2)* $\mathbb{U}_n(x_1, ..., x_n) \subseteq \sum_{j=2}^{n} \mathbb{U}_n(x_1, ..., x_1, x_j)$*,*
*(3)* $\mathbb{U}_n(x_1, ..., x_n) \subseteq \sum_{j=1}^{n} \mathbb{U}_n(x_j, a, ..., a)$*,*
*(4)* $\mathbb{U}_n(x_1, x_2, ..., x_2) \subseteq (n-1)\mathbb{U}_n(x_1, ..., x_1, x_2)$*.*

Proposition 3. *Let* $(X, \mathbb{U}_n)$ *be a* $U_n$*-hypermetric space, then* $\{0\} \subseteq \mathbb{U}_n(x_1, ..., x_n)$ *for all* $x_1, ..., x_n \in X$*.*

*Proof.* By the condition $(U4)$ of the definition of a $U_n$-hypermetric space, we have $\{0\} = \mathbb{U}_n(x_1, ..., x_1) \subseteq \mathbb{U}_n(x_1, ..., x_n)$. □

Proposition 4. *Every* $U_n$*-hypermetric space* $(X, \mathbb{U}_n)$ *defines a* $U_2$*-hypermetric space* $(X, \mathbb{U}_2)$ *as follows:*

$$\mathbb{U}_2(x, y) = \mathbb{U}_n(x; y, \ldots, y) + \mathbb{U}_n(y; x, \ldots, x); \quad for\ all \quad x, y \in X.$$

Proposition 5. *Let* $e$ *be an arbitrary positive real value number, and* $(X, d)$ *be a metric space. We define an induced hypermetric,*

$$\mathbb{U}_2^e : X \times X \to P^*(\mathbb{R}_+^0)$$

$$\mathbb{U}_2^e(x, y) = \begin{cases} \big(d(x,y) - e, d(x,y) + e\big) \cup \{0\} & ; \quad x \neq y, \quad d(x,y) \geq e \\ \big(d(x,y) - e, d(x,y) + e\big) \cap \mathbb{R}_+^0 & ; \quad x \neq y, \quad d(x,y) < e \\ \{0\} & ; \quad x = y. \end{cases}$$

*Then* $(X, \mathbb{U}_2^e)$ *is a* $U_2$*-hypermetric space.*

## Main results

Let $(X, \mathbb{U}_n)$ be a $U_n$-hypermetric space and $\widetilde{X}$ be a partition of $X$. For each point $p \in X$, we denote $\widetilde{p}$ a point in $\widetilde{X}$ containing $p$, and we denote the equivalent relation induced by the relation by $\sim$.

Definition 3. *Let* $(X, \mathbb{U}_n)$ *be a* $U_n$*-hypermetric space. Let* $p_1, \ldots, p_n \in X$*, and consider* $\widetilde{p_1}, \ldots, \widetilde{p_n} \in \widetilde{X}$*. A quotient* $U_n$*-hypermetric of the points of* $\widetilde{X}$ *induced by* $\mathbb{U}_n$ *is the function*
$\widetilde{\mathbb{U}}_n : \widetilde{X}^n \longrightarrow P^*(\mathbb{R}^+)$ *given by* $\widetilde{\mathbb{U}}_n(\widetilde{p_1}, \ldots, \widetilde{p_n}) = \bigcap_{p_i \in \widetilde{p_i}} \mathbb{U}_n(p_1, \ldots, p_n)$.

Proposition 6. *The quotient $U_n$-hypermetric induced by $\mathbb{U}_n$ is well-defined and is a $U_n$-hypermetric on $\widetilde{X}$.*

*Proof.* $\widetilde{\mathbb{U}}_n$ satisfies all the properties $(U1)-(U4)$.

$$\widetilde{\mathbb{U}}_n(\widetilde{p}_1,\ldots,\widetilde{p}_n) \subseteq \widetilde{\mathbb{U}}_n(\widetilde{p}_1,\widetilde{q},\ldots,\widetilde{q}) + \widetilde{\mathbb{U}}_n(\widetilde{q},\widetilde{p}_2,\ldots,\widetilde{p}_n)$$

$$\bigcap_{p_i\in\widetilde{P}_i} \mathbb{U}_n(p_1,\ldots,p_n) \subseteq \bigcap_{\substack{p_i\in\widetilde{P}_i\\ q\in\widetilde{q}}} \Big(\mathbb{U}_n(p_1,q,\ldots,q) + \mathbb{U}_n(q,p_2,\ldots,p_n)\Big)$$

$$= \bigcap_{\substack{p_i\in\widetilde{P}_i\\ q\in\widetilde{q}}} \mathbb{U}_n(p_1,q,\ldots,q) + \bigcap_{\substack{p_i\in\widetilde{P}_i\\ q\in\widetilde{q}}} \mathbb{U}_n(q,p_2,\ldots,p_n)$$

$$= \bigcap_{\substack{p_i\in\widetilde{P}_i\\ q\in\widetilde{q}}} \Big(\mathbb{U}_n(p_1,q,\ldots,q) + \mathbb{U}_n(q,p_2,\ldots,p_n)\Big).$$

□

Let $(X,\mathbb{U}_n)$ be a $U_n$-hypermetric space of a dimension $n>2$. For any arbitrary $a$ in $X$, define the function $\mathbb{U}_{n-1}$ on $X^{n-1}$ by

$$\mathbb{U}_{n-1}(x_1,\ldots,x_{n-1}) := \mathbb{U}_n(x_1,\ldots,x_{n-1},a).$$

Then we have the following result.

Proposition 7. *The function $\mathbb{U}_{n-1}$ defines a $U_{n-1}$-hypermetric on $X$.*

*Proof.* We will verify that $\mathbb{U}_{n-1}$ satisfies the five properties of a $U_{n-1}$-hypermetric. □

Proposition 8. *Let $f: X\to Y$ be an injection from a set $X$ to a set $Y$. If $\mathbb{U}_n : X^n \to P^*(\mathbb{R}^0_+)$ is a $U_n$-hypermetric on the set $Y$, then $\overline{\mathbb{U}}_n : X^n \to P^*(\mathbb{R}^0_+)$, given by the formula $\overline{\mathbb{U}}_n(x_1,\ldots,x_n) = \mathbb{U}_n(f(x_1),\ldots,f(x_n))$ for all $x_1,\ldots,x_n\in X$, is a $U_n$-hypermetric on the set $X$.*

Proposition 9. *Let $(X, \mathbb{U}_n)$ be any $U_n$-hypermetric space. Let $\lambda$ be any positive real number. Then $(X, \mathbb{U}_n^\lambda)$ is also a $U_n$-hypermetric space where $\mathbb{U}_n^\lambda(x_1, \ldots, x_n) := \{A \cap [0, \lambda) | A \in U_n(x_1, \ldots, x_n)\}$.*

So, on the same $X$ many $U_n$-hypermetrics can be defined, as a result of the procedure in which the same set $X$ is endowed with different metric structures. Another structure in the next proposition is useful for scaling the $U_n$-hypermetric, so we need the following explanation.

For any non-empty subset $A$ of $\mathbb{R}_+^0$, and $\lambda \in \mathbb{R}^+$ we define a set $\lambda.A$ to be $\lambda.A := \{\lambda.a \mid a \in A\}$.

Proposition 10. *Let $(X, \mathbb{U}_n)$ be any $U_n$-hypermetric space. Let $\Lambda$ be any positive real number. We define $\dot{\mathbb{U}}_n^\Lambda(x_1, \ldots, x_n) = \lambda.\mathbb{U}_n(x_1, \ldots, x_n)$. Then $(X, \dot{\mathbb{U}}_n^\lambda)$is also a $U_n$-hypermetric space.*

A sequence $\{x_m\}$ in a $U_n$-hypermetric space $(X, \mathbb{U}_n)$ is said to converge to a point $s$ in $X$, if for any $\epsilon > 0$ there exists a natural number $N$ such that for every $m_1, \ldots, m_{n-1} \geq N$,

$$\mathbb{U}_n(x_{m_1}, ..., x_{m_{n-1}}, s) \subseteq [0, \epsilon),$$

then we write,

$$\lim_{m_1, ..., m_{n-1} \longrightarrow +\infty} \mathbb{U}_n(x_{m_1}, ..., x_{m_{n-1}}, s) = \{0\}.$$

We say that a sequence $\{x_m\}$ has a cluster point $x$ if there exists a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ that converges to $x$.

Proposition 11. *Let $(X, \mathbb{U}_n)$ and $(X', \mathbb{U}'_n)$ be two $U_n$-hypermetric spaces. Then a function $f : X \to X'$ is $U_n$-continuous at a point $x \in X$, if and only if it is $U_n$-sequentially continuous at $x$; that is, whenever sequence $\{x_m\}$ is $U_n$-convergent to $x$ one has $\{f(x_m)\}$ which is $U_m$ convergent to $f(x)$.*

Definition 4. *Let $(X, \mathbb{U}_n)$ be a $U_n$-hypermetric space, and $A \subseteq X$. The set $A$ is $U_n$-compact if for every $U_n$-sequence $\{x_m\}$ in $A$, there exists a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $U_n$-converges to $x_0 \in A$.*

Proposition 12. *Let $(X, \mathbb{U}_n)$ and $(X', \mathbb{U}'_n)$ be two $U_n$-hypermetric spaces and $f : X \to X'$ a $U_n$-continuous function on $X$. If $X$ is $U_n$-compact, then $f(X)$ is $U_n$-compact.*

Definition 5. *Let* $(X, \mathbb{U}_n)$ *be a* $U_n$*-hypermetric space, then for* $x_0 \in X$, $r > 0$, *the* $U_n$*-hyperball with a center* $x_0$ *and a radius* $r$ *is*

$$B_{U_n}(x_0, r) = \{y \in X : \mathbb{U}_n(x_0, y, ..., y) \subseteq [0, r)\}.$$

Proposition 13. *Let* $(X, \mathbb{U}_n)$ *be a* $U_n$*-hypermetric space, then for* $x_0 \in X$, $r > 0$,
*(i) If* $\mathbb{U}_n(x_0, x_2, ..., x_n) \subseteq [0, r)$, *then* $x_2, ..., x_n \in B_{U_n}(x_0, r)$,
*(ii) If* $y \in B_{U_n}(x_0, r)$, *then there exists,* $\delta > 0$ *such that* $B_{U_n}(y, \delta) \subseteq B_{U_n}(x_0, r)$.

*Proof.* The proof of (i) is trivial. In (ii) it suffices to show that for every $U_n$-hyperball $B_{U_n}(x, r)$ and every $y \in B_{U_n}(x, r))$, there exists $\delta > 0$ such that, $y \in B_{U_n}(y, \delta) \subseteq B_{U_n}(x, r)$. So let $y \in B_{U_n}(x, r)$. Then $\mathbb{U}_n(x, \ldots, x\,; y) - \mathbb{U}_n(x, \ldots, x\,; x) \subseteq [0, r)$. Set

$$[0, \delta) := [0, r) - \mathbb{U}_n(x, \ldots, x\,;\, y) + \mathbb{U}_n(x, \ldots, x\,;\, x).$$

then $\delta > 0$, and hence $y \in B_{U-n}(y, \delta)$.

Now let $z \in B_{U_n}(y, \delta)$ , i.e, $\mathbb{U}_n(y, \ldots, y\,;\, z) - \mathbb{U}_n(y, \ldots, y\,;\, y) \subseteq [0, \delta)$, then

$$\begin{aligned}
\mathbb{U}_n(x, \ldots, x\,;\, z) &\subseteq \mathbb{U}_n(x, \ldots, x\,;\, y) + \mathbb{U}_n(y, \ldots, y\,;\, z) - \mathbb{U}_n(y, \ldots, y\,;\, y) \\
&- \mathbb{U}_n(x, \ldots, x\,;\, x) \\
&\subseteq \mathbb{U}_n(x, \ldots, x\,;\, y) - \mathbb{U}_n(x, \ldots, x\,;\, x) + [0, \delta) \\
&\subseteq [0, r).
\end{aligned}$$

Thus, $z \in B_{U_n}(x, r)$, and hence $B_{U_n}(y, \delta) \subseteq B_{U_n}(x, r)$. □

Proposition 14. *The set of all* $\mathbb{U}_n$*-balls,* $\mathcal{B}_n = \{B_{U_n}(x, r) : x \in X, r > 0\}$, *forms a basis for a topology* $\mathcal{T}(U_n)$ *on* $X$.

Definition 6. *Let* $(X, \mathbb{U}_n)$ *be a* $U_n$*-hypermetric space. The sequence* $\{x_n\} \subseteq X$ *is* $U_n$*-convergent to* $x$ *if it* $U_n$*-converges to* $x$ *in the* $U_n$*-hypermetric topology,* $\mathcal{T}(U_n)$.

Proposition 15. *Let* $(X, \mathbb{U}_n)$ *be a* $U_n$*-hypermetric space. Then for a sequence* $\{x_m\} \subseteq X$, *and a point* $x \in X$ *the following are equivalent:*
*(1)* $\{x_m\}$ *is* $\mathbb{U}_n$*-convergent to* $x$,
*(2)* $\mathbb{U}_n(x_m, ..., x_m, x) \to 0$,
*(3)* $\mathbb{U}_n(x_m, x, ..., x) \to 0$.

Definition 7. *Let* $(X, \mathbb{U}_n)$, $(Y, \mathbb{V}_m)$ *be universal hypermetric spaces of the dimensions* $n$, $m$, *respectively, a function* $f : X \longrightarrow Y$ *is* $U_{n,m}$*-continuous at a point* $x_0 \in X$, *if* $f^{-1}(B_{V_m}(f(x_0), r)) \in \mathcal{T}(U_n)$, *for all* $r > 0$.

*We say* $f$ *is* $U_{n,m}$*-continuous if it is* $U_{n,m}$*-continuous at all points of* $X$*; that is, continuous as a function from* $X$ *with the* $\mathcal{T}(U_n)$*-topology to* $Y$ *with the* $\mathcal{T}(V_m)$*-topology.*

In the sequel, for simplicity we assume that $n = m$. Since $U_n$-hypermetric topologies are metric topologies, we have:

Definition 8. *Let* $(X, \mathbb{U}_n)$ *and* $(Y, \mathbb{V}_n)$ *be two* $U_n$*-hypermetric spaces and* $f : (X, \mathbb{U}_n) \to (Y, \mathbb{V}_n)$ *be a function. The function* $f$ *is called* $U_n$*-continuous at a point* $a \in X$ *if and only if, for given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $x_1, \ldots, x_{n-1} \in X$ *and the subset relation* $\mathbb{U}_n(a, x_1, \ldots, x_{n-1}) \subseteq [0, \delta)$ *implies that* $\mathbb{V}_n(f(a), f(x_1), \ldots, f(x_{n-1})) \subseteq [0, \epsilon)$.

*A function* $f$ *is* $U_n$*-continuous on* $X$ *if and only if it is* $U_n$*-continuous at all* $a \in X$.

Proposition 16. *Let* $(X, \mathbb{U}_n)$, $(Y, \mathbb{V}_n)$ *be* $U_n$*-hypermetric spaces, a function* $f : X \longrightarrow Y$ *is* $U_n$*-continuous at point* $x \in X$ *if and only if it is* $U_n$*-sequentially continuous at* $x$*; that is, whenever* $\{x_n\}$ *is* $U_n$*-convergent to* $x$ *we have that* $(f(x_n))$ *is* $U_n$*-convergent to* $f(x)$.

Proposition 17. *Let* $(X, \mathbb{U}_n)$ *be a* $U_n$*-hypermetric space. Then the function* $\mathbb{U}_n(z_1, z_2, ..., z_n)$ *is jointly* $U_n$*-continuous in all* $n$ *of its variables.*

Definition 9. *A map* $f : X \longrightarrow Y$ *between* $U_n$*-hypermetric spaces* $(X, \mathbb{U}_n)$ *and* $(Y, \mathbb{U}'_n)$ *is an iso-hypermetry when* $\mathbb{U}_n(x_1, ..., x_n) = \mathbb{U}'_n(f(x_1), ..., f(x_n))$ *for all* $x_1, \ldots, x_n \in X$. *If the iso-hypermetry is injective, we call it iso-hypermetric embedding. A bijective iso-hypermetry is called an iso-hypermetric isomorphism.*

We discuss now about the concept of completeness of $U_n$-hypermetric spaces.

Definition 10. *Let* $(X, \mathbb{U}_n)$ *be a* $U_n$*-hypermetric space, then a sequence* $\{x_m\} \subseteq X$ *is said to be* $U_n$*-Cauchy if for every* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $\mathbb{U}_n(x_{m_1}, x_{m_2}, ..., x_{m_n}) < \varepsilon$ *for all* $m_1, m_2, ..., m_n \geq N$.

The next proposition follows directly from the definitions.

Proposition 18. *In a $U_n$-hypermetric space, $(X, \mathbb{U}_n)$, the following are equivalent.*
$(i)$ *The sequence $\{x_m\}$ is $U_n$-Cauchy.*
$(ii)$ *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathbb{U}_n(x_l, x_m, ..., x_m) < \varepsilon$, for all $l, m \geq N$.*
$(iii)$ *$\{x_m\}$ is a Cauchy sequence in the metric space $(X, d_U)$.*

Corollary 1. **(i)** *Every $U_n$-convergent sequence in a $U_n$-hypermetric space is $U_n$-Cauchy.*
**(ii)** *If a $U_n$-Cauchy sequence in a $U_n$-hypermetric space $(X, \mathbb{U}_n)$ contains a $U_n$-convergent subsequence, then the sequence itself is $U_n$-convergent.*

Definition 11. *A $U_n$-hypermetric space $(X, \mathbb{U}_n)$ is said to be $U_n$-complete if every $U_n$-Cauchy sequence in $(X, \mathbb{U}_n)$ is $U_n$-convergent in $(X, \mathbb{U}_n)$.*

Proposition 19. *A $U_n$-hypermetric space $(X, \mathbb{U}_n)$ is $U_n$-complete if and only if $(X, d_U)$ is a complete metric space.*

Definition 12. *Let $(X, \mathbb{U}_n)$ and $(Y, \mathbb{U}_n{}')$ be two $U_n$-hypermetric spaces. A function $f : X \longrightarrow Y$ is called a $U_n$-contraction if there exists a constant $k \in [0, 1)$ such that $\mathbb{U}_n{}'(f(x_1), ..., f(x_n)) = k\mathbb{U}_n(x_1, ..., x_n)$ for all $x_1, \ldots, x_n \in X$.*

It follows that $f$ is $U_n$-continuous because $\mathbb{U}_n(x_1, ..., x_n) \subseteq [0, \delta)$ with $k \neq 0$ and $\delta := \epsilon/k$ implies $\mathbb{U}_n{}'(f(x_1), ..., f(x_n)) \subseteq [0, \epsilon)$.

Theorem 1. *Let $(X, \mathbb{U}_n)$ be a $U_n$-complete space and let $T : X \to X$ be a $U_n$-contraction map. Then $T$ has a unique fixed point $T(x) = x$.*

*Proof.* We consider $x_{m+1} = T(x_m)$, with $x_0$ being any point in $X$. We have, by repeated use of the rectangle inequality and the application of contraction property, the following:

$$\mathbb{U}_n(x_m, x_{m+1}, \ldots, x_{m+1}) \subseteq k^m \mathbb{U}_n(x_0, x_1, \ldots, x_1),$$

for all $m, s_1 \in \mathbb{N}$ which $m < s_1$ and $k \in [0,1)$.

$$\begin{array}{rcl}
\mathbb{U}_n(x_m, x_{s_1}, \ldots, x_{s_1}) & \subseteq & \mathbb{U}_n(x_m, x_{m+1}, \ldots, x_{m+1}) \\
& + & \mathbb{U}_n(x_{m+1}, x_{m+2}, \ldots, x_{m+2}) \\
& + & \mathbb{U}_n(x_{m+2}, x_{m+3}, \ldots, x_{m+3}) \\
& + & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
& + & \frac{k^m(1-k^{s-m})}{1-k}\mathbb{U}_n(x_0, x_1, \ldots, x_1) \\
& \subseteq & (k^m + k^{m+1} + \ldots + k^{s_1-1})\mathbb{U}_n(x_0, x_1, \ldots, x_1) \\
& = & \frac{k^m(1-k^{s_1-m})}{1-k}\mathbb{U}_n(x_0, x_1, \ldots, x_1).
\end{array}$$

Then we have

$$\lim_{m\,,\,s_1 \to +\infty} \mathbb{U}_n(x_m, x_{s_1}, \ldots, x_{s_1}) = \{0\},$$

since

$$\lim_{m\,,\,s_1 \to +\infty} \frac{k^m(1-k^{s_1-m})}{1-k}\mathbb{U}_n(x_0, x_1, \ldots, x_1) = \{0\}.$$

For $m \leq s_1 \leq s_2 \in \mathbb{N}$ and $(U5)$ implies that

$$\begin{array}{rcl}
\mathbb{U}_n(x_m, x_{s_1}, x_{s_2} \ldots, x_{s_2}) & \subseteq & \mathbb{U}_n(x_m, x_{s_1}, \ldots, x_{s_1}) \\
& + & \mathbb{U}_n(x_{s_1}, x_{s_1}, \ldots, x_{s_2}),
\end{array}$$

now taking the limit as $m, s_1, s_2 \to +\infty$, we get

$$\mathbb{U}_n(x_m, x_{s_1}, x_{s_2} \ldots, x_{s_2}) \to \{0\}.$$

Now for $m \leq s_1 \leq s_2 \leq \ldots \leq s_{n-1} \in \mathbb{N}$, we will have

$$\mathbb{U}_n(x_m, x_{s_1}, \ldots, x_{s_n}) \to \{0\}; \quad whenever, \quad m, s_1, \ldots, s_{n-1} \to +\infty,$$

then $\{x_m\}$ is a Cauchy sequence. By completeness of $(X, \mathbb{U}_n)$, there exists $a \in X$ such that $\{x_n\}$ is $\mathbb{U}_n$-convergent to $a$. The fact that the limit $x_m$ is a fixed point of $T$ follows the $\mathbb{U}_n$-continuity of $T$, and

$$Ta = T \lim_{m\to+\infty} x_m = \lim_{m\to+\infty} Tx_m = \lim_{m\to+\infty} x_{m+1} = a.$$

Finally, if $a$ and $b$ are two fixed points, then

$$\begin{array}{rcl}
\{0\} \subseteq \mathbb{U}_n(a, b, \ldots, b) & = & \mathbb{U}_n\Big(T(a), T(b), \ldots, T(b)\Big) \\
& \subseteq & k\mathbb{U}_n(a, b, \ldots, b).
\end{array}$$

Since $k < 1$, we have $\mathbb{U}_n(a, b, \ldots, b) = \{0\}$, so $a = b$ and the fixed point is unique. □

## Conclusions

In this article, we have put forward a development of the results of $U_n$-hypermetric spaces, covering a variety of topics relevant for understanding their properties including completeness and the fixed-point theorem. We believe this work may be relevant from both the theoretical standpoint and the point of view of applications in contemporary problems such as those of clusterings which often appear in practice.

## *References*

Cohen, L.W. & Goffman, C. 1949. The topology of ordered Abelian groups. *Transactions of the American Mathematical Society*, 67, pp.310-319. Available at: https://doi.org/10.1090/S0002-9947-1949-0032648-5.

Dehghan Nezhad, A. & Aral, Z. 2011. The topology of GB-metric spaces. *International Scholarly Research Notices*, art.ID:523453. Available at: https://doi.org/10.5402/2011/523453.

Dehghan Nezhad, A. & Khajuee, N. 2013. Some new results on complete $\mathbb{U}_n^*$-metric space. *Journal of Nonlinear Sciences and Applications*, 6(3), pp.216-226. Available at: http://dx.doi.org/10.22436/jnsa.006.03.07.

Dehghan Nezhad, A., Khajuee, N. & Mustafa, Z. 2017. Some new results on Universal metric spaces. *Thai Journal of Mathematics*, 15(2), pp.429-449 [online]. Available at: http://thaijmath.in.cmu.ac.th/index.php/thaijmath/article/view/623 [Accessed: 10 May 2021].

Dehghan Nezhad, A. & Mazaheri, H. 2010. New results in G-best approximation in G-metric spaces. *Ukrainian Mathematical Journal* 62(4), pp.648-654. Available at: https://doi.org/10.1007/s11253-010-0377-8.

Dhage, B.C., Pathan, A.M. & Rhoades, B.E. 2000. A general existence principle for fixed point theorems in $d$-metric spaces. *International Journal of Mathematics and Mathematical Sciences*, 23(7), art.ID:695952, pp.441-448. Available at: https://doi.org/10.1155/S0161171200001587.

Diminnie, C., Gahler S. & White, A. 2017. 2-inner product spaces, II1. *Demonstratio Mathematica*, 10(1), pp.169-188. Available at: https://doi.org/10.1515/dema-1977-0115.

Gähler, S. 1963. 2-metrische Räume und ihre topologische Struktur. *Mathematische Nachrichten*, 26(1-4), pp.115-148. Available at: https://doi.org/10.1002/mana.19630260109.

Gähler, S. 1964. Lineare 2-normietre Räume. *Mathematische Nachrichten*, 28(1-2), pp.1-43. Available at: https://doi.org/10.1002/mana.19640280102.

Gähler, S. 1966. Zur geometric 2-metrische räume. *Revue Roumaine de Mathématiques Pures et Appliquées*, 11, pp.664-669.

Ha, K.S., Cho, Y.J., Kim, S.S. & Khan, M.S. 1990. Strictly Convex Linear 2-Normed Spaces, *Mathematische Nachrichten*, 146(1), pp.7-16. Available at: https://doi.org/10.1002/mana.19901460102.

Kelly, J.B. 1975. Hypermetric spaces. In: *Kelly, L.M. (eds) The Geometry of Metric and Linear Spaces. Lecture Notes in Mathematics*, 490, pp.17-31. Berlin, Heidelberg: Springer. Available at: https://doi.org/10.1007/BFb0081128.

Khan, K.A. 2012. On the possibitity of $n$-topological spaces. *Internatial Journal of Mathematical Archive*, 3(7), pp.2520-2523 [online]. Available at: http://www.ijma.info/index.php/ijma/article/view/1442 [Accessed: 10 May 2021].

Khan, K.A. 2014. Generalized $n$-metric spaces and fixed point theorems. *Journal of Nonlinear and Convex Analysis*, 15(6), pp.1221-1229 [online]. Available at: http://www.yokohamapublishers.jp/online2/jncav15.html [Accessed: 10 May 2021].

Mustafa, Z. & Sims, B. 2003. Some remarks concerning D-metric spaces. In: *Proceeding of the International Conferences on Fixed Point Theory and Applications*, Valencia (Spain) 13-19 July 2003, Eds. J. G. Falset and Enrique Llorens Fuster and B. Sims, pp.189-198. Available at: https://carma.newcastle.edu.au/brailey/Research_papers/SOME%20REMARKS%20CONCERNING%20D-METRIC%20SPACES%20Mustafa%20and%20Sims.pdf [Accessed: 10 May 2021].

Mustafa, Z. & Sims, B. 2006. A new approach to generalized metric spaces. *Journal of Nonlinear Convex Analysis*, 7(2), pp.289-297 [online]. Available at: https://carma.edu.au/brailey/Research_papers/A%20new%20Approach%20to%20Generalized%20Metric%20Spaces.pdf [Accessed: 10 May 2021].

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## О НОВОЙ ВЕРСИИ РЕЗУЛЬТАТОВ В ОБЛАСТИ $U_n$-ГИПЕРМЕТРИЧЕСКИХ ПРОСТРАНСТВ

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РУБРИКА ГРНТИ: 27.00.00 МАТЕМАТИКА:
27.25.17 Метрическая теория функций,
27.39.15 Линейные пространства, снабженные топологией, порядком и другими структурами
ВИД СТАТЬИ: оригинальная научная статья

*Резюме:*

*Введение/цель: Целью данной статьи является представление концепции универсального $U_n$-гиперметрического пространства. Обобщено $n$-мерное $(n \geq 2)$ гиперметрическое расстояние на произвольном непустом множестве $X$, при этом данная так называемая гиперметрика вычисляет расстояние между всеми n точками пространства. В статье обсуждается концепция полноты в отношении гиперметрики, а также теорема о неподвижной точке, которые играют важную роль в разных направлениях прикладной математики.*

*Методы: В статье применялись теоретические методы функционального анализа.*

*Результаты: Представлена концепция универсального $U_n$-гиперметрического пространства. Описаны универсальные свойства $U_n$-гиперметрических пространств.*

*Выводы: Новая версия результатов в области $U_n$-гиперметрических пространств может применяться в различных дисциплинах, в которых требуется степень кластеризации.*

*Ключевые слова: $U_n$-гиперметрические пространства, OG-метрика, G-метрика.*

## НОВА ВЕРЗИЈА РЕЗУЛТАТА $U_n$-ХИПЕРМЕТРИЧКИХ ПРОСТОРА

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ОБЛАСТ: математика
ВРСТА ЧЛАНКА: оригинални научни рад

*Сажетак:*

*Увод/Циљ: У раду је представљен концепт универзалног $U_n$-хиперметричког простора. Генерализује се $n$-димензионално $(n \geq 2)$ хиперметричко растојање на произвољном непразном скупу $X$. Притом, ова тзв. хиперметрика изражава колико је међусобно растојање свих $n$ тачака простора. Анализира се појам комплетности, у односу на хиперметрику, као и теорема непокретне тачке, која има значајну улогу у примењеној математици на разним пољима.*

*Методе: Примењене су стандардне теоријске методе функционалне анализе.*

*Резултати: Представљен је концепт генерализованог $U_n$-хиперметричког простора. Описане су и универзалне особине $U_n$-хиперметричких простора.*

*Закључци: Нова верзија резултата $U_n$-хиперметричких простора може имати примену у разнородним дисциплинама у којима је захтевано да се квантификује степен груписања.*

*Кључне речи: $U_n$-хиперметрички простори, OG-метрика, G-метрика.*

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