# INTERPOLATIVE GENERALISED MEIR-KEELER CONTRACTION

*Shobha* Jain[a], *Vuk* N. Stojiljković[b], *Stojan* N. Radenović[c]

[a]Shri Vaishnav Vidyapeeth Vishwavidyalaya, Department of Mathematics, Indore, Madhya Pradesh, Republic of India,
e-mail: shobajain1@yahoo.com,
ORCID iD: https://orcid.org/0000-0002-9253-8689

[b]University of Novi Sad, Faculty of Science, Novi Sad, Republic of Serbia,
e-mail: vuk.stojiljkovic999@gmail.com, **corresponding author**,
ORCID iD: https://orcid.org/0000-0002-4244-4342

[c]University of Belgrade, Faculty of Mechanical Engineering, Belgrade, Republic of Serbia,
e-mail: radens@beotel.net,
ORCID iD: https://orcid.org/0000-0001-8254-6688

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*Abstract*:

*Introduction/purpose: The aim of this paper is to introduce the notion of an interpolative generalised Meir-Keeler contractive condition for a pair of self maps in a fuzzy metric space, which enlarges, unifies and generalizes the Meir-Keeler contraction which is for only one self map. Using this, we establish a unique common fixed point theorem for two self maps through weak compatibility. The article includes an example, which shows the validity of our results.*

*Methods: Functional analysis methods with a Meir-Keeler contraction.*

*Results: A unique fixed point for self maps in a fuzzy metric space is obtained.*

*Conclusions: A fixed point of the self maps is obtained.*

*Key words: Fuzzy metric space, common fixed points, weak compatibility, Interpolative generalised Meir-Keelar contraction.*

## Introduction

In $1965$ L. Zadeh (Zadeh, 1965) introduced the theory of fuzzy sets. Later on, in $1978$, the concept of a fuzzy metric space was introduced by Kramosil and Michalek in (Kramosil & Michalek, 1975), which was modified by George and Veeramani (George & Veeramani, 1994) in order to obtain a Hausdorff topology for this class of fuzzy metric spaces. Then in year 1988, Grabiec (Grabiec, 1988) gave a fuzzy version of the Banach (Banach, 1922) contraction principle in the setting of a fuzzy metric space. Over the past years, various authors have tried to generalize the fixed point theorem by modifying and varying the contractive condition, see, e.g., (Gregori & Sapena, 2002), (Jain & Jain, 2021), (Mihet, 2008), (Saha et al, 2016), (Tirado, 2012) and (Wardowski, 2013) in the sense of George and Veeramani. In $2019$, Zheng and Wang (Zheng & Wang, 2019) introduced a Meir-Keeler contraction in the setting of a fuzzy metric (Schweizer & Sklar, 1983) space and proved some fixed point results for a self map.

Inspired with the interpolative theory, Karapinar and Agrawal (Karapinar & Agarwal, 2019) introduced the notion of an interpolative Rus-Reich-$\acute{C}$iri$\acute{c}$ type contraction via the simulation function in a metric space. Motivated by this paper, we introduce an interpolative generalised Meir-Keeler contraction (Gregori & Minana, 2014) for two self maps (Rhoades, 2001) in the setting of a fuzzy metric space, which enlarges, unifies and generalizes the existing Meir-Keelar contraction in a fuzzy metric (Mihet, 2010) space through weak compatibility (Banach, 1922).

The structure of the paper is as follows:

After the preliminaries, we introduce a interpolative generalised Meir-Keeler contraction in the setting of a fuzzy metric space. Then we study the Meir-Keeler contractive mapping due to Zheng and Wang (Zheng & Wang, 2019). In section $4$, the existence of a unique common fixed point of an interpolative generalised Meir-Keeler contractive mapping has been established through weak compatibility followed by an example.

## Preliminaries

Definition 1. *(George & Veeramani, 1994) A mapping* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous triangular norm (t-norm for short) if* $*$ *is continuous and satisfies the following conditions:*

*(i)* $*$ *is commutative and associative, i.e.* $a * b = b * a$ *and* $a * (b * c) = (a * b) * c$*, for all* $a, b, c \in [0,1]$;

*(ii)* $1 * a = a$, *for all* $a \in [0,1]$;

*(iii)* $a * c \leq b * d$, *for* $a \leq b, c \leq d$ *for* $a, b, c, d \in [0,1]$.

The well-known examples of the t-norm are the minimum t-norm $*_m$, $a *_m b = min\{a, b\}$ written as $*_m$ and the product t-norm $*$, $a * b = ab$.

DEFINITION 2. *(George & Veeramani, 1994) A fuzzy metric space is an ordered triple $(X, M, *)$ such that $X$ is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$;*

*(GV1)* $M(x, y, t) > 0$;

*(GV2)* $M(x, y, t) = 1$ *if and only if* $x = y$;

*(GV3)* $M(x, y, t) = M(y, x, t)$;

*(GV4)* $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;

*(GV5)* $M(x, y, .) : (0, +\infty) \to (0, 1]$ *is continuous.*

Note that in view of the condition (GV2) we have $M(x, x, t) = 1$, for all $x \in X$ and $t > 0$ and $M(x, y, t) < 1$, for all $x \neq y$ and $t > 0$.
The following notion was introduced by George and Veeramani in (George & Veeramani, 1994).

DEFINITION 3. *(George & Veeramani, 1994) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be M-Cauchy, or simply Cauchy, if for each $\epsilon \in (0, 1)$ and each $t > 0$ there exists an $n_0 \in N$, such that $M(x_n, x_m, t) > 1 - \epsilon$, for all $n, m \geq n_0$. Equivalently, $\{x_n\}$ is Cauchy if $\lim_{n,m\to+\infty} M(x_n, x_m, t) = 1$, for all $t > 0$.*

LEMMA 1. *(Grabiec, 1988) Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is non-decreasing for all $x, y \in X$.*

THEOREM 1. *(George & Veeramani, 1994) Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}_{n\in N}$ in $X$ converges to $x \in X$ if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$.*

DEFINITION 4. *(George & Veeramani, 1994) $(X, M, *)$ (or simply X) is called M-complete if every M-Cauchy sequence in X is convergent.*

LEMMA 2. *(Saha et al, 2016) If $*$ is a continuous t-norm $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences such that $\alpha_n \to \alpha, \beta_n \to \beta$ and $\gamma_n \to \gamma$ as $n \to +\infty$ then*

$$\overline{\lim}_{k\to+\infty}(\alpha_k * \beta_k * \gamma_k) = \alpha * \overline{\lim}_{k\to+\infty}\beta_k * \gamma,$$

*and*

$$\underline{\lim}_{k\to+\infty}(\alpha_k * \beta_k * \gamma_k) = \alpha * \underline{\lim}_{k\to+\infty}\beta_k * \gamma.$$

LEMMA 3. *(Saha et al, 2016) Let $\{f(k,.) : (0,+\infty) \to (0,1]; k = 0,1,2,...,\}$ be a sequence of functions such that $f(k,.)$ is continuous and monotone increasing for each $k \geq 0$. Then $\overline{\lim}_{k\to+\infty} f(k,t)$ is a left continuous function in t and $\underline{\lim}_{k\to+\infty} f(k,t)$ is a right continuous function in it.*

Denote $\triangle = \{\delta|\delta : (0,1] \to (0,1]\}$ where $\delta$ is right continuous.

DEFINITION 5. *(Zheng & Wang, 2019) Let $(X, M, *)$ be a fuzzy metric space. A mapping $f : X \to X$ is said to be a fuzzy Meir-Keeler contractive mapping with respect to $\delta \in \triangle$ if the following condition holds:*

$$\text{for all } \epsilon \in (0,1), \epsilon - \delta(\epsilon) < M(x,y,t) \leq \epsilon \ \ implies \ \ M(fx, fy, t) > \epsilon, \quad (1)$$

*for all $x, y \in X, t > 0$.*

DEFINITION 6. *(Jain et al, 2009) Two self maps $f$ and $g$ in a fuzzy metric space $(X, M, *)$ are said to be weakly compatible if they commute at their coincidence points i.e. for $x \in X, fx = gx = y$ implies $gy = fy$.*

## Interpolative generalised Meir-Keeler contraction

DEFINITION 7. *Let $(X, M, *)$ be a fuzzy metric space. A pair $(f, g)$ of self maps in $X$ is said to be an interpolative generalised Meir-Keeler contractive if there exists $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ and for all $x, y \in X, t > 0$*

$$\text{for all } \epsilon \in (0,1), \epsilon - \delta(\epsilon) < M(x,y) \leq \epsilon \ \ implies \ \ M(fx, fy, t) > \epsilon, \quad (2)$$

*where*

$$M(x,y) = (M(fx,gx,t))^{\alpha}(M(fy,gy,t))^{\beta}(M(gx,gy,t))^{(1-\alpha-\beta)}.$$

REMARK 1. From equation (2) for all $x \neq y \in X, t > 0$ the pair $(f, g)$ is a strict contraction i.e.

$$M(fx, fy, t) > M(x,y).$$

Thus for $x \neq y$.

$$M(fx,fy,t) > (M(fx,gx,t))^{\alpha}(M(fy,gy,t))^{\beta}(M(gx,gy,t))^{(1-\alpha-\beta)} \quad (3)$$

REMARK 2. Taking $g = I$, the identity map in equation (2)we obtain

$$\text{for all } \epsilon \in (0,1), \epsilon - \delta(\epsilon) < M(x,y) \leq \epsilon \ \ implies \ \ M(fx,fy,t) > \epsilon, \qquad (4)$$

where

$$M(x,y) = (M(fx,x,t))^{\alpha}(M(fy,y,t))^{\beta}(M(x,y,t))^{(1-\alpha-\beta)}.$$

which is an interpolative generalised Meir-Keeler contraction, for a self map $f$.

REMARK 3. Taking $\alpha = 0, \beta = 0$ in equation (4) then $M(x,y) = M(x,y,t)$ and we have

$$\text{for all } \epsilon \in (0,1), \epsilon - \delta(\epsilon) < M(x,y,t) \leq \epsilon \ \ implies \ \ M(fx,fy,t) > \epsilon, \qquad (5)$$

which is precisely the Meir-Keeler contraction, for a self map given by Zheng and Wang (Zheng & Wang, 2019).

LEMMA 4. *(Zheng & Wang, 2019) If $\delta \in \triangle$, then for $t \in (0,1)$, there exists $k = k(t) \in N$ such that*
*$t + \frac{\delta(t)}{k} < 1$ and $\delta\left(t + \frac{\delta(t)}{k}\right) - \frac{\delta(t)}{k} > 0.$*

Before we prove the main result, we prove the following lemma:

LEMMA 5. *Let $(f,g)$ be a pair of an interpolative Meir-Keeler contractive mapping with respect to $\delta \in \triangle$ and $f(X) \subseteq g(X)$. Construct a sequence $\{y_n\}$, by $fx_n = gx_{n+1} = y_n$, for $n = 0,1,2,\ldots$. Then $\lim_{n\to+\infty} M(y_n, y_{n+1}, t) = 1$.*

*Proof.* Suppose if possible on the contrary that $\lim_{n\to+\infty} M(y_n, y_{n+1}, t) = p(< 1)$. For $\alpha, \beta \in [0,1)$ we have

$$\begin{aligned}
\lim_{n\to+\infty} M(x_n, x_{n+1}) &= \lim_{n\to+\infty} \left\{ \begin{array}{c} ((M(fx_n, gx_n, t))^{\alpha}(M(fx_{n+1}, gx_{n+1}, t))^{\beta} \\ (M(gx_n, gx_{n+1}, t))^{(1-\alpha-\beta)} \end{array} \right\} \\
&= \lim_{n\to+\infty} \left\{ \begin{array}{c} (M(y_n, y_{n-1}, t))^{\alpha}(M(y_{n+1}, y_n, t))^{\beta} \\ (M(y_{n-1}, y_{n+}, t))^{(1-\alpha-\beta)} \end{array} \right\} \\
&= \lim_{n\to+\infty} \left(p^{\alpha}p^{\beta}p^{(1-\alpha-\beta)}\right) \\
&= p.
\end{aligned}$$

By using lemma (4), for $p < 1$ and $\delta \in \triangle$ we can find $k = k(p) \in N$ such that

$p + \frac{\delta(p)}{k} < 1$ and $\delta\left(p + \frac{\delta(p)}{k}\right) - \frac{\delta(p)}{k} > 0$.

Since $\lim_{n\to+\infty} M(x_n, x_{n+1}) = p$, we can find $n_0$ such that when $n > n_0$,

$$M(x_n, x_{n+1}) > p + \frac{\delta(p)}{k} - \delta\left(p + \frac{\delta(p)}{k}\right). \tag{6}$$

Also, there exists $n_1$ such that whenever $n > n_1$,

$$M(x_n, x_{n+1}) < p + \frac{\delta(p)}{k}. \tag{7}$$

Let $n > \max\{n_0, n_1\}$. Then both equations (6), and ( 7) hold for such $n$. Taking $\epsilon = p + \frac{\delta(p)}{k}$, in equation (2) we get

$$M(fx_n, fx_{n+1}, t) > p + \frac{\delta(p)}{k},$$

i. e. $M(y_n, y_{n+1}, t) > p + \frac{\delta(p)}{k}$, which contradicts the fact that $\lim_{n\to+\infty} M(y_n, y_{n+1}, t) = p$. Therefore, $\lim_{n\to+\infty} M(y_n, y_{n+1}, t) = 1$. □

## Main results

Our first new result is the next one:

THEOREM 2. *: Let $f$ and $g$ be self maps in a fuzzy metric space $(X, M, *)$ satisfying the following conditions:*

*(4.11)* $f(X) \subseteq g(X)$;

*(4.12) The pair $(f, g)$ is an interpolative generalised Meir-Keeler contraction;*

*(4.13) $f(X)$ is complete;*

*(4.14) The pair $(f, g)$ is weakly compatible.*

*Then $f$ and $g$ have a unique common fixed point in $X$ if and only if there exists $x_0 \in X$ such that $\bigwedge_{t>0} M(x_0, f(x_0), t) > 0$.*

*Proof.* : Suppose the pair $(f, g)$ has a unique common fixed point $u$ then $u = fu = gu$. Therefore, $M(u, fu, t) = 1, \forall t > 0$. Hence

$\bigwedge_{t>0} M(u, fu, t) > 0.$

Conversely, suppose that there exists $x_0 \in X$ such that $\bigwedge_{t>0} M(x_0, f(x_0), t) > 0$. Construct a sequence $\{y_n\}$, by defining $fx_n = gx_{n+1} = y_n$, for $n = 0, 1, 2, \ldots$. First we show that if the two maps $f$ and $g$ have a common fixed point then it is unique. Let $u$ and $v$ be two common fixed points of $f$ and $g$. Then $u = fu = gu$ and $v = fv = gv$. We show that $u = v$.

Suppose, on the contrary that $u \neq v$, then $fu \neq fv$. Now

$$\begin{aligned} M(u,v) &= (M(fu,gu,t))^{\alpha}(M(fv,gv,t))^{\beta}(M(gu,gv,t))^{(1-\alpha-\beta)}, \\ &= (M(u,u,t))^{\alpha}(M(v,v,t))^{\beta}(M(u,v,t))^{(1-\alpha-\beta)}, \\ &= (M(u,v,t))^{(1-\alpha-\beta)}. \end{aligned}$$

Now

$$\begin{aligned} M(u,v,t) &= M(fu,fv,t), \\ &> M(u,v), \text{ using (3)} \\ &= (M(u,v,t))^{(1-\alpha-\beta)} \end{aligned}$$

i. e. $M(u,v,t) > (M(u,v,t))^{(1-\alpha-\beta)}$,
implies

$$(M(u,v,t))^{(\alpha+\beta)} > 1,$$

which is not true as the left hand quantity is less than 1.So $u = v$. Thus, if the pair $(f, g)$ has a common fixed point then it is unique.

**Step 1** To see the existence of a common fixed point of the self maps $f$ and $g$, we consider the following cases.

**CASE I** Suppose any two terms of the sequence $\{y_n\}$ are equal i. e. for some $n \in N, y_n = y_{n+1}$. As $y_n = fx_n = gx_{n+1} = fx_{n+1} = gx_{n+2} = y_{n+1}$ we have $fx_{n+1} = gx_{n+1}$. Let $fx_{n+1} = gx_{n+1} = z$. So $x_{n+1}$ is a point of coincidence of the pair $(f, g)$. As the pair $(f, g)$ is weakly compatible we have $fz = gz$. Now we show that $fz = z$. Suppose, if possible on the contrary, that $fz \neq z$ so $fz \neq fx_{n+1}$. By using equation (3) we have

$$\begin{aligned} M(z,fz,t) &= M(fx_{n+1},fz,t), \\ &> (M(fx_{n+1},gx_{n+1},t))^{\alpha}(M(fz,gz,t))^{\beta}(M(gx_{n+1},gz,t))^{(1-\alpha-\beta)}, \end{aligned}$$

$$
\begin{aligned}
&= (M(z,z,t))^{\alpha}(M(fz,fz,t))^{\beta}(M(z,fz,t))^{(1-\alpha-\beta)},\\
&= (M(z,fz,t))^{(1-\alpha-\beta)}.
\end{aligned}
$$

i.e.
$M(z,fz,t) > (M(z,fz,t))^{(1-\alpha-\beta)}$, for all $t > 0$ implies $(M(z,fz,t))^{(\alpha+\beta)} > 1$, which is not possible. Hence $fz = z$. Therefore, $z$ is a common fixed point of the pair $(f,g)$ in this case.

So we can assume the consecutive terms of the sequence $\{y_n\}$ are distinct.

Again, to see the existence of a common fixed point in other cases, we first show that all the terms of the sequence $\{y_n\}$ are distinct.

**CASE II** Suppose $y_n = y_m$, for some $m > (n+1)$, then as all the consecutive terms of the sequence $\{y_n\}$ are distinct, we claim that $y_{n+1} = y_{m+1}$. Suppose if possible on the contrary $y_{n+1} \neq y_{m+1}$ then $y_n \neq y_{n+1} \neq y_{m+1} \neq y_m$ implies $y_n \neq y_m$ which contradicts our assumption. So we have $y_{n+1} = y_{m+1}$. Also and

$$
\begin{aligned}
M(y_{n+1},y_{n+2},t) &= M(fx_{n+1},fx_{n+2},t)\\
&> \left\{\begin{array}{c}(M(fx_{n+1},gx_{n+1},t))^{\alpha}(M(fx_{n+2},gx_{n+2},t))^{\beta}\\(M(gx_{n+1},gx_{n+2},t))^{(1-\alpha-\beta)}\end{array}\right\}\\
&= \left\{\begin{array}{c}(M(y_{n+1},y_n,t))^{\alpha}(M(y_{n+2},y_{n+1},t))^{\beta}\\(M(y_n,y_{n+1},t))^{(1-\alpha-\beta)}\end{array}\right\}\\
&= (M(y_{n+1},y_n,t))^{1-\beta}(M(y_{n+2},y_{n+1},t))^{\beta}
\end{aligned}
$$

i. e.

$$M(y_{n+1},y_{n+2},t) > (M(y_{n+1},y_n,t))^{1-\beta}(M(y_{n+2},y_{n+1}t))^{\beta}.$$

Thus

$$M(y_n,y_{n+1},t) < M(y_{n+1},y_{n+2},t). \tag{8}$$

So

$$M(y_n,y_{n+1},t) < M(y_{n+1},y_{n+2},t) < M(y_{n+2},y_{n+3},t) < ... < M(y_m,y_{m+1},t).$$

i. e. $M(y_n,y_{n+1},t) < M(y_n,y_{n+1},t)$, which is not possible. So this case does not arise.

Thus, we conclude that for distinct $n, m \in N, y_n \neq y_m$. Therefore, the elements of the sequence $\{y_n\}$ are distinct. From equation (8) we have

$$M(y_n, y_{n+1}, t) < M(y_{n+1}, y_{n+2}, t),$$

for all $t > 0$. Thus, $\{M(y_n, y_{n+1}, t)\}$, for each $t > 0$, is a strictly increasing sequence, which is bounded above by $1$. Therefore, by lemma 5, for $t > 0$,

$$\lim_{n \to +\infty} M(y_n, y_{n+1}, t) = 1, \tag{9}$$

Now we prove that the sequence $\{y_n\}$ is M-Cauchy. Suppose if possible on the contrary that it is not true; then there exist $\eta \in (0, 1), t_0 > 0$ and the sequences $\{p(n)\}, \{q(n)\}$ ($p(n)$ being the smallest ones of the index)

$$n < p(n) < q(n), M(y_{p(n)}, y_{q(n)}, t_0) \leq 1 - \eta, M(y_{p(n)-1}, y_{q(n)}, t_0) > 1 - \eta. \tag{10}$$

**STEP 2**: In this step, we show that $\lim_{n \to \infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0) = 1 - \eta$. Now for all $n \geq 1, 0 < \lambda < t_0/2$, we obtain,

$$\begin{aligned} 1 - \eta \;& \geq M(y_{p(n)}, y_{q(n)}, t_0), \qquad \text{using (10)} \\ & \geq M(y_{p(n)}, y_{p(n)-1}, \lambda) * M(y_{p(n)-1}, y_{q(n)-1}, t_0, \lambda - 2) * \\ & * M(y_{q(n)-1}, y_{q(n)}, \lambda). \end{aligned} \tag{11}$$

Let

$$h_1(t) = \overline{\lim_{n \to +\infty}} M(y_{p(n)-1}, y_{q(n)-1}, t), t > 0.$$

Taking the limit supremum on both sides of equation (11), and using the properties of M and $*$, and by lemma 3, we obtain

$$\begin{aligned} 1 - \eta \;& \geq 1 * \overline{\lim}_{n \to +\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0 - 2\lambda) * 1 \qquad \text{using (10)} \\ & = h_1(t_0 - 2\lambda). \end{aligned}$$

Since M is bounded with the range in $(0, 1]$, continuous and non-decreasing in the third variable t, it follows from lemma 3, that $h_1$ is continuous from the left. Therefore, for $\lambda \to 0$ , we obtain

$$h_1(t_0) = \overline{\lim_{n \to +\infty}} M(y_{p(n)-1}, y_{q(n)-1}, t_0) \leq 1 - \eta. \tag{12}$$

Let

$$h_2(t) = \underline{\lim}_{n \to +\infty} M(y_{p(n)-1}, y_{q(n)-1}, t), t > 0.$$

Again, for all $n \geq 1, \lambda > 0$

$$M(y_{p(n)-1}, y_{q(n)-1}, t_0 + \lambda) \;\; \geq \;\; M(y_{p(n)-1}, y_{q(n)}, t_0) * M(y_{q(n)}, y_{q(n)-1}, \lambda)$$

$$> \quad (1-\eta) * M(y_{q(n)}, y_{q(n)-1}, t_0) \quad \text{using (10)}$$

Taking the limit infimum as $n \to +\infty$ in the above inequality, we obtain

$$\begin{aligned} h_2(\lambda + t_0) &= \underline{\lim}_{n\to+\infty} M(y_{q(n)-1}, y_{q(n)-1}, \lambda + t_0), \\ &\geq (1-\eta) * \underline{\lim}_{n\to+\infty} M(y_{p(n)}, y_{q(n)-1}, \lambda), \qquad \text{using (9)} \\ &= (1-\eta) * 1 \\ &= 1-\eta. \end{aligned}$$

Since M is bounded with the range in $(0,1]$ , continuous and non-decreasing in the third variable t, it follows from lemma 3, that $h_2$ is continuous from the right. So letting $\lambda \to 0$ , we obtain

$$\underline{\lim}_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0) \geq (1-\eta). \tag{13}$$

Combining the inequalities (12) and (13), we get

$$\lim_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0) = (1-\eta). \tag{14}$$

**STEP 3** In this step, we show that $\lim_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0) = 1-\eta$. From equation (10) we have

$$\overline{\lim}_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0) \leq 1-\eta. \tag{15}$$

Also for all $n \geq 1$ and $\lambda > 0$ we have

$$\begin{aligned} M(y_{p(n)}, y_{q(n)}, t_0 + 2\lambda) \geq\ & M(y_{p(n)}, y_{p(n)-1}, \lambda) * M(y_{p(n)-1}, y_{q(n)-1}, t_0) * \\ * \ & M(y_{q(n)-1}, y_{q(n)}, \lambda) \end{aligned}$$

Taking the limit infimum as $n \to +\infty$ in the above inequality, using (9), (14) and the properties of M and $*$ and by lemma 2, we obtain

$$\begin{aligned} \underline{\lim}_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0 + 2\lambda) &\geq 1 * \underline{\lim}_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0) * 1 \\ &= 1-\eta, \quad \text{using (14)} \end{aligned} \tag{16}$$

Since M is bounded with the range in $(0,1]$, continuous and non-decreasing in the third variable t, it follows from lemma 3 that $\underline{\lim}_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0)$ is a continuous function of t from the right.

Therefore, for $\lambda \to 0$ , we obtain

$$\underline{\lim}_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0) \geq (1-\eta). \quad (17)$$

Combining inequalities (15) and (17), we get

$$\lim_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0) = (1-\eta). \quad (18)$$

**STEP 4** In this step, we show that the sequence $\{y_n\}$ is an M-Cauchy sequence.

Using equations (9) and (14) at $t = t_0$, we have

$$\begin{aligned} M(x_{p(n)}, x_{q(n)}) &= \left\{ \begin{array}{c} (M(fx_{p(n)}, gx_{p(n)}, t_0))^{\alpha}(M(fx_{q(n)}, gx_{q(n)}, t_0))^{\beta} \\ (M(gx_{p(n)}, gx_{q(n)}, t_0))^{(1-\alpha-\beta)} \end{array} \right\} \\ &= \left\{ \begin{array}{c} (M(y_{p(n)}, y_{p(n)-1}, t_0))^{\alpha}(M(y_{q(n)}, y_{q(n)-1}, t_0))^{\beta} \\ (M(y_{p(n)-1}, y_{q(n)-1}, t_0))^{(1-\alpha-\beta)} \end{array} \right\} \end{aligned}$$

. (19)

Therefore

$$\lim_{t=t_0, n\to+\infty} M(x_{p(n)}, x_{q(n)}) = (1-\eta)^{(1-\alpha-\beta)}. \quad (20)$$

And

$$\begin{aligned} M(x_{p(n)}, x_{q(n)}) &= \left\{ \begin{array}{c} (M(fx_{p(n)}, gx_{p(n)}, t_0))^{\alpha}(M(fx_{q(n)}, gx_{q(n)}, t_0))^{\beta} \\ (M(gx_{p(n)}, gx_{q(n)}, t_0)^{(1-\alpha-\beta)} \end{array} \right\} \\ &> M(fx_{p(n)}, fx_{q(n)}, t_0) \\ &= M(y_{p(n)}, y_{q(n)}, t_0). \end{aligned}$$

$$M(y_{p(n)}, y_{q(n)}, t_0) > M(x_{p(n)}, x_{q(n)}) \quad (21)$$

For $n \to +\infty$ and using equations (20) and (21), we have

$$(1-\eta) \geq (1-\eta)^{(1-\alpha-\beta)},$$

implies that $(1-\eta)^{(\alpha+\beta)} \geq 1.$, which is not possible as$(1-\eta) < 1$. So,$\{y_n\}$ is an M-Cauchy sequence in $g(X)$ which is M-complete. Therefore, there exists $z \in g(X)$ such that

$$\{y_n\} \to z. \tag{22}$$

i. e.

$$\{fx_n\} \to z \quad \text{and} \quad \{gx_{n+1}\} \to z. \tag{23}$$

As $z \in g(X)$ there exists $v \in X$ such that

$$z = gv. \tag{24}$$

**STEP 5** Now we show that $gv = fv$. Suppose, on the contrary, that $fv \neq gv (= w)$. Then exists a positive integer $n_0$ such that $gv \neq gx_n$, for all $n \geq n_0$.

$$\begin{aligned} M(x_n, v) &= (M(fx_n, gx_n, t))^{\alpha}(M(fv, gv, t))^{\beta}(M(gx_n, gv, t)^{(1-\alpha-\beta)} \\ &= (M(y_n, y_{n-1}, t))^{\alpha}(M(fv, z, t))^{\beta}(M(gx_n, z, t)^{(1-\alpha-\beta)}. \end{aligned}$$

Now

$$\begin{aligned} M(fx_n, fv, t) &> M(x_n, v) \\ &= (M(y_n, y_{n-1}, t))^{\alpha}(M(fv, z, t))^{\beta}(M(gx_n, z, t))^{(1-\alpha-\beta)}. \end{aligned}$$

For $n \to +\infty$ and using equations (9), (23) and (24) we get $M(z, fv, t) \geq [M(fv, z, t)]^{\beta}$ i. e. $M(fv, z, t)]^{1-\beta} > 1$, which is not possible if $fv \neq z$. Hence $fv = u$ and we have

$$fv = gv = z. \tag{25}$$

As the pair of self maps $(f, g)$ is weakly compatible, we have

$$fz = gz. \tag{26}$$

**STEP 6** Now we show that $fz = z$. Suppose, on the contrary that $fz \neq z$. Then $gz \neq z$.

$$\begin{aligned} M(z, v) &= (M(fz, gz, t))^{\alpha}(M(fv, gv, t))^{\beta}(M(gz, gv, t)^{(1-\alpha-\beta)} \\ &= (M(fz, z, t))^{(1-\alpha-\beta)} \qquad \text{using (25, 26)} \end{aligned}$$

and

$$M(fz, z, t) = M(fz, fv, t)$$

$$\begin{aligned} &> \ M(z,v), \qquad \text{using (25)} \\ &= \ [M(fz,z,t]^{(1-\alpha-\beta)}. \end{aligned}$$

i. e. $M(fz,z,t)^{(\alpha+\beta)} > 1$ which is not possible as the left hand side is less than 1.Thus, $fu = gu = u$. □

Taking $g = I$ in Theorem 2, then the sequence $\{x_n\} = \{x_0, fx_0, \cdot\cdot\cdot f^n x_0, , \cdots\}$ becomes a Picard sequence for the self map $f$ and we have

COROLLARY 1. *Let $f$ be an interpolative fuzzy Meir-Keeler contractive map on a M-complete fuzzy metric space $(X, M, *)$.Then the map $f$ has a unique fixed point in $X$.*

REMARK 4. If we take $\alpha = 0$ and $\beta = 0$ in the above corollary, we obtain Theorem $3.1$ of Zheng and Wang (Zheng & Wang, 2019).

EXAMPLE 1. (of Theorem 4.1) Let $X = [0,1]$ . Define a self map $f : X \to X$ by $f(x) = \frac{x}{2}$, and $g(x) = x$, the identity map on $X$. Taking $M(x,y) = \frac{1}{1+d(x,y)}$, then $(X, M, .)$ is a complete stationary fuzzy metric space with the product t-norm. Define $\delta$ as follows:

$$\delta(t) = \begin{cases} \frac{1}{12}; & \text{if } 0 < t < \frac{3}{4}, \\ \frac{1}{n(n+2)}, & \text{if } \frac{n-1}{n} \leq t \leq \frac{n}{n+1}, \text{for } n \geq 4. \end{cases}$$

Then $\delta \in \triangle$.

Taking $\alpha = 0 = \beta$. observe that for all values of $x, y \in X, f(x), f(y) \in [0, \frac{1}{3})$. We show that the quadruple $(X, M, \delta, f)$ is an interpolative Meir-Keeler contractive. For this we prove the following condition:

$$forall \ \ \epsilon \in (\frac{3}{4}, 1), \epsilon - \delta(\epsilon) < M(x,y) \leq \epsilon \Longrightarrow M(fx, fy) > \epsilon.$$

If $\epsilon \in (\frac{3}{4}, 1) \Longrightarrow \frac{n-1}{n} \leq \epsilon \leq \frac{n}{n+1}$, for $n \geq 4$, so $\delta(t) = \frac{1}{n(n+2)}$.

Therefore, the inequality $\epsilon - < M(x,y) \leq \epsilon$ gives
$(\frac{n-1}{n}) - \frac{1}{n(n+2)} < \epsilon - \delta(\epsilon) < \frac{1}{1+d(x,y)} \leq \epsilon < \frac{n}{n+1}$. Therefore $\frac{1}{n} \leq d(x,y) < \frac{2}{n}$

which implies that $x, y \in [0,1]$. Hence

$$M(fx, fy) = \frac{1}{1 + d(fx, fy)} = \frac{1}{1 + \frac{d(x,y)}{3}} > \frac{1}{1 + \frac{1}{n}} = \frac{n}{n+1} > \epsilon.$$

Thus, the quadruple $(X, M, \delta, f)$ is an interpolative Meir-Keeler contractive and $x = 0$ is the unique fixed point of the map $f$.

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## ИНТЕРПОЛЯЦИОННОЕ ОБОБЩЕННОЕ СЖАТИЕ МЕИРА-КЕЛЛЕРА

*Собха* Джейн[а], *Вук* Н. Стоилькович[б], *Стојан* Н. Раденович[в]

[а] Университет Шри Вайшнав Видьяпит Вишвавидьялайя в Индауре, математический факультет, кафедра математики, г. Индаур, Мадхья-Прадеш, Республика Индия

[б] Нови-Садский университет, факультет естественных наук, г. Нови-Сад, Республика Сербия, **корреспондент**

[в] Белградский университет, Машиностроительный факультет, г. Белград, Республика Сербия

РУБРИКА ГРНТИ: 27.25.17 Метрическая теория функций,
27.33.00 Интегральные уравнения,
27.39.29 Приближенные методы функционального анализа

ВИД СТАТЬИ: оригинальная научная статья

*Резюме:*

*Введение/цель: Цель данной статьи заключается в введении понятия интерполяционного обобщенного условия сжатия Меира-Келлера для отображений в нечетком метрическом пространстве, которое расширяет, объединяет и обобщает многообразие Меира-Келлера, предназначенное только для одного отображения. При применении устанавливается единая теорема о совместной неподвижной точке для двух отображений через слабую совместимость. В статье приведен пример, доказывающий достоверность результатов исследования.*

*Методы: Методы функционального анализа с сокращением Меира-Келлера.*

*Результаты: Получена уникальная неподвижная точка для отображений в нечетком метрическом пространстве.*

*Выводы: Получена неподвижная точка собственных отображений.*

*Ключевые слова: нечеткое метрическое пространство, общие фиксированные точки, слабая совместимость, интерполяционное обобщенное сокращение Меира-Келлера.*



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## ИНТЕРПОЛАТИВНА УОПШТЕНА МЕИР-КЕЛЕРОВА КОНТРАКЦИЈА

*Собха* Јаин[а], *Вук* Н. Стојиљковић[б], *Стојан* Н. Раденовић[в]

[а] Shri Vaishnav Vidyapeeth Vishwavidyalaya, Катедра математике, Индоре, Мадја Прадеш, Република Индија

[б] Универзитет у Новом Саду, Природно-математички факултет, Нови Сад, Република Србија, **аутор за преписку**

[в] Универзитет у Београду, Машински факултет, Београд, Република Србија

ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

*Сажетак:*

*Увод/циљ: Циљ овог рада је да се уведе појам интерполативног генерализованог Меир-Келеровог контрактивног услова за пресликавања у фузиметричком простору. Он увећава, обједињује и генерализује Меир-Келерову контракцију и служи за само једно пресликавање. Користећи га, успостављамо јединствену заједничку теорему фиксне тачке за два пресликавања кроз слабу компатибилност. Рад садржи пример који показује валидност наших резултата.*

*Методе: Методе функционалне анализе са Меир-Келеровом контракцијом.*

*Резултати: Јединствена фиксна тачка за пресликавања у фузипростору је добијена.*

*Закључак: Фиксна тачка пресликавања самог у себе је добијена.*

*Кључне речи: фузиметрички простор, заједничке фиксне тачке, слаба компатибилност, интерполативна генерализована Меир-Келерова контракција.*

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EDITORIAL NOTE: The third author of this article, Stojan N. Radenović, is a current member of the Editorial Board of the *Military Technical Courier*. Therefore, the Editorial Team has ensured that the double blind reviewing process was even more transparent and more rigorous. The Team made additional effort to maintain the integrity of the review and to minimize any bias by having another associate editor handle the review procedure independently of the editor – author in a completely transparent process. The Editorial Team has taken special care that the referee did not recognize the author's identity, thus avoiding the conflict of interest.

КОММЕНТАРИЙ РЕДКОЛЛЕГИИ: Третий автор данной статьи Стоян Н. Раденович является действующим членом редколлегии журнала *«Военно-технический вестник»*. Поэтому редколлегия провела более открытое и более строгое двойное слепое рецензирование. Редколлегия приложила дополнительные усилия для того чтобы сохранить целостность рецензирования и свести к минимуму предвзятость, вследствие чего второй редактор-сотрудник управлял процессом рецензирования независимо от редактора-автора, таким образом процесс рецензирования был абсолютно прозрачным. Редколлегия во избежание конфликта интересов позаботилась о том, чтобы рецензент не узнал кто является автором статьи.

РЕДАКЦИЈСКИ КОМЕНТАР: Трећи аутор овог чланка Стојан Н. Раденовић је актуелни члан Уређивачког одбора *Војнотехничког гласника*. Због тога је уредништво спровело транспарентнији и ригорознији двоструко слепи процес рецензије. Уложило је додатни напор да одржи интегритет рецензије и необјективност сведе на најмању могућу меру тако што је други уредник сарадник водио процедуру рецензије независно од уредника аутора, при чему је тај процес био апсолутно транспарентан. Уредништво је посебно водило рачуна да рецензент не препозна ко је написао рад и да не дође до конфликта интереса.

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