

## CERTAIN INTEGRALS INVOLVING GENERALIZED MITTAG-LEFFLER TYPE FUNCTIONS

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DOI: 10.5937/vojtehg70-40296; <https://doi.org/10.5937/vojtehg70-40296>

FIELD: Mathematics

ARTICLE TYPE: Original scientific paper

### *Abstract:*

*Introduction/purpose: Certain integrals involving the generalized Mittag-Leffler function with different types of polynomials are established.*

*Methods: The properties of the generalized Mittag-Leffler function are used in conjunction with different kinds of polynomials such as Jacobi, Legendre, and Hermite in order to evaluate their integrals.*

*Results: Some integral formulae involving the Legendre function, the Bessel Maitland function and the generalized hypergeometric functions are derived.*

*Conclusions: The results obtained here are general in nature and could be useful to establish further integral formulae involving other kinds of polynomials.*

*Key words: Mittag-Leffler function, Generalized hypergeometric function, Bessel Maitland function, Jacobi polynomials, Hermite polynomials.*

## Introduction

This paper follows the lines of the companion paper (Haq et al, 2019) involving the generalized Galuè-type Struve function in which the same topics are dealt here with the generalized Mittag-Leffler functions. As it is well known, a special function:

$$\mathcal{E}_v(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(vk + 1)}, \quad z, v \in \mathbb{C}, \Re(v) > 0, \quad (1)$$

and its general form

$$\mathcal{E}_{v,\omega}(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(vk + \omega)}, \quad z, v \in \mathbb{C}, \Re(v) > 0, \Re(\omega) > 0, \quad (2)$$

are called Mittag-Leffler functions (Erdelyi et al, 1953a),  $\mathbb{C}$  being the set of complex numbers. The former was established by Mittag-Leffler (Mittag-Leffler, 1903) in connection with his method of summation of some divergent series. Certain properties of this function were studied and investigated. The function defined by (2) appeared for the first time in the work of Wiman (Wiman, 1905). The functions given by equations (1) and (2) are entire functions of order  $\mu = \frac{1}{v}$  and of type  $\sigma = 1$  (see for example (Erdelyi et al, 1953b)). By means of the series representations, a generalization of the functions defined by equations (1) and (2) is introduced by Prabhakar (Prabhakar, 1971) as:

$$\mathcal{E}_{v,\omega}^\rho(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{\Gamma(vk + \omega) k!}, \quad v, \omega, \rho \in \mathbb{C}, \Re(v) > 0, \Re(\omega) > 0, \quad (3)$$

where

$$(\rho)_k = \rho(\rho + 1) \dots (\rho + k - 1) = \frac{\Gamma(\rho + k)}{\Gamma(\rho)},$$

whenever  $\Gamma(\rho)$  is defined,  $(\rho)_0 = 1, \rho = 0$ . It is an entire function of order  $\mu = (1/v)[\Re(v)^{\Re(v)}]^{-1/v}$ . For various properties of this function with applications, see Prabhakar (Prabhakar, 1971). Further generalization of the

Mittag-Leffler function  $\mathcal{E}_{v,\omega}^\rho(z)$  was considered earlier by Shukla and Prajapati (Shukla & Prajapati, 2007) which is given as:

$$\mathcal{E}_{v,\omega}^{\rho,q}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_{kq} z^k}{\Gamma(vk + \omega)k!},$$

with  $z, \omega, \rho \in \mathbb{C}$ ,  $\Re(v) > \max(0, \Re(q) - 1)$ ,  $\Re(q) > 0$ , (4)

which is the special case when  $q \in (0, 1)$  and  $\min\{\Re(\omega), \Re(\rho)\} > 0$ .

In continuation of this study, Salim and Faraj (Salim & Faraj, 2012; Nadir et al, 2014) introduced a new generalization of the Mittag-Leffler function which was given as:

$$\mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_{kq} z^k}{\Gamma(vk + \omega)(\delta)_{pk}}, \tag{5}$$

$(\min\{\Re(v), \Re(\omega), \Re(\rho), \Re(\delta)\} > 0; \quad p, q > 0; \quad z, v, \omega, \rho, \delta, \in \mathbb{C}).$

Numerous generalizations and cases of the Mittag-Leffler function have been studied and investigated, see for details (Singh & Rawat, 2013; Wright, 1935b; Faraj et al, 2013; Dorrego & Cerutti, 2012; Srivastava & Tomovski, 2009; Saxena et al, 2011; Khan & Ahmed, 2012).

Integral formulae involving the Mittag-Leffler functions have been developed by many authors, see for example, (Prajapati & Shukla, 2012; Prajapati et al, 2013; Gehlot, 2021; Purohit et al, 2011). In this sequel, here, we aim to establish certain new generalized integral formulae involving the new generalization of the Mittag-Leffler function. The main result presented here is general enough to be specialized to give many interesting integral formulae which are derived as special cases.

### Integrals with the Jacobi polynomials

The Jacobi polynomials  $P_n^{\varrho,\sigma}(y)$  (Rainville, 1960; Srivastava & Manocha, 1984) may be defined by

$$P_n^{(\varrho,\sigma)}(y) = \frac{(1 + \varrho)_n}{n!} {}_2F_1 \left[ \begin{matrix} -n, 1 + \varrho + \sigma + n; \\ \varrho + 1; \end{matrix} \frac{1 - y}{2} \right]. \tag{6}$$

When  $\varrho = \sigma = 0$ , the polynomial in (6) becomes the Legendre polynomial (Rainville, 1960). From (6), it follows that  $P_n^{(\varrho,\sigma)}(y)$  is a polynomial of

the degree  $n$  and that

$$P_n^{(\varrho, \sigma)}(y) = \frac{(1 + \varrho)_n}{n!}. \quad (7)$$

Here, we obtain the following integrals.

**THEOREM 1.** *If  $p, q > 0$ ,  $z, v, \omega, \rho, \delta, \in \mathbb{C}$ ,  $\Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$  and  $\Re(\xi) > -1, \varrho > -1, \sigma > -1$  then the following integral formula holds true*

$$\begin{aligned} & \int_{-1}^1 y^\xi (1-y)^\varrho (1+y)^{\eta+hk} P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1+y)^h] dy \\ &= \frac{(-1)^n 2^{\varrho+\eta+1} \Gamma(\eta+hk+1) \Gamma(n+\varrho+1) \Gamma(\eta+hk+\sigma+1)}{n! \Gamma(\eta+hk+\sigma+n+1) \Gamma(\eta+hk+\varrho+n+2)} \\ & \times \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}(2^h z) \times {}_3F_2 \left[ \begin{matrix} -\xi, \eta+hk+\sigma+1, \eta+hk+1; \\ \eta+hk+\sigma+n+1, \eta+hk+\varrho+n+2; \end{matrix} 1 \right]. \end{aligned} \quad (8)$$

**Proof.** Naming the left-hand side (LHS) of (8) as  $I_1$  and using the definition (5), we have

$$\begin{aligned} I_1 &= \int_{-1}^1 y^\xi (1-y)^\varrho (1+y)^\eta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1+y)^h] dy \\ I_1 &= \int_{-1}^1 y^\xi (1-y)^\varrho (1+y)^\eta P_n^{(\varrho, \sigma)}(y) \sum_{k=0}^{\infty} \frac{(\rho)_{kq} [z(1+y)^h]^k}{\Gamma(vk+\omega)(\delta)_{pk}} dy, \end{aligned}$$

interchanging the order of integration and summation which is permissible under the conditions of the theorem, the above expression becomes

$$\sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk+\omega)(\delta)_{pk}} \int_{-1}^1 y^\xi (1-y)^\varrho (1+y)^{\eta+hk} P_n^{(\varrho, \sigma)}(y) dy. \quad (9)$$

Apply the following formula (Saxena, 2008) on (9)

$$\begin{aligned} & \int_{-1}^1 y^\xi (1-y)^\varrho (1+y)^\eta P_n^{(\varrho, \sigma)}(y) dy = \\ & \frac{(-1)^n 2^{\varrho+\eta+1} \Gamma(\eta+1) \Gamma(n+\varrho+1) \Gamma(\eta+\sigma+1)}{n! \Gamma(\eta+\sigma+n+1) \Gamma(\eta+\varrho+n+2)} \end{aligned}$$

$$\times {}_3F_2 \left[ \begin{matrix} -\xi, \eta + \sigma + 1, \eta + 1; \\ \eta + \sigma + n + 1, \eta + \varrho + n + 2; \end{matrix} 1 \right], \quad (10)$$

provided that  $\varrho > -1$  and  $\sigma > -1$ , and we get the desired result.

**THEOREM 2.** *If  $p, q > 0$ ,  $z, v, \omega, \rho, \delta \in \mathbb{C}$ ,  $\Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$  and  $\Re(\xi) > -1, \varrho > -1, \sigma > -1$  then the following integral formula holds true*

$$\begin{aligned} & \int_{-1}^1 (1-y)^\eta (1+y)^\sigma P_n^{(\varrho, \sigma)}(y) P_n^{(\mu, \theta)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1-y)^h] dy \\ &= \frac{2^{\eta+\sigma+1} \Gamma(1+\mu+m) \Gamma(1+\varrho+n)}{m!n!} \\ & \times \sum_{k=0}^{\infty} \frac{(-m)_k (1+\mu+\theta+m)_k}{\Gamma(1+\mu+k)(k!)} \sum_{l=0}^{\infty} \frac{(-m)_l (1+\mu+\theta+m)_l}{\Gamma(1+\mu+k)(l!)} \\ & \times \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}(2^h z) \mathbb{B}(1+\eta+hk+k+l, 1+\sigma). \end{aligned} \quad (11)$$

Proof. Denoting the LHS of (11) by  $I_2$  and using definition (5), we get

$$\begin{aligned} I_2 &= \int_{-1}^1 (1-y)^\eta (1+y)^\sigma P_n^{(\varrho, \sigma)}(y) P_n^{(\mu, \theta)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1-y)^h] dy \\ I_2 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk+\omega)(\delta)_{pk}} \int_{-1}^1 (1-y)^{\eta+hk} (1+y)^\sigma P_n^{(\varrho, \sigma)}(y) P_n^{(\mu, \theta)}(y) dy. \end{aligned} \quad (12)$$

Now, using (6) in (12), we get

$$\begin{aligned} I_2 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk+\omega)(\delta)_{pk}} \frac{(1+\mu)_m}{m!} \sum_{k=0}^{\infty} \frac{(-m)_k (1+\mu+\theta+m)_k}{(1+\mu)_k 2^k k!} \\ & \times \int_{-1}^1 (1-y)^{\eta+hk+k} (1+y)^\sigma P_n^{(\varrho, \sigma)}(y) dy. \end{aligned} \quad (13)$$

Again using (6) in (13), we attain

$$\begin{aligned} I_2 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk+\omega)(\delta)_{pk}} \frac{\Gamma(1+\mu+m) \Gamma(1+\varrho+n)}{m!n!} \\ & \sum_{k=0}^{\infty} \frac{(-m)_k (1+\mu+\theta+m)_k}{\Gamma(1+\mu+k) 2^k (k!)} \sum_{l=0}^{\infty} \frac{(-m)_l (1+\mu+\theta+m)_l}{\Gamma(1+\mu+l) 2^l (l!)} \end{aligned}$$

$$\times \int_{-1}^1 (1-y)^{\eta+hk+k+l} (1+y)^\sigma P_n^{(\varrho, \sigma)}(y) dy, \quad (14)$$

but by the formula found in (Rainville, 1960; Srivastava & Manocha, 1984)

$$\int_{-1}^1 (1-y)^{\eta+n} (1+y)^{\sigma+n} dy = 2^{2n+\sigma+\eta+1} \mathbb{B}(1+\eta+n, 1+\sigma+n), \quad (15)$$

using it in (14), we get the required result.

**THEOREM 3.** *If  $p, q > 0$   $z, v, \omega, \rho, \delta \in \mathbb{C}$ ,  $\Re(v) > 0$ ,  $\Re(\omega) > 0$ ,  $\Re(\rho) > 0$ ,  $\Re(\delta) > 0$  and  $\varrho > -1$ ,  $\sigma > -1$ , then*

$$\begin{aligned} & \int_{-1}^1 (1-y)^\mu (1+y)^\theta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1-y)^h (1+y)^t] dy \\ &= \frac{2^{\mu+\theta+1}}{n!} \frac{(1+\varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\varrho+\sigma+n)_k}{(1+\varrho)_k (k!)} \\ & \times \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}(2^{h+t} z) \mathbb{B}(1+\mu+hk+k, 1+\theta+tk). \end{aligned} \quad (16)$$

**Proof.** Denoting the LHS of (16) by  $I_3$ ,

$$\begin{aligned} I_3 &= \int_{-1}^1 (1-y)^\mu (1+y)^\theta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1-y)^h (1+y)^t] dy \\ &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk+\omega) (\delta)_{pk}} \int_{-1}^1 (1-y)^{\mu+hk} (1+y)^{\theta+tk} P_n^{(\varrho, \sigma)}(y) dy. \end{aligned} \quad (17)$$

Now, using (6) in (17) we have

$$\begin{aligned} I_3 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk+\omega) (\delta)_{pk}} \frac{(1+\varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\varrho+\sigma+n)_k}{(1+\varrho)_k 2^k k!} \\ & \times \int_{-1}^1 (1-y)^{n+\mu+hk+k-n} (1+y)^{n+\theta+tk-n} dy, \end{aligned} \quad (18)$$

further, using (15) in (18) we attain the desired result.

**THEOREM 4.** *If  $p, q > 0$   $z, v, \omega, \rho, \delta \in \mathbb{C}$ ,  $\Re(v) > 0$ ,  $\Re(\omega) > 0$ ,  $\Re(\rho) > 0$ ,  $\Re(\delta) > 0$  and  $\varrho > -1$ ,  $\sigma > -1$ , then*

$$\int_{-1}^1 (1-y)^\mu (1+y)^\theta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1+y)^{-h}] dy$$

$$\begin{aligned}
 &= \frac{2^{\mu+\theta+1}}{n!} \frac{(1+\varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\varrho+\sigma+n)_k}{(1+\varrho)_k (k!)} \\
 &\quad \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^{-h}z) \mathbb{B}(1+\mu+k, 1+\theta-hk). \tag{19}
 \end{aligned}$$

Proof. Denoting the LHS of (19) by  $I_4$ ,

$$\begin{aligned}
 I_4 &= \int_{-1}^1 (1-y)^\mu (1+y)^\theta P_n^{(\varrho,\sigma)}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(1+y)^{-h}] dy \\
 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq}(z)^k}{\Gamma(vk+\omega)(\delta)_{pk}} \int_{-1}^1 (1-y)^\mu (1+y)^{\theta-hk} P_n^{(\varrho,\sigma)}(y) dy. \tag{20}
 \end{aligned}$$

Now, using (6) in (20) we attain

$$\begin{aligned}
 I_4 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq}(z)^k}{\Gamma(vk+\omega)(\delta)_{pk}} \frac{(1+\varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\varrho+\sigma+n)_k}{(1+\varrho)_k 2^k k!} \\
 &\quad \times \int_{-1}^1 (1-y)^{n+\mu+k-n} (1+y)^{n+\theta-hk-n} dy \tag{21}
 \end{aligned}$$

further, using (15) in (21) we attain the required result.

**THEOREM 5.** If  $p, q > 0$   $z, v, \omega, \rho, \delta, \in \mathbb{C}$ ,  $\Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$  and  $\varrho > -1, \sigma > -1$ , then

$$\begin{aligned}
 &\int_{-1}^1 (1-y)^\mu (1+y)^\theta P_n^{(\varrho,\sigma)}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(1-y)^h(1+y)^{-t}] dy \\
 &= \frac{2^{\mu+\theta+1}}{n!} \frac{(1+\varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\varrho+\sigma+n)_k}{(1+\varrho)_k (k!)} \\
 &\quad \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^{h-t}z) \mathbb{B}(1+\mu+hk+k, 1+\theta-tk). \tag{22}
 \end{aligned}$$

Proof. Denoting the LHS of (22) by  $I_5$ ,

$$\begin{aligned}
 I_5 &= \int_{-1}^1 (1-y)^\mu (1+y)^\theta P_n^{(\varrho,\sigma)}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(1-y)^h(1+y)^{-t}] dy \\
 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq}(z)^k}{\Gamma(vk+\omega)(\delta)_{pk}} \int_{-1}^1 (1-y)^{\mu+hk} (1+y)^{\theta-tk} P_n^{(\varrho,\sigma)}(y) dy, \tag{23}
 \end{aligned}$$



now using (6) in (23) we attain

$$I_5 = \sum_{k=0}^{\infty} \frac{(\rho)_{kq}(z)^k}{\Gamma(vk + \omega)(\delta)_{pk}} \frac{(1 + \varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1 + \varrho + \sigma + n)_k}{(1 + \varrho)_k 2^k k!} \times \int_{-1}^1 (1 - y)^{n+\mu+hk+k-n} (1 + y)^{n+\theta-tk-n} dy \quad (24)$$

further, using (15) in (24) we attain the required result.

### Some special cases

If we replace  $\eta$  by  $\xi - 1$  and put  $\varrho = \sigma = \mu = \theta = 0$  then the integral  $I_2$  transforms into the following integral involving the Legendre polynomial (Rainville, 1960)

$$I_6 = \int_{-1}^1 (1 - y)^{\xi-1} P_n(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(1 - y)^h] dy$$

$$I_6 = \sum_{k=0}^{\infty} \frac{2^\xi (-m)_k (1 + m)_k}{(k!)^2} \times \sum_{l=0}^{\infty} \frac{(-n)_l (1 + n)_l}{l!^2} \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^h z) \mathbb{B}(1 + \xi + hk + k + l, 1). \quad (25)$$

If  $\sigma = \varrho = 0$ ,  $\mu$  is replaced by  $\mu - 1$  and  $\theta$  by  $\theta - 1$ , then the integral  $I_3$  transforms into the following integral involving the Legendre polynomial (Rainville, 1960)

$$I_7 = \int_{-1}^1 (1 - y)^{\mu-1} (1 + y)^{\theta-1} P_n(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(1 - y)^h (1 + y)^t] dy$$

$$I_7 = \sum_{k=0}^{\infty} \frac{2^{\mu+\theta-1} (-n)_k (1 + n)_k}{(k!)^2} \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^{h+t} z) \mathbb{B}(1 + \mu + hk + k, \theta + tk). \quad (26)$$

If  $\varrho = \sigma = 0$ ,  $\mu$  is replaced by  $\mu - 1$  and  $\theta$  by  $\theta - 1$  then the integral  $I_3$  transforms into the following integral involving the Legendre polynomial (Rainville, 1960)

$$I_8 = \int_{-1}^1 (1 - y)^{\mu-1} (1 + y)^{\theta-1} P_n(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(1 - y)^h (1 + y)^{-t}] dy$$



$$I_8 = \sum_{k=0}^{\infty} \frac{2^{\mu+\theta-1}(-n)_k(1+n)_k}{(k!)^2} \times \mathcal{E}_{\nu,\omega,p}^{\rho,\delta,q}(2^{h-t}z)\mathbb{B}(1+\mu+hk+k,\theta-tk). \quad (27)$$

### Integral with the Bessel Maitland function

The special case of the Wright function (Erdelyi et al, 1953b), see also (Wright, 1935a,b) written in the form

$$\phi(-; A, a; z) = {}_0\psi_1 \left[ \begin{matrix} -; \\ (A, a); \end{matrix} z \right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(Ak+a)} \frac{z^k}{k!}, \quad (28)$$

with complex  $z, a \in \mathbb{C}$  and real  $A \in \mathbb{R}$ . When  $A = \eta, a = \nu + 1$  and  $z$  is replaced by  $-z$ , then the function  $\phi(\eta, \nu + 1; -z)$  is defined by  $J_{\nu}^{\eta}(z)$

$$\phi(\eta, \nu + 1; -z) = J_{\nu}^{\eta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\eta k + \nu + 1)} \frac{(-z)^k}{k!}, \quad (29)$$

and such a function is known as the Bessel Maitland function, or the generalized Bessel function, or the Wright generalized Bessel function, see (Mcbride, 1995).

**THEOREM 6.** *If  $p, q > 0, z, v, \omega, \rho, \delta \in \mathbb{C}, \Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0, \rho - \rho\tau > -1, \rho > 0, 0 < \tau < 1$  and  $\Re(\mu + 1) > 0$ , then the following integral formula holds true.*

$$\int_{-1}^1 (y)^{\mu} J_{\nu}^{\tau}(y) \mathcal{E}_{\nu,\omega,p}^{\rho,\delta,q}[z(y)^{\rho}] dy = \frac{\Gamma(\mu + \rho k + 1)}{\Gamma(1 + \nu - \tau - \tau(\mu + \rho k))} \times \mathcal{E}_{\nu,\omega,p}^{\rho,\delta,q}(z). \quad (30)$$

Proof. Naming the LHS of (30) as  $I_9$ , we obtain

$$I_9 = \int_{-1}^1 (y)^{\mu} J_{\nu}^{\tau}(y) \mathcal{E}_{\nu,\omega,p}^{\rho,\delta,q}[z(y)^{\rho}] dy$$

$$I_9 = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(\nu k + \omega)(\delta)_{pk}} \times \int_0^{\infty} (y^{\mu+\rho k}) J_{\nu}^{\tau}(y) dy. \quad (31)$$

Now we know the formula, see (Saxena, 2008)

$$\int_0^{\infty} (y^{\mu}) J_{\nu}^{\tau}(y) dy = \frac{\Gamma(\mu + 1)}{\Gamma(1 + \nu - \tau - \tau\mu)}, \quad (32)$$

provided  $\Re(\mu) > -1, 0 < \tau < 1$ .

Using (32) in (31), we attain

$$I_9 = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(\nu k + \omega)(\delta)_{pk}} \times \frac{\Gamma(\mu + \varrho k + 1)}{\Gamma(1 + \nu - \tau - \tau(\mu + \varrho k))},$$

hence proved.

### Integrals with the Legendre functions

The Legendre functions are solution of Legendre's differential equation, see (Erdelyi et al, 1953a)

$$(1 - z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + [(\nu(\nu + 1)) - \omega^2(1 - z^2)^{-1}] w = 0, \quad (33)$$

where  $z, \nu, \omega$  are unrestricted.

Under the substitution  $w = (z^2 - 1)^{\omega/2} \nu$  in (5.1) becomes

$$(1 - z^2) \frac{d^2 \nu}{dz^2} - 2(\omega + 1)z \frac{d\nu}{dz} + [(\nu + \omega)(\nu + \omega + 1)] \nu = 0, \quad (34)$$

and with  $\lambda = 1/2 - z/2$  as the independent variable, this differential equation becomes

$$\lambda(1 - \lambda) \frac{d^2 \nu}{d\lambda^2} + (\omega + 1)(1 - 2\lambda) \frac{d\nu}{d\lambda} + [(\nu - \omega)(\nu + \omega + 1)] \nu = 0. \quad (35)$$

This is the Gauss hypergeometric type equation with  $a = \omega - \nu, b = \nu + \omega + 1, c = \omega + 1$ .

Hence it follows that the function

$$W = P_\nu^\omega(z) = \frac{1}{\Gamma(1 - \omega)} \left( \frac{z + 1}{z - 1} \right)^{\omega/2} {}_2F_1 \left[ \begin{matrix} -\nu, \nu + 1; \\ 1 - \omega; \end{matrix} 1/2 - z/2 \right], \quad (36)$$

for  $|1 - z| < 2$  is a solution of (33).

The function  $P_\nu^\omega(z)$  is known as the Legendre function of the first kind (Erdelyi et al, 1953a). It is one valued and regular on the  $z$ -plane, supposed cut along the real axis from 1 to  $-\infty$ .

**THEOREM 7.** *If  $p, q > 0$   $z, v, \omega, \rho, \delta \in \mathbb{C}$ ,  $\Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$  and  $\theta > 0$  and  $\eta$  is a positive integer then*

$$\int_0^1 (y)^{\theta-1} (1-y^2)^{\eta/2} P_\nu^\eta(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^q] dy$$

$$= \frac{(-1)^\eta \sqrt{\pi} 2^{-\theta-\eta} \Gamma(\nu + \eta + 1)}{\Gamma(1 - \eta + \nu)} \times$$

$$\sum_{k=0}^{\infty} \frac{\Gamma(\theta + \rho k)}{\Gamma(1/2 + \frac{\theta + \rho k}{2} + \eta/2 - \nu/2) \Gamma(1 + \frac{\theta + \rho k}{2} + \eta/2 + \nu/2)} \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(z/2^\theta). \quad (37)$$

**Proof.** Denoting the LHS of (37) by  $I_{10}$ ,

$$I_{10} = \int_0^1 (y)^{\theta-1} (1-y^2)^{\eta/2} P_\nu^\eta(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^q] dy$$

$$I_{10} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk + \omega)(\delta)_{pk}} \times \int_0^1 y^{\theta-1+\rho k} (1-y^2)^{\eta/2} P_\nu^\eta(y) dy. \quad (38)$$

Now the integral in (38) can be solved by using the formula (Erdelyi et al, 1953a)

$$\int_0^1 y^{\theta-1} (1-y^2)^{\eta/2} P_\nu^\eta(y) dy$$

$$= \frac{(-1)^\eta \sqrt{\pi} 2^{-\theta-\eta} \Gamma(\theta) \Gamma(\nu + \eta + 1)}{(\Gamma(1/2 + \theta/2 + \eta/2 - \nu/2) \Gamma(1 + \theta/2 + \eta/2 + \nu/2) (1 - \eta + \nu))}, \quad (39)$$

provided  $\Re(\theta) > 0, \eta = 1, 2, 3, \dots$

Now (38) becomes

$$I_{10} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk + \omega)(\delta)_{pk}} \times$$

$$\frac{(-1)^\eta \sqrt{\pi} 2^{-(\theta+\rho k)-\eta} \Gamma(\theta + \rho k) \Gamma(\eta + \nu + 1)}{\Gamma(1/2 + \frac{\theta + \rho k}{2} + \eta/2 - \nu/2) \Gamma(1 + \frac{\theta + \rho k}{2} + \eta/2 + \nu/2) \Gamma(1 - \eta + \nu)},$$

which is the desired result.

**THEOREM 8.** *If  $p, q > 0$   $z, v, \omega, \rho, \delta \in \mathbb{C}$ ,  $\Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$  and  $\theta > 0$  and  $\eta$  is a positive integer then*

$$\int_0^1 (y)^{\theta-1} (1-y^2)^{-\eta/2} P_\nu^\eta(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^q] dy$$

$$= \sqrt{\pi} 2^{-\theta+\eta} \sum_{k=0}^{\infty} \frac{\Gamma(\theta + \rho k)}{\Gamma(1/2 + \frac{\theta+\rho k}{2} - \eta/2 - \nu/2) \Gamma(1 + \frac{\theta+\rho k}{2} - \eta/2 - \nu/2)} \times \mathcal{E}_{\nu, \omega, p}^{\rho, \delta, q}(z/2^\theta). \quad (40)$$

Proof. Denoting the LHS of (40) by  $I_{11}$ ,

$$I_{11} = \int_0^1 (y)^{\theta-1} (1-y^2)^{-\eta/2} P_\nu^\eta(y) \mathcal{E}_{\nu, \omega, p}^{\rho, \delta, q}[z(y)^e] dy$$

$$I_{11} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(\nu k + \omega)(\delta)_{pk}} \times \int_0^1 y^{\theta-1+\rho k} (1-y^2)^{-\eta/2} P_\nu^\eta(y) dy, \quad (41)$$

now the integral in (41) can be solved by using the formula (Erdelyi et al, 1953a)

$$\int_0^{\infty} y^{\theta-1} (1-y^2)^{-\eta/2} P_\nu^\eta(y) dy$$

$$= \frac{\sqrt{\pi} 2^{-\theta+\eta} \Gamma(\theta)}{\Gamma(1/2 + \theta/2 - \eta/2 - \nu/2) \Gamma(1 + \theta/2 - \eta/2 - \nu/2)}, \quad (42)$$

provided  $\Re(\theta) > 0, \eta = 1, 2, 3, \dots$

Again (41) becomes

$$I_{11} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(\nu k + \omega)(\delta)_{pk}} \times \frac{\sqrt{\pi} 2^{-(\theta+\rho k)+\eta} \Gamma(\theta + \rho k)}{\Gamma(1/2 + \frac{\theta+\rho k}{2} - \eta/2 - \nu/2) \Gamma(1 + \frac{\theta+\rho k}{2} - \eta/2 - \nu/2)}.$$

### Integrals with the Hermite polynomials

The Hermite polynomials  $H_n(y)$ , see (Rainville, 1960; Srivastava & Manocha, 1984) may be defined by means of the relation

$$\exp(2yt - t^2) = \sum_{k=0}^{\infty} \frac{H_k(y) t^k}{k!}, \quad (43)$$

valid for all finite  $y$  and  $t$ . Since

$$\exp(2yt - t^2) = \exp(2yt) \exp(-t^2)$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \frac{(2y)^n t^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2y)^{n-2k} t^n}{(n-2k)!k!}.
 \end{aligned}$$

It follows from (43) that

$$H_n(y) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2y)^{n-2k} t^n}{(n-2k)!k!}. \tag{44}$$

The examination of equation (44) shows that  $H_n(y)$  is a polynomial of degree precisely  $n$  in  $y$  and that

$$H_n(y) = 2^n y^n + \pi_{n-2}(y) \tag{45}$$

in which  $\pi_{n-2}(y)$  is a polynomial of the degree  $(n-2)$  in  $y$ .

**THEOREM 9.** *If  $p, q > 0$ ,  $z, v, \omega, \rho, \delta \in \mathbb{C}$ ,  $\Re(v) > 0$ ,  $\Re(\omega) > 0$ ,  $\Re(\rho) > 0$ ,  $\Re(\delta) > 0$  and  $h > 0$ ,  $\Re(\mu) = 0, 1, 2, \dots$  then*

$$\begin{aligned}
 &\int_{-\infty}^{\infty} (y)^{2\mu} \exp(-y^2) H_{2\nu}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^{-2h}] dy \\
 &= \sqrt{\pi} 2^{2(\nu-\mu)} \sum_{k=0}^{\infty} \frac{\Gamma(2\mu - hk + 1)}{\Gamma(\mu - hk - \nu + 1)} \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^{2h}z). \tag{46}
 \end{aligned}$$

**Proof.** Denoting the LHS of (9) by  $I_{12}$ , we have

$$\begin{aligned}
 I_{12} &= \int_{-\infty}^{\infty} (y)^{2\mu} \exp(-y^2) H_{2\nu}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^{-2h}] dy \\
 I_{12} &= \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk + \omega)(\delta)_{pk}} \times \int_{-\infty}^{\infty} y^{2\mu-2hk} (\exp)^{-y^2} H_{2\nu}(y) dy, \tag{47}
 \end{aligned}$$

now the integral in (47) can be solved by using the formula (Saxena, 2008)

$$\int_{-\infty}^{\infty} y^{2\mu} (\exp)^{-y^2} H_{2\nu}(y) dy = \frac{\sqrt{\pi} 2^{2(\nu-\mu)} \Gamma(2\mu + 1)}{\Gamma(\mu - \nu + 1)}. \tag{48}$$

Again (47) becomes

$$I_{12} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk + \omega)(\delta)_{pk}} \times \frac{\sqrt{\pi} 2^{2(\nu-(2\mu+2hk))} \Gamma(2\mu - 2hk + 1)}{\Gamma(\mu - hk - \nu + 1)}.$$

**THEOREM 10.** *If  $p, q > 0$ ,  $z, v, \omega, \rho, \delta \in \mathbb{C}$ ,  $\Re(v) > 0$ ,  $\Re(\omega) > 0$ ,  $\Re(\rho) > 0$ ,  $\Re(\delta) > 0$  and  $h > 0$ ,  $\Re(\mu) = 0, 1, 2, \dots$  then*

$$\int_{-\infty}^{\infty} (y)^{2\mu} \exp(-y^2) H_{2\nu}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^{2h}] dy = \sqrt{\pi} 2^{2(\nu-\mu)} \sum_{k=0}^{\infty} \frac{\Gamma(2\mu + 2hk + 1)}{\Gamma(\mu + hk - \nu + 1)} \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^{-2h}z). \quad (49)$$

**Proof.** Denoting the LHS of (10) by  $I_{13}$ , we have

$$I_{13} = \int_{-\infty}^{\infty} (y)^{2\mu} \exp(-y^2) H_{2\nu}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^{2h}] dy$$

$$I_{13} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk + \omega)(\delta)_{pk}} \times \int_{-\infty}^{\infty} y^{2\mu+2hk} (\exp)^{-y^2} H_{2\nu}(y) dy, \quad (50)$$

using the formula mentioned in (48), then the above expression (50), we get the desired result.

### Integrals with the generalized hypergeometric functions

A generalized hypergeometric function (Rainville, 1960) may be defined by

$${}_pF_q \left[ \begin{matrix} (\varrho)_1, (\varrho)_2, \dots, (\varrho)_p; \\ (\sigma)_1, (\sigma)_2, \dots, (\sigma)_q; \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p (\varrho_i)_n}{\prod_{j=1}^q (\sigma_j)_n} \frac{z^n}{n!}, \quad (51)$$

in which no denominator parameter  $\sigma_j$  is allowed to be zero or a negative integer. If any numerator parameter  $\varrho_i$  in (51) is zero or a negative integer, the series terminates.

**THEOREM 11.** *The following integral formula holds true,*

$$\int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^\varrho(t-y)^\sigma] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^u(t-y)^v] dy = \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(zt^{u+v}) t^{\mu+\theta-1} \sum_{k=0}^{\infty} f(k) t^{(\varrho+\sigma)k} \times \mathbb{B}(\mu + uk + \varrho k, \theta + vk + \sigma k), \quad (52)$$

where

$$f(k) = \frac{(l_1)_k, \dots, (l_p)_k}{(m_1)_k, \dots, (m_q)_k} \tag{53}$$

provided

- (1)  $\Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$  and  $p, q > 0,$
- (2)  $\Re(\varrho) \geq 0, \Re(v) \geq 0$  (both are not zero simultaneously),
- (3)  $\varrho$  and  $\sigma$  are positive integers such that  $\varrho + \sigma \geq 1.$

Proof. Representing the LHS of (11) by  $I_{15}$ , we have

$$I_{15} = \int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^\varrho(t-y)^\sigma] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^u(t-y)^v] dy$$

$$I_{15} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k t^{\theta+vk-1}}{\Gamma(vk + \omega)(\delta)_{pk}} \times$$

$$\int_0^t y^{\mu+uk-1} (1-x/t)^{\theta+vk-1} {}_pF_q[(l_p); (m_q) : ay^\varrho(t-y)^\sigma] dy,$$

putting  $x = st$  and  $dx = tds$ , then we get

$$I_{15} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} (zt^{u+v})^k t^{\mu+\theta-1}}{\Gamma(vk + \omega)(\delta)_{pk}} \times$$

$$\int_0^1 (s)^{\mu+uk-1} (1-s)^{\theta+vk-1} {}_pF_q[(l_p); (m_q) : at^{\varrho+\sigma} s^\varrho (1-s)^\sigma] ds.$$

The remaining theorems could be proved in a completely analogous fashion.

**THEOREM 12.** *The following integral formula holds true,*

$$I_{16} = \int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^\varrho(t-y)^\sigma] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^{-u}(t-y)^{-v}] dy$$

$$I_{16} = \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(zt^{-u-v}) t^{\mu+\theta-1} \sum_{k=0}^{\infty} f(k) t^{(\varrho+\sigma)k} \times \mathbb{B}(\mu - uk + \varrho k, \theta - vk + \sigma k), \tag{54}$$

where  $f(k)$  is defined in (53) provided

- (1)  $\Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$  and  $p, q > 0,$
- (2)  $\Re(\varrho) \geq 0, \Re(v) \geq 0$  (both are not zero simultaneously),
- (3)  $\varrho$  and  $\sigma$  are positive integers such that  $\varrho + \sigma \geq 1.$

**THEOREM 13.** *The following integral formula holds true,*

$$I_{17} = \int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^\varrho(t-y)^\sigma] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^u(t-y)^{-v}] dy$$

$$I_{17} = \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(zt^{u-v}) t^{\mu+\theta-1} \sum_{k=0}^{\infty} f(k) t^{(\varrho+\sigma)k} \times \mathbb{B}(\mu + uk + \varrho k, \theta - vk + \sigma k),$$

(55)

where  $f(k)$  is defined in (53) provided

- (1)  $\Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$  and  $p, q > 0,$
- (2)  $\Re(\varrho) \geq 0, \Re(v) \geq 0$  (both are not zero simultaneously),
- (3)  $\varrho$  and  $\sigma$  are positive integers such that  $\varrho + \sigma \geq 1.$

**THEOREM 14.** *The following integral formula holds true,*

$$I_{18} = \int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^\varrho(t-y)^\sigma] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^{-u}(t-y)^v] dy$$

$$I_{18} = \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(zt^{-u+v}) t^{\mu+\theta-1} \sum_{k=0}^{\infty} f(k) t^{(\varrho+\sigma)k} \times \mathbb{B}(\mu - uk + \varrho k, \theta + vk + \sigma k),$$

(56)

provided

- (1)  $\Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$  and  $p, q > 0,$
- (2)  $\Re(\varrho) \geq 0, \Re(v) \geq 0$  (both are not zero simultaneously),
- (3)  $\varrho$  and  $\sigma$  are positive integers such that  $\varrho + \sigma \geq 1.$

## Conclusions

Certain new generalized integral formulae involving the Generalized Mittag-Leffler Type functions with many types of polynomials were established in this study. The results obtained here are general in nature and yield to many interesting formulae which are derived as particular cases.



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### НЕКОТОРЫЕ ИНТЕГРАЛЫ С ОБОБЩЕННЫМИ ФУНКЦИЯМИ ТИПА МИТТАГ-ЛЕФФЛЕРА

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РУБРИКА ГРНТИ: 27.23.17 Дифференциальное и  
интегральное исчисление  
27.23.21 Интегральные преобразования.  
Операционное исчисление  
27.23.25 Специальные функции

ВИД СТАТЬИ: оригинальная научная статья

#### **Резюме:**

*Введение/цель:* В данной статье установлены определенные интегралы, включающие обобщенную функцию Миттага-Леффлера с различными типами многочленов.

*Методы:* Свойства обобщенной функции Миттага-Леффлера используются в сочетании с различными видами многочленов, такими как Якоби, Лежандр и Эрмит для оценки их интегралов.

*Результаты:* Получены некоторые интегральные формулы, включающие функцию Лежандра, функцию Бесселя Мейтланда и обобщенные гипергеометрические функции.

*Выводы:* Полученные результаты носят общий характер и могут быть полезны для установления дальнейших ин-



тегральных формул, включающих другие виды многочленов.

*Ключевые слова:* функция Миттаг-Леффлера, обобщенная гипергеометрическая функция, функция Бесселя-Мейтленда, многочлены Якоби, многочлены Эрмита.

#### НЕКИ ИНТЕГРАЛИ КОЈИ УКЉУЧУЈУ ГЕНЕРАЛИЗОВАНЕ МИТАГ-ЛЕФЛЕРОВЕ ФУНКЦИЈЕ

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ОБЛАСТ: математика

КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

**Сажетак:**

*Увод/циљ:* Дефинисани су неки интегрални облици укључују генерализовану Миттаг-Лефлерову функцију са различитим врстама полинома.

*Метод:* Својства генерализоване Миттаг-Лефлерове функције користе се на различитим врстама полинома, као што су Јакобијеви, Лежандрови, Ермитови, како би одредили њихове интеграле.

*Резултати:* Изведене су неке интегралне формуле које укључују Лежандрову функцију, Бесел-Мејтландову функцију и генерализоване хипергеометријске функције.

*Закључак:* Добијени резултати су опште природе и могли би бити корисни за утврђивање других интегралних формула које укључују друге врсте полинома.

*Кључне речи:* Миттаг-Лефлерова функција, генерализована хипергеометријска функција, Бесел-Мејтландова функција, Јакобијеви полиноми, Ермитови полиноми.

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Paper received on / Дата получения работы / Датум пријема чланка: 22.08.2022.  
Manuscript corrections submitted on / Дата получения исправленной версии работы / Датум достављања исправки рукописа: 10.10.2022.

Paper accepted for publishing on / Дата окончательного согласования работы / Датум коначног прихватања чланка за објављивање: 12.10.2022.

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