Numerical methods and their application in dynamics of structures

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Abstract:

Introduction/purpose: The aim of this paper is to analyse the numerical methods for solving differential equations of dynamic equilibrium in technical problems.

Methods: The paper gives an overview of the following numerical methods: the method of central difference, the method of linear acceleration, the Newmark method, and the Wilson θ method.

Results: Various problems in applying numerical methods in dynamics of structures have been solved.

Conclusion: It has been shown that the application of numerical methods has a fundamental importance in dynamics of structures.

Key words: numerical methods, method of central difference, method of linear acceleration, Newmark method, Wilson θ method.

Introduction

Numerical methods have played a very significant role in the development of technical sciences. Today, these methods, above all, have great importance and wide application in engineering (e.g. in mechanical and civil engineering). They represent one approach to solving problems in higher mathematics with the help of computers. The main advantage of numerical methods is that the solution can be obtained even in cases where it is not possible to obtain an analytical solution. Numerical methods provide solutions that are always approximations, but mostly accurate enough from the aspect of engineering accuracy.

Numerical methods are processed in numerous papers and books, see (Hoffman, 2001; Rao, 2001; Wilson, 2001; Bathe, 2014; Subbaraj & Dokainish, 1989; Newmark, 1959; Noh & Bathe, 2019; Jin et al, 2004; Liu et al, 2018). Wilson in the ref. (2001) investigated dynamics of structures using numerical integration. The paper of Liu et al. (2018) gives an improvement of the Wilson-θ and Newmark-β methods for quasi-periodic solutions of nonlinear dynamical systems.

Various problems are solved by numerical methods (e.g. solving large systems of linear equations, solving systems of nonlinear equations, solving all types of partial differential equations, solving eigenvalues and eigenvectors, etc.).

In this paper, the emphasis is placed on applying numerical methods in dynamics of structures. Using numerical methods, it is possible to obtain the dynamic response of a structure excited due to various influences. Depending on the specific problem, excitation can be given in the form of mathematical functions or deterministic (very complex problems). Dynamic responses of structures include the following dynamic parameters: dynamic internal forces, dynamic strains, modal parameters, dynamic displacements, dynamic velocities, dynamic accelerations, and others. Problems from dynamics of structures have been solved by numerical methods in numerous papers, e.g. (Wu, 2008; Bamer et al, 2021; Esen, 2017; Tapia Andrade & Torres Berni, 2021).

Numerical methods

Many numerical methods for analysis, simulation and design of engineering processes and systems are described in (Hofman, 2001; Rao, 2001).

Equations of oscillation of a dynamic system in a closed form can only be solved in the case of linear systems and under the action of simpler forms of load which can be formulated analytically. For more complex loads which are defined as a discrete function (e.g. moving load, seismic load, and others), it is necessary to use numerical methods even in the analysis of simpler linear systems. The inclusion of nonlinearity is achieved by numerical approximation. Numerical solutions, in principle, are based on iterative methods. The most famous iterative method is Runge-Kutta.

To determine the dynamic response of a structure excited by various influences, it is necessary to solve a system of differential equations:

$$\left[\mathbf{M}\right]\left\{\ddot{\mathbf{U}}\right\}+\left[\mathbf{C}\right]\left\{\dot{\mathbf{U}}\right\}+\left[\mathbf{K}\right]\left\{\mathbf{U}\right\}=\left\{\mathbf{P(t)}\right\}. \qquad (1)$$

In Eq. (1), the notations [**M**], [**C**] and [**K**] are the matrices of consistent mass, damping and stiffness of the system. The notations {$\ddot{\mathbf{U}}$}, {$\dot{\mathbf{U}}$} and {**U**} are the acceleration, velocity and displacement vectors of the system. The notation {**P(t)**} represents the vector of external forces in the nodes of the system. For more information on structural dynamics, see (Dhatt & Touzot, 1984; Wilson, 2001; Clough & Penzien, 2015).

There are various methods for numerically determining dynamic responses of structures. In this paper, the emphasis is placed on direct integral numerical methods. To solve the problem of a dynamic response of structures, the following four direct integral methods will be presented:

- method of central difference,
- method of linear acceleration,
- Newmark method, and
- Wilson θ method.

The first two methods are explicit while the second two methods are implicit. Explicit methods generally require a small time step Δ*t*, but their solution does not require a relatively long time. On the other hand, implicit methods allow a relatively large time step Δ*t*, but their solution requires a much longer time. It is important to choose the appropriate time step Δ*t*. It affects the stability and accuracy of the solution, but also the total calculation time. Too large a step can lead to inaccurate results and an unstable response of the structure. On the other hand, too small a step can result in an unnecessarily long time to solve the problem.

Method of central difference

The central differential method is used in systems with fast and short-term oscillations and impulse loads. It is based on the approximation of differential expressions by difference. It is conditionally stable. It is excellent for small Δ*t*, while it is unstable for large Δ*t*. In order to obtain results of satisfactory accuracy, the time interval must be chosen so that it is small enough, i.e. $\Delta t<T_n/\pi$ (e.g. $\Delta t=0.1T_n$), where T_n is the oscillation period of the structure that coincides with the largest oscillation mode. More details on this method are given in (Jin et al, 2004; Wilson, 2001). The main steps of the central difference method are shown in Algorithm 1:

Algorithm 1 - Method of central difference
1: $\ddot{u}_0=\dfrac{p_o-c\dot{u}_o-ku}{m}$
2: Select Δt
3: $u_{-1}=u_0-\Delta t\dot{u}_0+\dfrac{(\Delta t)^2}{2}\ddot{u}_0$
4: $k=\dfrac{m}{(\Delta t)^2}+\dfrac{c}{2\Delta t}$

5: $a = \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}$
6: $b = k - \frac{2m}{(\Delta t)^2}$
7: $\hat{p}_i = p_i - au_{i-1} - bu_i$
8: $u_{i+1} = u_i + \Delta u_i,\ \dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i,\ \ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}_i$

Method of linear acceleration

The linear acceleration method is one of the most well-known step-by-step integration methods. This method gives excellent results with relatively little effort in computation. Its main characteristics are: the acceleration changes linearly during the interval, the damping and stiffness coefficients of the system remain constant during the time interval, the accuracy of the method depends on the size of the selected time step, and sudden changes of the stiffness and damping function must be taken into account. It is very good for small Δ*t*, while it is unstable for large Δ*t*. A detailed description of this method is given in (Bathe, 2014; Wilson, 2001). The procedure of the linear acceleration method is shown in the form of Algorithm 2:

Algorithm 2 - Method of linear acceleration
1: $\ddot{u}_0 = \frac{p_o - c\dot{u}_o - ku}{m}$
2: Select Δt
3: $\Delta\hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i$
4: $\hat{k} = k + \frac{4m}{\Delta t^2} + \frac{2c}{\Delta t}$
5: $\Delta u_i = \frac{\Delta\hat{p}_i}{k}$
6: $\Delta\dot{u}_i = \frac{2}{\Delta t}\Delta u_i - 2\dot{u}_i$
7: $\Delta\ddot{u}_i = \frac{4}{(\Delta t)^2}\left(\Delta u_i - \Delta t\dot{u}_i\right) - 2\ddot{u}_i$
8: $u_{i+1} = u_i + \Delta u_i,\ \dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i,\ \ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}_i$

Newmark method

The Newmark method of direct integrations is a special case of the linear acceleration method. It is the most well-known implicit method for solving the dynamic equilibrium equation, which involves the choice of two parameters. It is used in systems with fast and short-term oscillations. Newmark established this method in 1959 (Newmark, 1959). In this method, the parameters β and γ are introduced, defining the change of acceleration over a time interval, the stability and accuracy of the method. The method is unconditionally stable. The time step Δt is chosen more for accuracy than for stability, if the condition $\gamma \geq 0.5$ and $\beta \geq 0.25(\gamma+05)^2$. The negative characteristic of this method is reflected in the numerical extension of the period. It is described in more detail in (Bathe, 2014; Newmark, 1959; Jin et al, 2004; Liu et al, 2018; Wilson, 2001; Anahory Simoes et al, 2023; Karimi et al, 2018; Hassan, 2019). Solving system oscillation equations using the Newmark method by direct integration is shown by Algorithm 3.

Algorithm 3 - Newmark method
1: $\ddot{u}_0 = \dfrac{p_o - c\dot{u}_o - ku}{m}$
2: Select Δt
3: $\hat{k} = k + \dfrac{m}{\beta(\Delta t)^2} + \dfrac{\gamma c}{\beta \Delta t}$
4: $a = \dfrac{m}{\beta \Delta t} + \dfrac{\gamma c}{\beta}$
5: $b = \dfrac{m}{2\beta} + \Delta t\left(\dfrac{\gamma}{2\beta} - 1\right)c$
6: $\Delta \hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i$
7: $\Delta u_i = \dfrac{\Delta \hat{p}_i}{k}$
8: $\Delta \dot{u}_i = \dfrac{\gamma}{\beta \Delta t}\Delta u_i - \dfrac{\gamma}{\beta}\dot{u}_i + \Delta t\left(\dfrac{\gamma}{2\beta} - 1\right)\ddot{u}_i$
9: $\Delta \ddot{u}_i = \dfrac{1}{\beta(\Delta t)^2}\Delta u_i - \dfrac{1}{\Delta t \beta}\dot{u}_i - \dfrac{1}{2\beta}\ddot{u}_i$
10: $u_{i+1} = u_i + \Delta u_i$, $\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i$, $\ddot{u}_{i+1} = \ddot{u}_i + \Delta \ddot{u}_i$

Wilson θ method

The Wilson θ method is applied to slowly oscillating systems. The time interval is defined by the condition $0 \leq T \leq \theta \cdot \Delta t$. The accuracy of the method depends on lengthening the period and decreasing the amplitude. The method is unconditionally stable if the extended time step meets the condition $\theta \geq 1.37$ (customary: $\theta = 1.4$). In this case, there is a very slight extension of the period and a slight decrease of the amplitude. It is used in systems with slow oscillations. The negative characteristics of this method are the extension of the period and damping. It is described in detail, see (Liu et al, 2018; Wilson, 2001; Mohammadzadeh et al, 2017). In this paper, the Wilson θ method is briefly described in Algorithm 4.

Algorithm 4 - Wilson θ method
1: $m\ddot{u}_0 = p_o - c\dot{u}_o - ku$
2: Select Δt i θ
3: $k = k + \frac{6m}{\theta(\Delta t)^2} + \frac{3c}{\theta \Delta t}$
4: $\Delta\hat{p}_i = p_{i+1} + (p_{i+2} - p_{i+1})(\theta - 1) - p_i$
5: $\overline{\Delta\hat{p}_i} = \hat{p}_i + m\left(\frac{6}{\theta\Delta t}\dot{u}_i + 3\ddot{u}_i\right) + C\left(3\dot{u}_i + \frac{\theta\Delta t}{2}\ddot{u}_i\right)$
6: $\bar{k}\hat{u}_i = \overline{\Delta\hat{p}_i}$
7: $\Delta\hat{\ddot{u}}_i = \frac{6}{\theta^2 \Delta t^2}\Delta\hat{u}_i - \frac{6}{\theta\Delta t}\dot{u}_i - \frac{\theta\Delta t}{2}\ddot{u}_i$
8: $\Delta\ddot{u}_i = \frac{\Delta\hat{\ddot{u}}_i}{\theta\Delta t}$
9: $\Delta\dot{u}_i = \ddot{u}_i \Delta t - 2\dot{u}_i$
10: $\Delta u_i = \dot{u}_i \Delta t + \frac{1}{2}\ddot{u}_i \Delta t^2 + \frac{1}{6}\Delta\ddot{u}_i \Delta t^2$
11: $u_{i+1} = u_i + \Delta u_i,\ \dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i,$ $m\ddot{u}_{i+1} = p_{i+1} + C\dot{u}_{i+1} - Ku_{i+1}$

Examples of codes in structural dynamics

There are several programming environments designed to perform complex mathematical operations. On the one hand, programming is done in different programming languages (e.g. Fortran, C, C++, C#, Java, and others). On the other hand, ready-made software packages (e.g. Mathematica, Matlab, Mathcad, and others) can be used. The author of this paper selected Wolfram Mathematica for solving the problem of dynamic behaviour of complex structures. Mathematica is a software package that shapes a fully integrated computing and communication environment. It is based on symbolic problem solving. It handles both complex analytical expressions and purely numerical values equally well. For more detail on Mathematica, see (Wolfram, 2003).

In the continuation of this paper, some examples of the numerical determination of dynamic parameters of real structures are illustrated, using programs written in the Mathematica software package. The author compiled and solved all the examples. The examples are given in order, from simple problems to complex ones.

Example problem 1

The sketch and the discrete model of the single-degree system (SDOF) of a bridge crane are shown in Figure 1.

Given: m=4485 kg, L=9.2 m, l=4.6 m, I=0.00264 m^4, E=2.1·10^{11} N/m^2, t_0 = 0 s, y_0=-0.06 m, y'_0=0 m/s, and T=2 s.

Find: Response of free undamped oscillations of the crane's main girder.

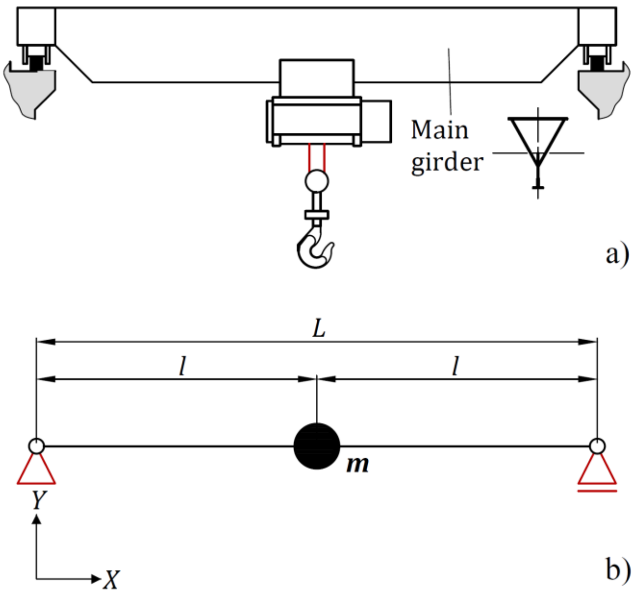

Figure 1 – a) Sketch of the bridge crane, b) Model with the concentrated mass of the bridge crane
Рис. 1 – а) Эскиз мостового крана, б) Модель мостового крана с сосредоточенными нагрузками
Слика 1 – а) Скица мосне дизалице, б) модел са концентрисаном масом мосне дизалице

Solution:

The computer code "RV-program_1" for solving free undamped oscillations of the main girder of the bridge crane was written in the program package (language) Mathematica based on the Newmark method.

```
        (*Input data*)
P[t_]:=0
m=4485;k=7.32×10^6;t0=0;q={-0.06};dq={0};T=2;
        (*Initialisation*)
ddq=LinearSolve[m,P[t0]-k.q];
Print["t=",t0,",q=",q,",dq=",dq,",ddq=",ddq]
Dt=0.01;
ut={t0};u={q};du={dq};ddu={ddq};
        (*Loops*)
Do[DP=P[t+Dt]-P[t];
DPhat=DP+m.(4 dq/Dt+2 ddq);
khat=k+4 m/dt^2;
Dq=LinearSolve[khat,DPhat];
dDq=2 Dq/Dt-2 qd;
ddDq=4(Dq-Dt dq-dt^2 ddq/2)/Dt^2;
q=q+Dq;
dq=dq+dDq;
ddq=ddq+ddDq;
ut=Append[ut,t];u=Append[u,q];
du=Append[du,dq];ddu=Append[ddu,ddq];
(*Print["t=",t,",q=",q,",dq=",dq,",ddq=",ddq]*),{t,t0+Dt,T,Dt}]
Print["Total node:",n=Length[u]]
Print["ut=",ut,",u=",u]
        (*Plot *)
ListPlot[{Table[{ut[[i]],u[[i]]},{i,n}]},PlotStyle>{Thickness[0.005],{Blue}}},
Frame>True,FrameLabel>{"t(s)","qdin(m)"},GridLines>Automatic,
Joined->True]
```

The displacement diagram of free undamped oscillations of the crane main girder is shown in Figure 2.

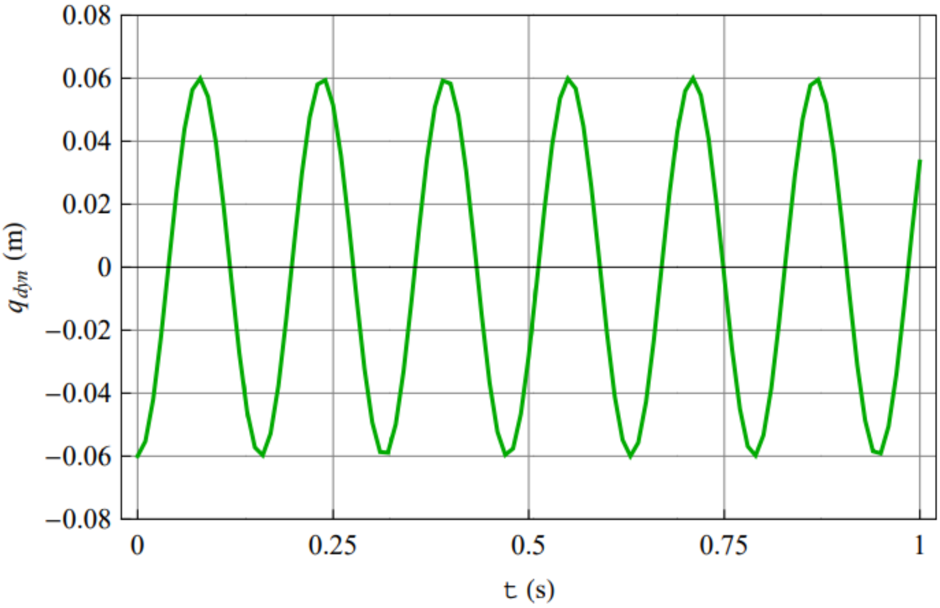

Figure 2 – Response – displacement of the middle of the bridge

Рис. 2 – Ответ - смещение середины моста

Слика 2 – Одговор – померање средине моста

Example problem 2

The sketch of the physical representation and the discrete model of the two-degree of freedom (2-DOF') system of a twin winch bridge crane are shown in Figure 3.

Given: m_1=10000 kg, m_2=10000 kg, L=16 m, a=3 m, c=9 m, I=0.00065 m^4, E=2.1·10^{11} N/m^2, t_0=0 s, y_{10}=-0.0453 m, y_{20}= -0.0582 m, y'_{10}=0 m/s, y'_{20}=0 m/s, and T=2 s.

Find: Eigenfrequencies and the response of free undamped oscillations of the crane main girder.

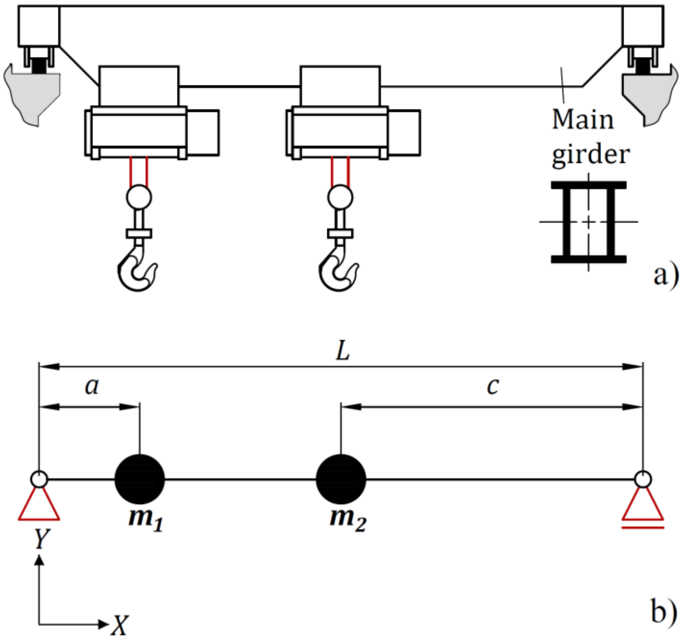

Figure 3 – a) Sketch of a bridge crane with two winches, b) Model with the concentrated masses of the carrying structure of the bridge crane
Рис. 3 – а) Эскиз мостового крана с двумя лебедками, б) Модель с сосредоточенными нагрузками несущей конструкции мостового крана
Слика 3 – а) Скица мосне дизалице са два витла, б) модел са концентрисаним масама носеће конструкције мосне дизалице

Solution:

The code "RV-program_2" for calculating the frequencies and periods of oscillations and for determining free damped oscillations of the main girder of a bridge crane with two winches was written in the Mathematica program package based on the Wilson θ method.

```
(*Input data 1*)
P[t_]:={0,0}
M={{10000,0},{0,10000}};C={{0,0},{0,0}};K=10^6{{3.64,-1.16667},
{-1.16667,1.12346}};
(*I - Frequencies and periods oscillation*)
ev=Sort[Eigenvalues[{K,M}]];
w=Sqrt[{ev}];w= w[[1]];n=2;
f=Table[w[[i]]/(2*\[Pi]),{i,n}];
T=Table[(2*\[Pi])/w[[i]],{i,n}];
Print["cfreq=",w,",cfreq=",f,",period=",T]
```

```
(*II - Response - displacements, speeds and accelerations*)
(*Input data 2*)
t0=0;q={-0.0453,-0.0582};q'10={0,0};T=2;
(*Initialisation*)
ddq=LinearSolve[M,P[t0]-K.q];
Print["t=",t0,",q=",q,",dq=",dq,",ddq=",ddq]
Dt =0.01;θ=1.4;Dτ=θ Dt;
ut={t0};u={q};du={dq};ddu={ddq};
(*Loops*)
Do[DP=P[t+Dt]+(P[t+2 Dt]−P[t+Dt])(θ−1)−P[t];
DPhat=DP+M.(6 dq/Dτ+3 ddq)+C.(3 dq+Dτ ddq/2);Khat=K+6 M/Dτ^2+3 C/Dτ;
Dq=LinearSolve[Khat,DPhat];
Dddq=(6 Dq/Dτ^2−6 dq/Dτ−3 ddq)/θ;
dDq=ddq Dt+Dddq Dt/2;
Dq=dq Dt+ddq Dt^2/2+Dddq Dt^2/6;
q=q+Dq;
dq=dq+Ddq;
ddq=LinearSolve[M,P[t]−C.dq−K.q];
ut=Append[ut,t];u=Append[u,q];
du=Append[du,dq];ddu=Append[ddu,ddq];
(*Print["t=",t,",q=",q,",dq=",dq,",ddq=",ddq]*),{t,t0+Dt,T,Dt}]
Print["Total node:",n=Length[u]]
Print["ut=",ut,",u=",u]
u1=u[[All,1]];u2=u[[All,2]];du1=du[[All,1]];du2=du[[All,2]];ddu1=ddu[[All,1]];
ddu2=ddu[[All,2]];
(*Plot*)
ListPlot[{Table[{ut[[i]],u1[[i]]},{i,n}],Table[{ut[[i]],u2[[i]]},{i,n}]},
PlotStyle->{{Thickness[0.005],Darker[Green]},{Dashed,Darker[Blue]}},
Frame->True,FrameLabel->{"t(s)","qdin(m)"},GridLines->Automatic,
Joined->True]
```

First, natural oscillation frequencies and oscillation periods of the crane main girder carrier were obtained: f=(1.30, 3.20) Hz; T=(0.77, 0.31) s.

The results of the execution of the program in the form of a diagram of the functions of the free undamped displacement of the crane's main girder, at the locations of the trolleys, are shown in Figure 4.

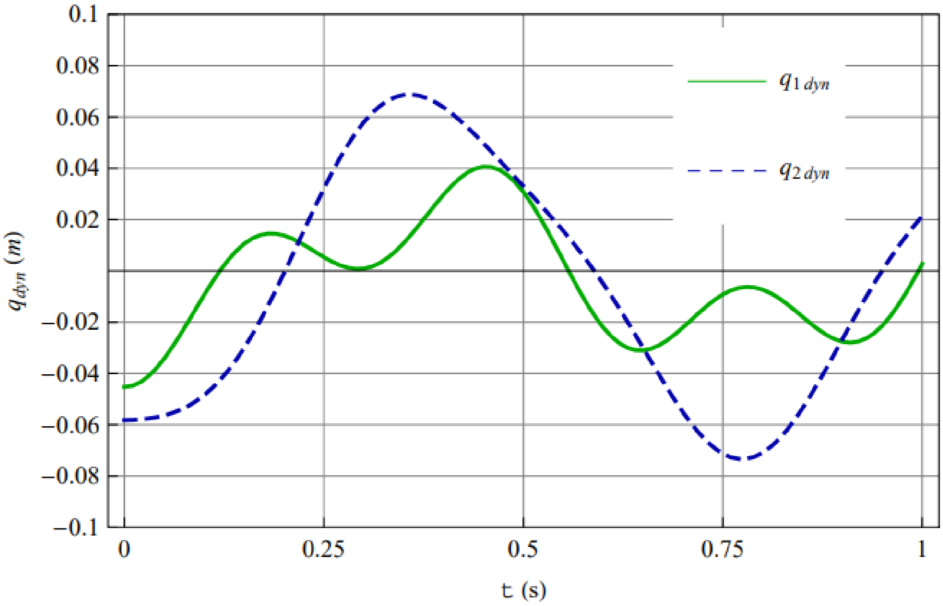

Figure 4 – Response - displacements in the locations of the trolley

Рис. 4 – Ответ - смещения в местах расположения тележки

Слика 4 – Одговор – померања на местима колица

Example problem 3

The physical representation and the discrete model with the single degree of freedom (SDOF') of the steel water tank are shown in Figure 5.

Given: m=15000 kg, k=18·10^6 N/m, c=20000 Ns/m; H=8 m, t_0 =0 s, T=1 s, and $P(t)$ – (see Figure 6).

Find: Determine the response of the vertical carrying column of the tank due to the action of the impact force $P(t)$.

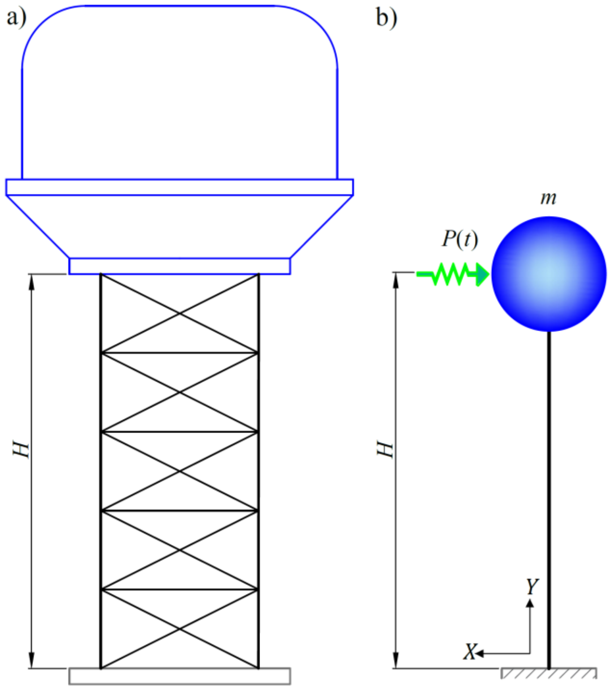

Figure 5 – a) Sketch of a steel water tank, b) Discrete model of a steel water tank - SDOF
Рис. 5 – а) Эскиз стального резервуара для воды, б) Дискретная модель стального резервуара для воды - SDOF
Слика 5 – а) Скица челичног воденог резервоара, б) дискретни модел челичног воденог резервоара – SDOF

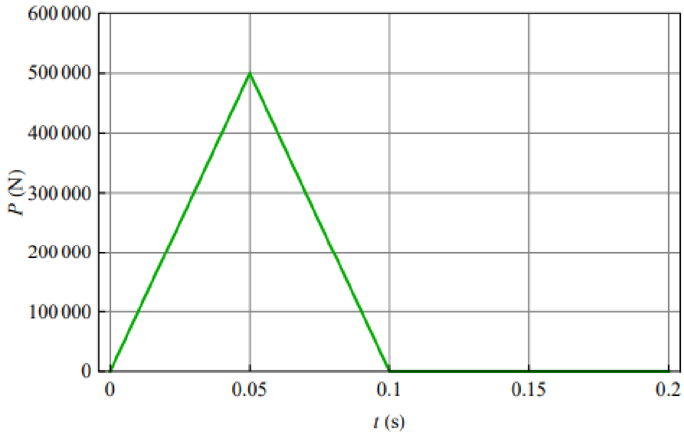

Figure 6 – Diagram of the change of the impact load in time
Рис. 6 – График изменения ударной нагрузки во времени
Слика 6 – Дијаграм промене ударног оптерећења у времену

Solution:

The code "RV-program_3" for solving the problem of oscillations of the vertical column of the tank excited by the impact force was written in the Mathematica programming language based on the Newmark method.

```
        (*Input data*)
P[t_]:=If[t<=0.05,5×10^5t/0.05,If[t<=0.1,0.15×10^5/0.05-5×10^5 t/0.05,0]]
m=15000;c=20000;k=18×10^6;t0=0;q={0};dq={0};T=0.5;
        (*Initialisation*)
ddq=LinearSolve[m,P[t0]-k.q];
Print["t=",t0,",q=",q,",dq=",dq,",ddq=",ddq]
Dt=0.01;
ut={t0};u={q};du={dq};ddu={ddq};
        (*Loops*)
Do[DP=P[t+Dt]-P[t];
DPhat=DP+m.(4 dq/Dt+2 ddq);
khat=k+4 m/Dt^2;
Dq=LinearSolve[khat,DPhat];
Ddq=2 Dq/Dt-2 qd;
Dddq=4(Dq-Dt dq-Dt^2 ddq/2)/Dt^2;
q=q+Dq;
dq=dq+Ddq;
ddq=ddq+Dddq;
ut=Append[ut,t];u=Append[u,q];du=Append[du,dq];ddu=
Append[ddu,ddq];
(*Print["t=",t,",q=",q,",dq=",dq,",ddq=",ddq]*),{t,t0+Dt,T,Dt}]
Print["Total node:",n=Length[u]]
Print["ut=",ut,",u=",u]
        (*Plot*)
ListPlot[{Table[{ut[[i]],u[[i]]},{i,n}]},PlotStyle>{Thickness[0.005],{Blue}}},Fra
me->True,FrameLabel->{"t(s)","qdin(m)"},GridLines->Automatic,Joined->
True]
```

The results in the form of a displacement diagram at the place of the concentrated mass are shown in Figure 7.

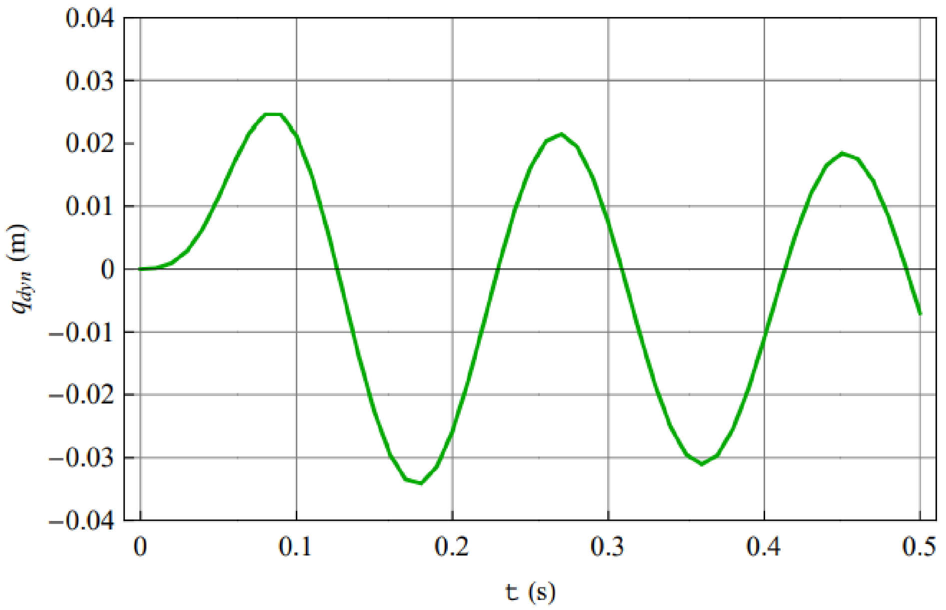

Figure 7 – Response - displacement at the tank place

Рис. 7 – Ответ - смещения в месте резервуара

Слика 7 – Одговор – померање на месту резервоара

Example problem 4

The physical description and the finite element model of the eight degree of freedom (8-DOF') overhead crane system are shown in Figure 8.

Given: m=8000 kg; L=16 m, l=2 m, A=0.0174 m^2, I=0.001723 m^4, E=2.1·10^{11} N/m^2, ρ =7850 kg/m^3; t_0=0 s, and T=10 s.

Find: Determine the response of the main girder of the bridge crane excited by the moving of the trolley.

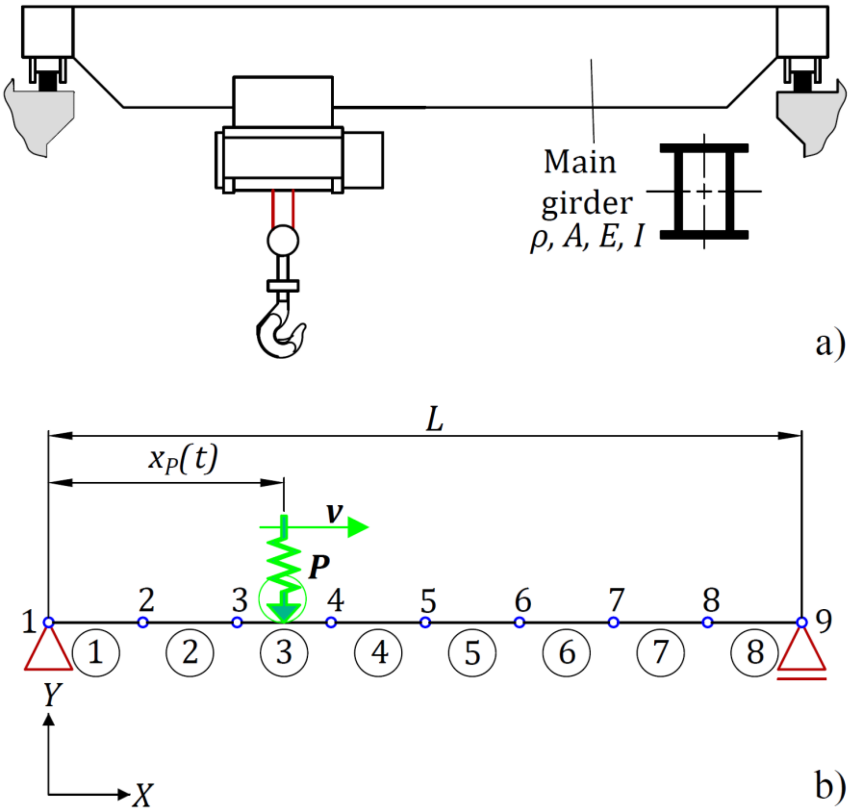

Figure 8 – a) Sketch of a bridge crane, b) Finite element model with the moving load of a bridge crane
Рис. 8 – а) Эскиз мостового крана, б) Конечно-элементная модель мостового крана с перемещением груза
Слика 8 – а) Скица мосне дизалице, б) завршноелементни модел са помичним оптерећењем мосне дизалице

Solution:

The code "RV-program_4" for solving the problem of the moving load of the bridge crane was programmed in the Mathematica programming language based on the Newmark method. Due to the scope of the program, a 6-step procedure is provided:

(i) Calculation of the stiffness [**K**] and inertia [**M**] matrices of the finite element model of the bridge crane;
(ii) Calculation of the eigenfrequencies of the FE model of the considered crane, using the algebraic equation det(K-ω^2M)=0;
(iii) Calculation of the global position of the moving load $x_p(t)$ in relation to the left end position of the main girder in the time step;

(iv) Dividing the total time domain T into n steps so that the corresponding time interval Δt is obtained;
(v) Calculation of the external force vector $\{\mathbf{P}(\mathbf{t})\}$ as a function of time t of the finite element model of the bridge crane; and
(vi) Solving equation (1) for the finite element model of the bridge crane for each time step n (n takes values from 1 to p).

In equation (1), the influence of structural damping is excluded. The code was programmed in Mathematica software based on the Newmark numerical method (Bathe, 2014).

The result in the form of a diagram of the displacement of node 5 of the main girder of the bridge crane during the moving of the trolley from the end left position to the middle of the bridge is shown in Figure 9.

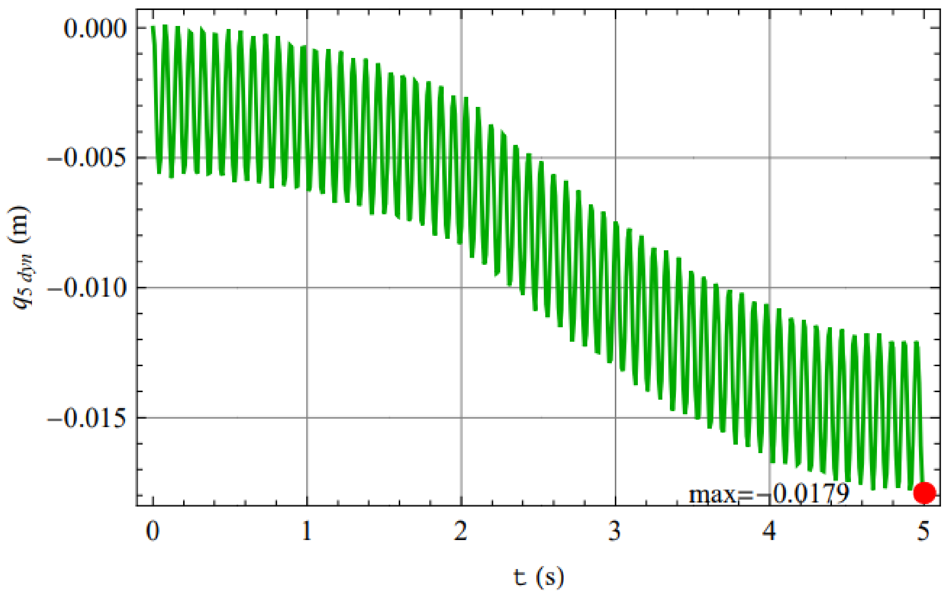

Figure 9 – Response – displacement, node 5 - q_{5dyn}
Рис. 9 – Ответ - смещение, узел 5 - $q_{5дин}$
Слика 9 – Одзив – померање, чвор 5-q_{5din}

Examples of problems in references

The dynamics of carrying structures of portal cranes using the Newmark numerical method was solved in papers (Vasiljević et al, 2016; Vasiljevic, 2020).

Conclusion

In this paper, four numerical methods with their algorithms and examples of their application in structural dynamics are briefly described. Critical comments were made on the possibilities of applying numerical methods in dynamics of structures.

Through some examples of real mechanical constructions, it is shown how to choose the size of the time step. The emphasis in practical examples is placed on the Newmark method and the Wilson θ method.

The importance of the paper is reflected in the importance of making regular decisions on choosing the most adequate numerical method in a specific situation. The paper can be useful to engineers-designers and researchers who deal with the problems of structural dynamics.

References

Anahory Simoes, A., Ferraro, S.J., Marrero, J.C. & Martín de Diego, D. 2023. A nonholonomic Newmark method. *Journal of Computational and Applied Mathematics*, 421, art.number:114873. Available at: https://doi.org/10.1016/j.cam.2022.114873.

Bamer, F., Shirafkan, N., Cao, X., Oueslati, A., Stoffel, G., Saxcé, M. & Markert, B. 2021. A Newmark space-time formulation in structural dynamics. *Computational Mechanics*, 67, pp.1331-1348. Available at: https://doi.org/10.1007/s00466-021-01989-4.

Bathe, K.-J. 2014. *Finite Element Procedures, 2nd edition.* New Jersey: Prentice-Hall. ISBN: 978-0-9790049-0-2.

Clough, R.W. & Penzien, J. 2015. *Dynamics of Structures, 2nd edition.* Berkeley, CA, USA: Computers & Structures, Inc. ISBN: 978-0923907518.

Dhatt, G. & Touzot, G.1984. *The Finite Element Method Displayed, 1st edition.* Norwich: John Wiley & Sons. ISBN: 978-0471901105.

Esen, I. 2017. A Modified FEM for Transverse and Lateral Vibration Analysis of Thin Beams Under a Mass Moving with a Variable Acceleration. *Latin American Journal of Solids and Structures*, 14(3), pp.485-511. Available at: https://doi.org/10.1590/1679-78253180.

Hassan, W.M. 2019. Numerical error assessment in nonlinear dynamic analysis of structures. *HBRC Journal*, 15(1), pp.1-31. Available at: https://doi.org/10.1080/16874048.2019.1619257.

Hoffman, J.D. 2001. *Numerical methods for Engineers and Scientists, 2nd edition.* New York, NY, USA: Marcel Dekker. ISBN: 0-8247-0443-6.

Jin, X, Ma, Q. & Li, S. 2004. Comparison of four numerical methods for calculating seismic dynamic response of sdof system. In: *13th World Conference on Earthquake Engineering*, Vancouver, B.C., Canada, Paper No. 2889, August 1-6 [online]. Available at: http://www.iitk.ac.in/nicee/wcee/article/13_2889.pdf [Accessed: 10 February 2023].

Karimi, Y., Rashahmadi, S. & Hasanzadeh, R. 2018. The Effects of Newmark Method Parameters on Errors in Dynamic Extended Finite Element Method Using Response Surface Method. *International Journal of Engineering*, 31(1), pp.50-57 [online]. Available at: https://www.ije.ir/article_73091.html [Accessed: 10 February 2023].

Liu, G., Lv, Z.R. & Chen, Y.M. 2018. Improving Wilson-θ and Newmark-β Methods for Quasi-Periodic Solutions of Nonlinear Dynamical Systems. *Journal of Applied Mathematics and Physics*, 6(8), pp.1625-1635. Available at: https://doi.org/10.4236/jamp.2018.68138.

Mohammadzadeh, S., Ghassemieh, M. & Park, Y. 2017. Structure-dependent improved Wilson-θ method with higher order of accuracy and controllable amplitude decay. *Applied Mathematical Modelling*, 52, pp.417-436. Available at: https://doi.org/10.1016/j.apm.2017.07.058.

Newmark, N.M. 1959. A Method of Computation for Structural Dynamics. *Journal of the Engineering Mechanics Division*, 85(3), pp.67-94. Available at: https://doi.org/10.1061/JMCEA3.0000098.

Noh, G. & Bathe, K.J. 2019. For direct time integrations: A comparison of the Newmark and ρ_∞-Bathe schemes. *Computers & Structures*, 225, art.number:106079, pp.1-12. Available at: https://doi.org/10.1016/j.compstruc.2019.05.015.

Rao, S.S. 2001. *Applied Numerical Methods for Engineers and Scientists, 1st edition.* Hoboken, NJ, USA: Prentice Hall. ISBN: 978-0130894809.

Subbaraj, K. & Dokainish, M.A. 1989. A survey of direct time integration methods in computational structural dynamics − Part II. Implicit methods. *Computers & Structures*, 32(6), pp.1387-1401. Available at: https://doi.org/10.1016/0045-7949(89)90315-5.

Tapia Andrade, A. & Torres Berni, W. 2021. Evaluation of the dynamic properties of a 2D-frame (MDOF) in a shake table. *Ingenius*, 26, pp.49-62. Available at: https://doi.org/10.17163/ings.n26.2021.05.

Vasiljevic, R. 2020. Transverse and longitudinal vibrations of a frame structure due to a moving trolley and the hoisted object using moving finite element. *Journal of Theoretical and Applied Mechanics*, 58(4), pp.825-839. Available at: https://doi.org/10.15632/jtam-pl/126755.

Vasiljević, R., Gašić, M. & Savković, M. 2016. Parameters Influencing the Dynamic Behaviour of the Carrying Structure of a Type H Portal Crane. *Strojniški Vestnik/Journal of Mechanical Engineering*, 62(10), pp.591-602. Available at: https://doi.org/10.5545/sv-jme.2016.3553.

Wilson, E.I. 2001. *Three-Dimensional Static and Dynamic Analysis of Structures, 3rd edition.* Berkeley, CA, USA: Computers and Structures Inc. ISBN: 978-0-923907-00-9.

Wolfram, S. 2003. *The Mathematica Book, 5th edition.* Champaign, IL, USA: Wolfram Media. ISBN: 978-1–57955–022–3.

Wu, J.-J. 2008. Transverse and longitudinal vibrations of a frame structure due to a moving trolley and the hoisted object using moving finite element. *International Journal of Mechanical Sciences*, 50(4), pp.613-625. Available at: https://doi.org/10.1016/j.ijmecsci.2008.02.001.

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Численные методы и их применение в динамике структур

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РУБРИКА ГРНТИ: 27.41.41 Алгоритмы решения задач вычислительной и дискретной математики,
30.15.27 Колебания механических систем,
30.03.19 Математические методы механики,
55.01.77 Методы исследования и моделирования. Математические и кибернетические методы
ВИД СТАТЬИ: обзорная статья

Резюме:

Введение/цель: Целью данной статьи является анализ численных методов решения дифференциальных уравнений динамического равновесия в технических задачах.

Методы: В статье приведен обзор численных методов: метод центральных разностей, метод линейного ускорения, метод Ньюмарка и θ - метод Вильсона.

Результаты: Решены различные задачи применения численных методов в динамике сооружений.

Выводы: Результаты исследования показали, что применение численных методов имеет ключевое значение в динамике сооружений.

Ключевые слова: численные методы, метод центральных разностей, метод линейного ускорения, метод Ньюмарка, θ - метод Вильсона.

Нумеричке методе и њихова примена у динамици конструкција

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КАТЕГОРИЈА (ТИП) ЧЛАНКА: прегледни рад

Сажетак:

Увод/циљ: У раду су анализиране нумеричке методе решавања диференцијалних једначина динамичке равнотеже у техничким проблемима.

Методе: Представљене су нумеричке методе: метода централних диференција, метода линеарног убрзања, Њумаркова метода и Вилсонова θ-метода.

Резултати: Решени су различити проблеми који се јављају при примени нумеричких метода у динамици конструкција.

Закључак: Показано је да примена нумеричких метода има фундаменталан значај у динамици конструкција.

Кључне речи: нумеричке методе, метода централних диференција, метода линеарног убрзања, Њумаркова метода, Вилсонова θ-метода.

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