Fixed point results for $\beta - F-$weak contraction mappings in complete S-metric spaces

Bulbul Khomdram[a], *Yumnam* Rohen[b],
Mohammad Saeed Khan[c], *Nicola* Fabiano[d]

[a] Dhanamanjuri University, D.M. College of Arts,
Department of Mathematics,
Imphal, Manipur, Republic of India,
e-mail: bulbulkhomdram@gmail.com,
ORCID iD: https://orcid.org/0000-0002-2609-838X

[b] Manipur University, Department of Mathematics,
Imphal, Manipur, Republic of India,
e-mail: ymnehor2008@yahoo.com,
ORCID iD: https://orcid.org/0000-0002-1859-4332

[c] Sefako Makgatho Health Sciences University,
Department of Mathematics and Applied Mathematics,
Ga-Rankuwa, Republic of South Africa,
e-mail: drsaeed9@gmail.com,
ORCID iD: https://orcid.org/0000-0003-0216-241X

[d] University of Belgrade, "Vinča" Institute of Nuclear Sciences - National Institute of the Republic of Serbia, Belgrade, Republic of Serbia,
e-mail: nicola.fabiano@gmail.com, **corresponding author**,
ORCID iD: https://orcid.org/0000-0003-1645-2071

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Abstract:

Introduction/purpose: This paper introduces the concept of $\beta - F$-weak contraction by using the concepts of $F-$weak contraction and $\alpha - \psi-$contraction.

Methods: The use of the $\beta - F$-weak contraction proves some fixed points theorems in the framework of $S-$metric spaces.

Results: The obtained results on fixed points in $S-$metric spaces generalize some known results in the literature.

Conclusions: The $\beta - F-$weak contraction generalizes some important contraction types and examines the existence of a fixed point in $S-$metric spaces. The results are used to solve a non-linear Fredholm integral equation.

Key words: fixed point, S-metric space, $\beta - F-$weak contraction, non-linear integral equation.

Introduction and preliminaries

It is well-known that the Banach contraction principle is regarded as one of the most important and useful results in metric fixed point theory. Because of its usefulness and simplicity, several authors generalized the Banach contraction principle in different directions. As one of the generalizations, Wardowski (Wardowksi, 2012) introduced the concept of F-contraction and proved a fixed point theorem that generalized the Banach contraction principle. The definition of F-contraction mapping is as follows:

DEFINITION 1. *(Wardowksi, 2012) Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \to \mathbb{R}$ such that*

(F1) *F is strictly increasing, that is, for all $\alpha, \beta \in (0, \infty)$ if $\alpha < \beta$ then $F(\alpha) < F(\beta)$;*

(F2) *For each sequence $\{\alpha_n\}$ of positive numbers, the following holds:*

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F3) *There exist $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+}(\alpha^k F(\alpha)) = 0$.*

Let (X, d) be a metric space. A map $T : X \to X$ is said to be an $F - contraction$ on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{1}$$

EXAMPLE 1. (Wardowksi, 2012) The following functions $F : (0, \infty) \to \mathbb{R}$ are the elements of $\mathcal{F}$:

1. $Fu = \ln u$,
2. $Fu = \ln(u^2 + u)$.

REMARK 1. (Wardowksi, 2012) From (1) and $(F1)$ it can be easily concluded that T is contractive, that is,

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Then, T is also continuous.

In 2014, Wardowski and Dung (Wardowski & Dung, 2014) extended the concept of F-contraction to F-weak contraction and obtained a variety of known contractions in the literature from it. The definition of F-weak contraction mapping is as follows:

Definition 2. *(Wardowski & Dung, 2014) Let (X, d) be a metric space. A map $T : X \to X$ is said to be an F-weak contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, the following holds:*

$$\tau + F(d(Tx, Ty)) \leq$$
$$F\Bigg(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}\Bigg).$$

For more articles related to F-contractions, see (Secelean, 2013; Dung & Hang, 2015; Piri & Kumam, 2014, 2016).

Recently, Gopal et al. (Gopal et al, 2016) extended the concept of F-contraction mappings to a weaker class of mappings called α-type F-contraction mappings and proved some results on fixed point theory. The consequences of their theorems generalized the results of Wardowski (Wardowksi, 2012), Hardy and Rogers (Hardy & Rogers, 1973), Ćirić (Ćirić, 1974). The definition of α-type F-contraction and α-type F-weak contraction mappings are as follows:

Definition 3. *(Gopal et al, 2016) Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be an α-type F-contraction on X if there exist $\tau > 0$ and two functions $F \in \mathcal{F}$ and $\alpha : X \times X \to \{-\infty\} \cup (0, \infty)$ such that for all $x, y \in X$ satisfying $d(fx, fy) > 0$, the following inequality holds*

$$\tau + \alpha(x, y)F(d(fx, fy)) \leq F(d(x, y)).$$

Definition 4. *(Gopal et al, 2016) Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be an α-type F-weak contraction on X if there exist $\tau > 0$ and two functions $F \in \mathcal{F}$ and $\alpha : X \times X \to \{-\infty\} \cup (0, \infty)$ such that for all $x, y \in X$ satisfying $d(fx, fy) > 0$, the following inequality holds*

$$\tau + \alpha(x, y)F(d(fx, fy)) \leq$$
$$F\Bigg(\max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\}\Bigg).$$

Subsequently, L.K. Dey et al. (Dey et al, 2019) introduced the notion of generalized $\alpha-F-$contraction and modified generalized $\alpha-F-$contraction mappings and presented a more generalized version of the results of Gopal et al. (Gopal et al, 2016).

Metric space and its applications have been extensively employed for decades in mathematics and various branches of applied sciences. For its effective applications and useful mathematical results, many researchers have attempted to give a more generalized and extended notion of metric space. As one of the generalizations, Sedghi et al. (Sedghi et al, 2012) introduced the concept of S-metric space as follows:

DEFINITION 5. *(Sedghi et al, 2012) Let X be a nonempty set. An S-metric on X is a function $S : X \times X \times X \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(1) $S(x, y, z) \geq 0,$

(2) $S(x, y, z) = 0$ *if and if* $x = y = z,$

(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a).$

The pair (X, S) is called S-metric space.

LEMMA 1. *(Sedghi et al, 2012) In an S-metric space, there exists $S(x, x, y) = S(y, y, x)$.*

DEFINITION 6. *(Sedghi et al, 2012) Let (X, S) be an S-metric space.*

(1) *A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and it is denoted by $\lim_{n\to\infty} x_n = x$.*

(2) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$.*

(3) *The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.*

DEFINITION 7. *(Sedghi & Dung, 2014) A mapping $T : X \to X$ is said to be S-continuous if $\{Tx_n\}$ is S-convergent to Tx, where $\{x_n\}$ is an S-convergent sequence converging to x.*

For more articles on S-metric space, see (Hieu et al, 2015; Özgür & Taş, 2016).

Definition 8. *(Alghamdi & Karapinar, 2013) Let $T : X \to X$ and $\beta : X \times X \times X \to [0, \infty)$, then T is said to be β-admissible if for all $x, y, z \in X$,*

$$\beta(x, y, z) \geq 1 \Rightarrow \beta(Tx, Ty, Tz) \geq 1.$$

Main results

In this article, $\mathfrak{F}$ denotes the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F_i) F is strictly increasing, that is, for all $u, v \in (0, \infty)$ if $u < v$ then $F(u) < F(v)$;

(F_{ii}) There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

Now, the definition of $\beta - F-$contraction and $\beta - F-$weak contraction mappings is presented as follows:

Definition 9. *Let (X, S) be an S-metric space and $h : X \to X$ be a mapping. Let $\beta : X \times X \times X \to [0, \infty)$ be a function and $F \in \mathfrak{F}$. The mapping h is said to be a $\beta - F-$ contraction on (X, S) if there exists $\tau > 0$ such that, for all $u, v \in X$ satisfying $S(hu, hu, hv) > 0$, the following condition holds:*

$$\tau + \beta(u, u, v)F(S(hu, hu, hv)) \leq F(S(u, u, v)).$$

Definition 10. *Let (X, S) be an S-metric space and $h : X \to X$ be a mapping. Let $\beta : X \times X \times X \to [0, \infty)$ be a function and $F \in \mathfrak{F}$. The mapping h is said to be a $\beta - F-$weak contraction on (X, S) if there exists $\tau > 0$ such that, for all $u, v \in X$ satisfying $S(hu, hu, hv) > 0$, the following condition holds:*

$$\tau + \beta(u, u, v)F(S(hu, hu, hv)) \leq F(M(u, u, v)), \tag{2}$$

where

$$\begin{aligned} M(u, u, v) &= \max\{S(u, u, v)), S(u, u, hu), S(v, v, hv), \\ &\quad \frac{1}{4}(S(u, u, hu) + S(u, u, hv) + S(v, v, hu))\}. \end{aligned}$$

REMARK 2. Every $\beta - F-$contraction is a $\beta - F-$weak contraction but the converse is not necessarily true.

EXAMPLE 2. Consider $X = [0,3]$ together with the S-metric $S(u,v,w) = |u-w| + |v-w|$, for all $u,v,w \in X$.

Let $h : X \to X$ be given by

$$h(u) = \begin{cases} 3, & \text{if } u \in [0,3); \\ 2, & \text{if } u = 3. \end{cases}$$

Then, for all $u,v \in [0,3]$ with $S(hu,hu,hv) > 0$ implies that either $u = 3$ or $v = 3$ but not both. So,

$$M(u,u,v) \geq S(3,3,h3) = 2.$$

Therefore, by choosing $\tau = \ln\sqrt{2}$, $F \in \mathfrak{F}$ as $Fv = \ln v$, for all $v > 0$ and $\beta : X \times X \times X \to [0,\infty)$ by

$$\beta(u,v,w) = \begin{cases} \frac{1}{2}, & \text{if } (u,v,w) \in A; \\ 2, & \text{if } (u,v,w) \in X^3 \backslash A, \end{cases}$$

where $A = \{(u,v,w) : u,v \in [0,3), w = 3 \text{ or } u,v = 3, w \in [0,3)\}$, it is clear that h is a $\beta - F-$weak contraction.

However, for $u = 3, v = 3$ and $w = \frac{5}{2}$, putting $Fv = \ln v$, for all $v > 0$, there is

$$\tau + \beta\big(3,3,\frac{5}{2}\big)F\big(S\big(h3,h3,h\frac{5}{2}\big)\big) = \tau + \beta\big(3,3,\frac{5}{2}\big)\ln 2,$$

and

$$F\bigg(S\big(3,3,\frac{5}{2}\big)\bigg) = \ln 1.$$

Clearly,

$$\tau + \beta\big(3,3,\frac{5}{2}\big)\ln 2 \nleq \ln 1$$

for every $\tau > 0$ and $\beta\big(3,3,\frac{5}{2}\big) \in [0,\infty)$. Thus, h is not a $\beta - F-$contraction.

Now, the main results are thus stated and proven.

THEOREM 1. *Let (X,S) be a complete S-metric space and $h : X \to X$ be a $\beta - F-$weak contraction satisfying the following conditions:*

(T1) h is β-admissible,

(T2) there exists $u_0 \in X$ such that $\beta(u_0, u_0, hu_0) \geq 1$,

(T3) h is S-continuous.

Then h has a fixed point.

Proof. By $(T2)$, there exists $u_0 \in X$ be such that $\beta(u_0, u_0, hu_0) \geq 1$. Define a sequence $\{u_n\}$ in X by $u_{n+1} = hu_n$ for all $n \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $u_{n_0} = hu_{n_0}$, for some $n_0 \in \mathbb{N}$, then u_{n_0} is a fixed point of h and the proof is complete. So, let us assume that $u_n \neq hu_n$ for all $n \in \mathbb{N}_0$.

From $(T1)$ and $(T2)$, it follows that

$$\beta(u_0, u_0, u_1) = \beta(u_0, u_0, hu_0) \geq 1 \Rightarrow \beta(hu_0, hu_0, hu_1) = \beta(u_1, u_1, u_2) \geq 1.$$

By induction,

$$\beta(u_n, u_n, u_{n+1}) \geq 1, \text{for all } n \in N_0.$$

Since $S(u_n, u_n, u_{n+1}) > 0$ and h is a $\beta - F-$weak contraction, for some $\tau > 0$, there exists

$$\tau + \beta(u_{n-1}, u_{n-1}, u_n)F(S(hu_{n-1}, hu_{n-1}, hu_n)) \leq F(M(u_{n-1}, u_{n-1}, u_n)), \tag{3}$$

where

$$\begin{aligned}
&M(u_{n-1}, u_{n-1}, u_n) = \max\{S(u_{n-1}, u_{n-1}, u_n), S(u_{n-1}, u_{n-1}, hu_{n-1}),\\
&S(u_n, u_n, hu_n), \frac{1}{4}(S(u_{n-1}, u_{n-1}, hu_{n-1}) + S(u_{n-1}, u_{n-1}, hu_n)\\
&+ S(u_n, u_n, hu_{n-1}))\} = \max\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1}),\\
&\frac{1}{4}(S(u_{n-1}, u_{n-1}, u_n) + S(u_{n-1}, u_{n-1}, u_{n+1}) + S(u_n, u_n, u_n))\}\\
&= \max\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1}), \frac{1}{4}(S(u_{n-1}, u_{n-1}, u_n)+\\
&2S(u_{n-1}, u_{n-1}, u_n) + S(u_n, u_n, u_{n+1}))\}
\end{aligned}$$

$$= \max\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1})\}$$

If $\max\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1})\} = S(u_n, u_n, u_{n+1})$, then (3) becomes

$$\tau + \beta(u_{n-1}, u_{n-1}, u_n)F(S(u_n, u_n, u_{n+1})) \leq F(S(u_n, u_n, u_{n+1})),$$

a contradiction. Therefore, it must be that

$$\max\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1})\} = S(u_{n-1}, u_{n-1}, u_n).$$

From (3), it follows that

$$\tau + \beta(u_{n-1}, u_{n-1}, u_n)F(S(u_n, u_n, u_{n+1})) \leq F(S(u_{n-1}, u_{n-1}, u_n)).$$

Therefore

$$\begin{aligned} F(S(u_n, u_n, u_{n+1})) &\leq \beta(u_{n-1}, u_{n-1}, u_n)F(S(u_n, u_n, u_{n+1})) \\ &\leq F(S(u_{n-1}, u_{n-1}, u_n)) - \tau \\ &< F(S(u_{n-1}, u_{n-1}, u_n)). \end{aligned} \tag{4}$$

By (F_i), it must be that

$$S(u_n, u_n, u_{n+1}) < S(u_{n-1}, u_{n-1}, u_n).$$

This shows that $\{\nu_n\}$, where $\nu_n = S(u_n, u_n, u_{n+1})$, is a decreasing sequence of non-negative real numbers, and hence

$$\lim_{n\to\infty} \nu_n = \nu \geq 0.$$

Next, it is shown that $\nu = 0$. On the contrary, it is assumed that $\nu > 0$. Then for every $n \in \mathbb{N}_0$, there exists

$$v \leq \nu_n.$$

Using (F_i) and (4), gives

$$\begin{aligned}
F(\nu) \leq F(\nu_n) &\leq F(\nu_{n-1}) - \tau \\
&\leq F(\nu_{n-2}) - 2\tau \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq F(\nu_0) - n\tau. \qquad (5)
\end{aligned}$$

Since $\lim_{n\to\infty}(F(\nu_0) - n\tau) = -\infty$, there exists $p_1 \in \mathbb{N}$ such that

$$F(\nu_0) - n\tau < F(\nu), \text{ for all } n > p_1. \qquad (6)$$

From (5) and (6), there follows

$$F(\nu) \leq F(\nu_0) - n\tau < F(\nu),$$

a contradiction. Therefore, there must be

$$\lim_{n\to\infty} \nu_n = 0.$$

By (F_{ii}), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \nu_n^k F(\nu_n) = 0. \qquad (7)$$

From (5) and (7), for all $n \in \mathbb{N}$, there follows

$$\lim_{n\to\infty} \nu_n^k (F(\nu_n) - F(\nu_0)) \leq \lim_{n\to\infty} (-\nu_n^k n\tau) \leq 0.$$

This implies that

$$\lim_{n\to\infty} (n\nu_n^k) = 0.$$

Therefore, $p_2 \in \mathbb{N}$ can be found, such that

$$\begin{aligned}
n\nu_n^k &\leq 1, \text{ for all } n \geq p_2 \\
\Rightarrow \nu_n &\leq \frac{1}{n^{\frac{1}{k}}}, \text{ for all } n \geq p_2.
\end{aligned}$$

Now,

$$
\begin{aligned}
S(u_n,u_n,u_m) &\leq 2S(u_n,u_n,u_{n+1})+S(u_{n+1},u_{n+1},u_m)\\
&\leq 2S(u_n,u_n,u_{n+1})+2S(u_{n+1},u_{n+1},u_{n+2})+...\\
&\quad ...+2S(u_{m-2},u_{m-2},u_{m-1})+S(u_{m-1},u_{m-1},u_m)\\
&\leq \sum_{q=n}^{\infty} 2S(u_q,u_q,u_{q+1})\\
&= 2\sum_{q=n}^{\infty}\nu_q\\
&\leq 2\sum_{q=n}^{\infty}\frac{1}{q^{\frac{1}{k}}}.
\end{aligned}
$$

Since $k \in (0,1)$, the series $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{k}}}$ is convergent. This implies that

$$\lim_{n,m\to\infty} S(u_n,u_n,u_m)=0.$$

This proves that $\{u_n\}$ is a Cauchy sequence. Since (X,S) is complete, there exists $\xi \in X$ such that $\lim_{n\to\infty} u_n=\xi$. Since h is S-continuous, there exists $\lim_{n\to\infty} hu_n = h\xi$.

Finally,

$$
\begin{aligned}
hu_n &= u_{n+1}\\
\Rightarrow \lim_{n\to\infty} hu_n &= \lim_{n\to\infty} u_{n+1}\\
\Rightarrow h\xi &= \xi.
\end{aligned}
$$

This proves that ξ is a fixed point of h.

□

In the following theorem, the continuity of h is replaced by the following condition:

$(\mathcal{H})$: *If* $\{u_n\}$ *is a sequence in* X *such that* $\beta(u_n,u_n,u_{n+1}) \geq 1$*, for all* $n\in\mathbb{N}_0$ *and* $u_n\to\xi$ *as* $n\to\infty$*, then* $\beta(u_n,u_n,\xi)\geq 1$*, for all* $n\in\mathbb{N}_0$

THEOREM 2. *Let* (X,S) *be a complete* S*-metric space and* $h: X\to X$ *be a* $\beta-F-$*weak contraction satisfying the following conditions:*

(T1) h *is* β*-admissible,*

(T2) *there exists* $u_0 \in X$ *such that* $\beta(u_0, u_0, hu_0) \geq 1$,

(T3) $(\mathcal{H})$ *holds,*

(T4) F *is continuous.*

Then h *has a fixed point.*

Proof. Following the proof of Theorem 1, it is known that $\{u_n\}$ defined by $u_{n+1} = hu_n$, is a Cauchy sequence with $\beta(u_n, u_n, u_{n+1}) \geq 1$, for all $n \in \mathbb{N}_0$ and it converges to some $\xi \in X$.

By (T_3), there exists

$$\beta(u_n, u_n, \xi) \geq 1, \text{ for all } n \in \mathbb{N}_0.$$

Next, it is shown that ξ is a fixed point of h. On the contrary, it is assumed that $h\xi \neq \xi$, that is, $S(\xi, \xi, h\xi) > 0$. Then, a number $m \in \mathbb{N}$ can be found, such that

$$\beta(u_n, u_n, h\xi) > 0, \text{ for all } n \geq m.$$

That is,

$$\beta(hu_{n-1}, hu_{n-1}, h\xi) > 0, \text{ for all } n \geq m.$$

Then, it is possible to find some $\tau > 0$ such that

$$\begin{aligned}\tau + F(S(u_n, u_n, h\xi)) &= \tau + F(S(hu_{n-1}, hu_{n-1}, h\xi)) \\ &\leq \tau + \beta(u_{n-1}, u_{n-1}, \xi)F(S(hu_{n-1}, hu_{n-1}, h\xi)) \\ &\leq F(M(u_{n-1}, u_{n-1}, \xi)). \end{aligned} \tag{8}$$

Now,

$$\begin{aligned}&M(u_{n-1}, u_{n-1}, \xi) = \max\{S(u_{n-1}, u_{n-1}, \xi), S(u_{n-1}, u_{n-1}, hu_{n-1}), S(\xi, \xi, h\xi), \\ &\frac{1}{4}(S(u_{n-1}, u_{n-1}, hu_{n-1}) + S(u_{n-1}, u_{n-1}, h\xi) + S(\xi, \xi, hu_{n-1}))\} \\ &= \max\{S(u_{n-1}, u_{n-1}, \xi), S(u_{n-1}, u_{n-1}, u_n), S(\xi, \xi, h\xi), \\ &\frac{1}{4}(S(u_{n-1}, u_{n-1}, u_n) + S(u_{n-1}, u_{n-1}, h\xi) + S(\xi, \xi, u_n))\}.\end{aligned}$$

Taking limit as $n \to \infty$ in (8) and using $(T4)$, yield

$$\tau + F(S(\xi, \xi, h\xi)) \leq F(S(\xi, \xi, h\xi)),$$

a contradiction. Therefore, it must be that $h\xi = \xi$, that is, ξ is a fixed point of h.

□

Next, the following condition is considered to ensure the uniqueness of the fixed point:

$(\mathcal{U})$ if $\xi, \eta \in Fix(h) = \{u \in X : hu = u\}$, then $\beta(\xi, \xi, \eta) \geq 1$.

THEOREM 3. *Adding the above condition $(\mathcal{U})$ to the hypothesis of Theorem 1 (respectively, Theorem 2) the uniqueness of the fixed point is obtained.*

Proof. Let $\xi, \eta \in Fix(h)$ with $\xi \neq \eta$. Then, $S(h\xi, h\xi, h\eta) = S(\xi, \xi, \eta) > 0$.
As a h is $\beta - F-$weak contraction, there exists $\tau > 0$ such that

$$\begin{aligned} \tau + F(S(\xi, \xi, \eta)) &= \tau + F(S(h\xi, h\xi, h\eta)) \\ &\leq \tau + \beta(\xi, \xi, \eta) F(S(h\xi, h\xi, h\eta)) \\ &\leq F(M(\xi, \xi, \eta)). \end{aligned} \tag{9}$$

Now,

$$\begin{aligned} M(\xi, \xi, \eta) &= \max\{S(\xi, \xi, \eta), S(\xi, \xi, h\xi), S(\eta, \eta, h\eta), \\ &\quad \frac{1}{4}(S(\xi, \xi, h\xi) + S(\xi, \xi, h\eta) + S(\eta, \eta, h\xi))\} \\ &= S(\xi, \xi, \eta). \end{aligned}$$

From (9), follows

$$\tau + F(S(\xi, \xi, \eta)) \leq F(S(\xi, \xi, \eta)),$$

a contradiction. Therefore, $\xi = \eta$.

□

From Remark 2, the following corollary is obtained:

Corollary 1. *Let (X, S) be a complete S-metric space and $h : X \to X$ be a $\beta - F-$contraction mapping satisfying the hypotheses of Theorem 3. Then h has a unique fixed point.*

Example 3. Consider $X = [0, 1]$ together with the S-metric $S(u, v, w) = |u - w| + |v - w|$, for all $u, v, w \in X$. Then, (X, S) is a complete S-metric space.

Let $h : X \to X$ be given by $hu = \frac{u}{10}$.

Also, let $F \in \mathfrak{F}$ as $Fv = \ln v$, for all $v > 0$.

Then, taking $\beta(u, v, w) = 1$, for all $u, v, w \in X$ and $\tau = \ln 10$ makes it clear that h is a $\beta - F-$weak contraction. Also, h satisfy all the hypotheses of Theorem 3. So, h has a unique fixed point. Clearly, $\xi = 0$ is the only fixed point of h.

Consequences

In this subsection, some known results in the literature are obtained as the consequences of these results. The examples are as follows:

(1) For all $x, y \in X$ and $0 \leq k < 1$,

$$S(Tx, Tx, Ty) \leq kS(x, x, y)$$

implies

$$\begin{aligned} S(Tx, Tx, Ty) &\leq k \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty), \\ &\quad \frac{1}{4}(S(x, x, Tx) + S(x, x, Ty) + S(y, y, Tx))\} \\ &= kM(x, x, y). \end{aligned}$$

If $S(Tx, Tx, Ty) > 0$, then

$$\tau + \ln S(Tx, Tx, Ty) \leq \ln(M(x, x, y)),$$

where $\tau = -\ln k > 0$.

Therefore, the contraction condition in Definition 2.13 of (Sedghi et al, 2012) becomes the condition (2) with $Fv = \ln v$, for all $v > 0$ and $\beta(u,v,w) = 1$, for all $u,v,w \in X$. This shows that Theorem 3 is a generalization of Theorem 3.1 of (Sedghi et al, 2012).

(2) For all $x, y \in X$ and $h \in [0,1)$,

$$S(Tx,Tx,Ty) \leq h \max\{S(Tx,Tx,x), S(Ty,Ty,y)\},$$

that is,

$$S(Tx,Tx,Ty) \leq h \max\{S(x,x,Tx), S(y,y,Ty)\}$$

implies

$$\begin{aligned} S(Tx,Tx,Ty) &\leq h \max\{S(x,x,y), S(x,x,Tx), S(y,y,Ty), \\ &\quad \frac{1}{4}(S(x,x,Tx) + S(x,x,Ty) + S(y,y,Tx))\} \\ &= hM(x,x,y). \end{aligned}$$

If $S(Tx,Tx,Ty) > 0$, then

$$\tau + \ln S(Tx,Tx,Ty) \leq \ln(M(x,x,y)),$$

where $\tau = -\ln h > 0$.

Therefore, the contraction condition in Corollary 2.10 of (Sedghi & Dung, 2014) becomes the condition (2) with $Fv = \ln v$, for all $v > 0$ and $\beta(u,v,w) = 1$, for all $u,v,w \in X$. This shows that Theorem 3 is a generalization of Corollary 2.10 of (Sedghi & Dung, 2014).

(3) For all $x, y \in X$ and $a, b, c \geq 0$ with $a + b + c < 1$,

$$S(Tx,Tx,Ty) \leq aS(x,x,y) + ab(Tx,Tx,x) + cS(Ty,Ty,y),$$

that is,

$$S(Tx,Tx,Ty) \leq aS(x,x,y) + ab(x,x,Tx) + cS(y,y,Ty)$$

implies

$$\begin{aligned} S(Tx,Tx,Ty) &\leq (a+b+c)\max\{S(x,x,y),S(x,x,Tx),S(y,y,Ty), \\ &\quad \frac{1}{4}(S(x,x,Tx)+S(x,x,Ty)+S(y,y,Tx))\} \\ &= (a+b+c)M(x,x,y). \end{aligned}$$

If $S(Tx,Tx,Ty) > 0$, then

$$\tau + \ln S(Tx,Tx,Ty) \leq \ln(M(x,x,y)),$$

where $\tau = -\ln(a+b+c) > 0$.

Therefore, the contraction condition in Corollary 2.12 of (Sedghi & Dung, 2014) becomes the condition (2) with $Fv = \ln v$, for all $v > 0$ and $\beta(u,v,w) = 1$, for all $u,v,w \in X$. This shows that Theorem 3 is a generalization of Corollary 2.12 of (Sedghi & Dung, 2014).

(4) Taking $\beta(u,v,w) = 1$ for all $u,v,w \in X$, we obtain Theorem 2.1 of (Ranjbar & Samei, 2019) from Corollary 1. Note that we are not using the condition $(F2)$ in our results.

Application

In this section, Theorem 3 is used to prove the existence and uniqueness of a solution of a non-linear Fredholm integral equation.

Let $X = (C[a,b],\mathbb{R})$ be the set of all continuous functions defined on $[a,b]$. Let the S-metric $S : X \times X \times X \to [0,\infty)$ be defined by

$$S(u,v,w) = \max_{s\in[a,b]} |u(s) - w(s)| + \max_{s\in[a,b]} |v(s) - w(s)|.$$

Then (X,S) is a complete S-metric space.

Now, the following non-linear Fredholm integral equation is considered:

$$v(t) = \varsigma(t) + \frac{1}{b-a}\int_a^b K(t,s,v(s))ds, \tag{10}$$

where $t,s \in [a,b]$. Assume that $K : [a,b]\times[a,b]\times X \to \mathbb{R}$ and $\varsigma : [a,b] \to \mathbb{R}$ are continuous.

Define the operator $T : X \to X$ by

$$Tv(t) = \varsigma(t) + \frac{1}{b-a}\int_a^b K(t,s,v(s))ds. \quad (11)$$

Note that (10) has a solution if and only if T has a fixed point.

THEOREM 4. *Let K be a continuous function satisfying*

$$\begin{aligned}|K(t,s,v(s)) - K(t,s,\varsigma(s))| \quad \leq \quad & k \max\Big\{|v(s)-\varsigma(s)|, |v(s)-Tv(s)|,\\ & |\varsigma(s)-T\varsigma(s)|, \frac{1}{4}(|v(s)-Tv(s)| +\\ & |v(s)-T\varsigma(s)| + |\varsigma(s)-Tv(s)|)\Big\},\end{aligned}$$

for all $v, \varsigma \in X$ with $v \neq \varsigma$; $s, t \in [a,b]$ and for some $k \in [0,1)$. Then the integral equation (10) has a unique solution.

Proof. Define $\beta : X \times X \times X \to [0,\infty)$ by $\beta(u,v,w) = 1$ for all $u, v, w \in X$. Then T is β-admissible. Take $F \in \mathfrak{F}$ as $Fu = \ln u$, for all $u > 0$.

Now,

$$\begin{aligned}2|Tv(t) - T\varsigma(t)| \quad = \quad & \frac{2}{b-a}\left|\int_a^b K(t,s,v(s))ds - \int_a^b K(t,s,\varsigma(s))ds\right|\\ \leq \quad & \frac{2}{b-a}\int_a^b \big|(K(t,s,v(s)) - K(t,s,\varsigma(s)))\big|ds\\ \leq \quad & \frac{2k}{b-a}\int_a^b \max\big\{|v(s)-\varsigma(s)|, |v(s)-Tv(s)|,\\ & |\varsigma(s)-T\varsigma(s)|, \frac{1}{4}(|v(s)-Tv(s)| +\\ & |v(s)-T\varsigma(s)| + |\varsigma(s)-Tv(s)|)\big\}ds.\end{aligned}$$

Taking the maximum on both sides, yields

$$\begin{aligned}& S(Tv, Tv, T\varsigma) = 2\max_{t\in[a,b]} |Tv(t) - T\varsigma(t)|\\ & \leq \frac{2k}{b-a}\max_{t\in[a,b]}\int_a^b \max\big\{|v(s)-\varsigma(s)|, |v(s)-Tv(s)|, |\varsigma(s)-T\varsigma(s)|,\\ & \frac{1}{4}(|v(s)-Tv(s)| + |v(s)-T\varsigma(s)| + |\varsigma(s)-Tv(s)|)\big\}ds\end{aligned}$$

$$\leq \frac{k}{b-a} \max\big(\max_{t\in[a,b]} \{2|\upsilon(s)-\varsigma(s)|, 2|\upsilon(s)-T\upsilon(s)|, 2|\varsigma(s)-T\varsigma(s)|,$$
$$\frac{1}{4}(2|\upsilon(s)-T\upsilon(s)|+2|\upsilon(s)-T\varsigma(s)|+2|\varsigma(s)-T\upsilon(s)|)\}\big) \int_a^b ds$$
$$= k\max\{S(\upsilon,\upsilon,\varsigma), S(\upsilon,\upsilon,T\upsilon), S(\varsigma,\varsigma,T\varsigma),$$
$$\frac{1}{4}(S(\upsilon,\upsilon,T\upsilon)+S(\upsilon,\upsilon,T\varsigma)+S(\varsigma,\varsigma,T\upsilon))\} = kM(\upsilon,\upsilon,\varsigma).$$

Taking the natural logarithm on both sides, gives

$$-\ln k + \ln S(T\upsilon, T\upsilon, T\varsigma) \leq \ln(M(\upsilon,\upsilon,\varsigma)).$$

So,

$$-\ln k + \beta(\upsilon,\upsilon,\varsigma)\ln S(T\upsilon, T\upsilon, T\varsigma) \leq \ln(M(\upsilon,\upsilon,\varsigma)).$$

Thus,

$$\tau + \beta(\upsilon,\upsilon,\varsigma)F(S(T\upsilon, T\upsilon, T\varsigma)) \leq F(M(\upsilon,\upsilon,\varsigma)),$$

where $-\ln k = \tau$.

This shows that T is a $\beta - F-$weak contraction. Thus, all the conditions of Theorem 3 are satisfied. Hence, the integral equation (10) has a unique solution.

□

Conclusions

In this paper, the concepts of $\beta - F-$contraction and $\beta - F-$weak contraction mappings are introduced and used to prove some fixed point results in the setting of S-metric space. Also, we obtain some known results in the literature as the consequences of our results. Also, some known results in the literature are obtained as the consequences of the results from this work. Finally, the obtained results are applied to prove the existence of a solution for a non-linear Fredholm integral equation.

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Resultados de punto fijo para mapeos de contracción débil $\beta - F-$ en espacios $S-$ métricos completos

Bulbul Khomdram[a], *Yumnan* Rohen[b],
Mohammad S. Khan[c], *Nicola* Fabiano[d]

[a] Universidad Dhanamanjuri, DM Facultad de Artes, Departamento de Matemáticas, Imphal, Manipur, República de la India

[b] Universidad de Manipur, Departamento de Matemáticas, Imphal, Manipur, República de la India

[c] Universidad de Ciencias de la Salud Sefako Makgatho, Departamento de Matemáticas y Matemáticas Aplicadas, Ga-Rankuwa, República de Sudáfrica

[d] Universidad de Belgrado, Instituto de Ciencias Nucleares ‘Vinča” - Instituto Nacional de la República de Serbia, Belgrado, República de Serbia, **autor de correspondencia**

CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: En este artículo presentamos el concepto de contracción $\beta - F-$débil utilizando los conceptos de contracción $F-$débil y contracción $\alpha - \psi$.

Métodos: Utilizando la contracción $\beta - F-$débil demostramos algunos teoremas de puntos fijos en el marco de espacios $S-$métricos.

Resultados: Los resultados obtenidos en puntos fijos en espacios $S-$métricos generalizan algunos resultados conocidos en la bibliografía.

Conclusión: La contracción débil $\beta - F$ generaliza algunos tipos de contracción importantes y examina la existencia de puntos fijos en espacios $S-$métricos. Los resultados se utilizan para resolver una ecuación integral de Fredholm no lineal.

Palabras claves: punto fijo, espacio S-métrico,

$\beta - F-$contracción débil, ecuación integral no lineal.

Результаты с фиксированной точкой для $\beta - F-$слабых сжимающих отображений в полных S-метрических пространствах

Бюльбюль Хомдрам[а], *Юмнам* Роэн[б],
Мохаммад С. Хан[в], *Никола* Фабиано[г]

[а] Университет Дханаманджури, Д.М. Колледж искусств, кафедра математики, г. Импхал, Манипур, Республика Индия

[б] Университет Манипура„ кафедра математики, г. Импхал, Манипур, Республика Индия

[в] Университет медицинских наук Сефако Макгато, кафедра математики и прикладной математики, г. Га-Ранкува, Республика Южная Африка

[г] Белградский университет, Институт ядерных исследований «Винча» − Институт государственного значения для Республики Сербия, г. Белград, Республика Сербия, **корреспондент**

РУБРИКА ГРНТИ: 27.25.17 Метрическая теория функций,
27.39.15 Линейные пространства, снабженные топологией, порядком и другими структурами

ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: В данной статье введено понятие $\beta - F-$ слабого сокращения, используя концепт $F-$слабого сокращения и $\alpha - \psi-$ сжатия.

Методы: С помощью $\beta - F-$слабого сжатия, доказываются некоторые теоремы о неподвижных точках в рамках $S-$метрических пространств.

Результаты: Результаты исследования о неподвижных точках в $S-$метрических пространствах обобщают некоторые известные в литературе результаты.

Выводы: $\beta - F-$слабое сжатие обобщает некоторые важные виды сокращений, исследуя существование неподвиж ной точки в $S-$метрических пространствах. Результаты статьи используются для решения нелинейного интегрального уравнения Фредгольма.

Ключевые слова: неподвижная точка, S-метрическое пространство, $\beta - F-$слабое сжатие, нелинейное интегральное уравнение.

Резултати фиксне тачке за $\beta - F-$слаба мапирања контракције у потпуним S-метричким просторима

Булбул Комдрам[а], *Јумнам* Рохен[б],
Мохамад Саед Кан[в], *Никола* Фабиано[г]

[а] Универзитет Данаманџури, Д.М. колеџ уметности, Одсек математике, Имфал, Манипур, Република Индија

[б] Универзитет Манипур, Одсек математике, Имфал, Манипур, Република Индија

[в] Универзитет здравствених наука Сефако Макгато, Департман за математику и примењену математику, Га-Ранкува, Јужноафричка Република

[г] Универзитет у Београду, Институт за нуклеарне науке "Винча" – Национални институт Републике Србије, Београд, Република Србија, **аутор за преписку**

ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: У овом раду уводи се појам $\beta - F-$слабе контракције користећи концепте $F-$слабе контракције и $\alpha - \psi-$контракције.

Методе: Коришћењем $\beta - F-$слабе контракције доказују се неке теореме о фиксним тачкама у оквиру $S-$метричких простора.

Резултати: Добијени резултати о фиксним тачкама у $S-$метричким просторима генерализују неке познате резултате у литератури.

Закључак: $\beta - F-$слаба контракција генерализује неке важне типове контракција и испитује постојање фиксне тачке у $S-$метричким просторима. Резултати се користе за решавање нелинеарне Фредхолмове интегралне једначине.

Кључне речи: фиксна тачка, $S-$метрички простор, $\beta - F-$слаба контракција, нелинеарна интегрална једначина.

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