New approach of Lebesgue integral in revised fuzzy cone metric spaces via unique coupled fixed point theorems

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Abstract:

Introduction/purpose: This article introduces the concept of revised fuzzy cone contraction by using the concept of a traiangular conorm and Revised Fuzzy Cone contractive conditions.

Methods: This article established new Revised Fuzzy Cone Contraction (RFC-C) type unique coupled Fixed Point theorems (FP theorems) in revised fuzzy cone metric spaces (RFCMS) by using the triangular property of RFCMS.

Results: The obtained results on fixed points in revised fuzzy cone metric spaces generalize some known results in the litrature and present illustrative examples to support the main work.

Conclusion: The RFC contractive conditions generalize some important contraction types and examine the existence of a fixed point in revised fuzzy cone metric spaces. In addition, the Lebesgue integral type mapping is applied to get the existence result of a unique coupled fixed point in RFCMS to validate the main work.

Key words: revised fuzzy metric, revised fuzzy cone, fixed point.

Introduction

In the year 1965, Zadeh (Zadeh, 1965) introduced the concept of fuzzy sets which permit the gradual assessment of the membership of elements in a set. To use this concept in topology, Kramosil & Michálek (1975) introduced the class of fuzzy metric spaces [FMS]. After that, George & Veeramani (1994) modified the concept of fuzzy metric spaces and defined a Hausdorff topology on this fuzzy space. After that In 2015, the notion of fuzzy cone metric space (FCM space) was introduced by Öner et al. (2015). Grabiec (1988), gave the well-known Banach contraction principle in the case of fuzzy metric spaces, in the sense of Kramosil and Michalek.

Indeed, Huang & Zhang (2007) rediscovered the idea of a Banach-valued metric space. Indeed, many mathematicians proposed it, but it became popular after Huangand Zhang's study. By adopting the theory that the underlying cone is normal, they demonstrated the convergence properties and some FP-theorems. In 2015, Öner et al. (2015) gave the idea of a fuzzy cone metric space (FCM-space), and they also presented some fundamental properties and "a single-valued Banach contraction theorem for FP with the assumption that all the sequences are Cauchy." After that, Li et al. (2021) settled some generalized fuzzy cone contractive type FP-results neglecting that "all the sequences are Cauchy" in a complete FCM-space. And later, Jabeen et al. (2020) presented some common FP theorems for three self-mappings, by taking into consideration the idea of weakly compatible in FCM-spaces with an integral type application.

Chen et al. (2020), gave the idea of coupled fuzzy cone contractive-type mappings. They proved "some coupled FP-theorems in FCM-spaces with non-linear integral type application." Latterly Rehman & Aydi (2021) presented the concept of rational type fuzzy cone contraction mappings in FCM-spaces. They used "the triangular property of fuzzy metric" as a fundamental tool and proved some common FP-theorems and give an application. Guo & Lakshmikantham (1987) proved "coupled FP results for the nonlinear operator with applications". Later, Bhaskar & Lakshmikantham (2006) present some coupled FP-theorems in the context of partially ordered metric spaces, and this work is also presented by Lakshmikantham & Ćirić (2009). The concept of a cone metric space is introduced by Huang & Zhang (2007) and they also proved FP results. Some more fixed point results in a cone metric space and Fuzzy Metric spaces can be found in (Janković et al, 2010; Javed et al, 2021;

Kadelburg et al, 2011; Karapinar, 2010; Rezapour & Hamlbarani, 2008; Shamas et al, 2021) and the references therein.

Alexander Šostak (2018) additionally represented the idea of George-Veeramani Fuzzy Metrics Revised [RFMS]. Presently, Olga Grigorenko et al. (2020) introduced "On t-conorm primarily based Fuzzy (Pseudo) metrics". In 2023, Muraliraj et al. (2023) proved some common coupled FP-results for commuting mappings in FMS. Muraliraj & Thangathamizh (2021b) introduced the concept of a Revised fuzzy modular metric space [RFMMS]. Moreover, Muraliraj & Thangathamizh (2023a) tend to prove that a Revised fuzzy cone topological space is pre-compact if and providing each sequence in it is a Cauchy subsequence. Further, we tend to show that $X_1 \times X_2$ may be a complete Revised fuzzy cone topological space if and providing X_1 and X_2 are complete Revised fuzzy cone metric areas. Finally, it is tried that each divisible Revised fuzzy cone topological space is second calculable and a mathematical space of a separable Revised fuzzy cone topological space is separable. Some more t-conorm results in various metric spaces can be found in (Kider, 2020, 2021; Muraliraj & Thangathamizh, 2021a, 2022, 2023b; Öner & Šostak, 2020; Parakath Nisha Bagam et al, 2024; Muraliraj et al, 2024) and the references therein.

This paper presents some unique coupled FP findings in RFCMS by taking the idea of Guo & Lakshmikantham (1987) and Chen et al. (2020). Furthermore, we have also presented an application of the two Lebesgue Integral Equations (LIE) for a common solution to uphold our work. This paper is organized as follows: Section 2 consists of preliminaries. Section 3 establishes some unique coupled FP-results in RFCMS with illustrative examples. Section 4 presents an application of Lebesgue integral mapping to get the existence result of unique coupled FP in RFCMS to hold up the main work. In Section 5, we discuss the conclusion of our work presenting the objectives and hypotheses of the research or intervention.

Priliemeries

Some fundamental definitions and lemmas are given in this section.

Definition 1 (George & Veeramani, 1994).

An operation $\odot : [0, 1]^2 \to [0, 1]$ is called a continuous t-conorm if

(i) $\odot$is associative, commutative, and continuous

(ii) $0 \odot q_1 = q_1$ and $q_1 \odot q_2 \leq q_3 \odot q_4$, whenever $q_1 \leq q_3$ and $q_2 \leq q_4, \forall\, q_1, q_2, q_3, q_4 \in [0, 1]$

(iii) The maximum; $q_1 \odot q_2 = max\,\{q_1, q_2\}$

(iv) The product; $q_1 \odot q_2 = q_1 + q_2 - q_1 q_2$

(vi) The Lukasiewicz; $q_1 \odot q_2 = min\{q_1 + q_2, 1\}$.

Definition 2 (Huang & Zhang, 2007**).**

Let E be a real Banach space and ϑ be the zero element of E, and p is a subset of E. Then, p is called a cone if,

(i)p is closed and nonempty, and $p \neq \{\vartheta\}$

(ii) $\alpha_1, \alpha_2 \in R, \alpha_1, \alpha_2 \geq 0$ and $\forall\ a, b \in p$, then $\alpha_1 a + \alpha_2 b \in p$

(iii) both $a \in p$ and $-a \in p$and then $a = \vartheta$

A partial ordering on a given cone $p \subset E$ is defined by $a \leq b \Leftrightarrow b - a \in p$. $a \prec b$ stands for $a \leq b$ and $a \neq b$, while $a \ll b$ stands for $b - a \in int\{p\}$. In this paper, all cones have nonempty interior.

Definition 3 (Šostak, 2018).

Let U be a set and $\odot: [0,1]^2 \to [0,1]$ is a continuous t-conorm. A RFMS, on the set U is a pair $(N_0, \odot)$ or simply N_0, where the mapping $N_0: U^2 \to [0,\ 1]$ satisfying the following conditions,

($\mathcal{RF}$ 1) $N_0(b_1, b_2, t) < 1$ and $N_0(b_1, b_2, t) = 0 \Leftrightarrow b_1 = b_2$

($\mathcal{RF}$ 2) $N_0(b_1, b_2, t) = N_0(b_2, b_1, t)$

($\mathcal{RF}$ 3) $N_0(b_1, b_2, t) \odot N_0(b_2, b_3, s) \geq N_0(b_1, b_3, t + s)$

($\mathcal{RF}$ 4) $N_0(b_1, b_2, -)$: $(0, \infty) \to 0,\ 1)$ is right continuous $\forall\ b_1, b_2, b_3 \in$ U and$t, s > 0$. Then, $(N_0, \odot)$ is said to be a RFM on $\mathfrak{M}$.

Definition 4 (Muraliraj & Thangathamizh, 2023).

A 3-tuple $(U, N_0, \odot)$ is said to be RFCMS if p is a cone of E, U is an arbitrary set, $\odot$ is a continuous t-conorm and $(N_0, \odot)$be a RFCM on $U^2 \times int\,(p)$ satisfying the following conditions; $\forall\ b_1, b_2, b_3 \in$ U and $t, s \in int(p)$.

i. $N_0(b_1, b_2, t) < 1$ and $N_0(b_1, b_2, t) = 0 \Leftrightarrow b_1 = b_2$

ii. $N_0(b_1, b_2, t) = N_0(b_2, b_1, t)$

iii. $N_0(b_1, b_2, t) \odot N_0(b_2, b_3, s) \geq N_0(b_1, b_3, t + s)$

iv. $N_0(b_1, b_2, -): int\,(p) \to [0,1]$ is continuous.

Definition 5 (Muraliraj & Thangathamizh, 2023)

Let $(U, N_0, \odot)$ is a RFCMS, $\exists\ b_1 \in$ U and $\{b_j\}$ be any sequence in U.

(i) $\{b_j\}$ converges to b_1 if for any $c \in (0,\ 1)$, t $\gg \theta$, and $\exists\ j_1 \in N$ such that $N_0(b_j, b_1, t) < c$, for j $\geq$ j_1. This can be written as $\lim_{j\to\infty} b_j = 0$, or $b_j \to b_1$ as $j \to \infty$.

(ii) (b_j) is Cauchy if for any $c \in (0,\ 1)$, t $\gg \theta$ and $\exists\ j_1 \in N$ such that $N_0(b_j, b_1, t) < c$ for $j, k \geq j_1$.

(iii) $(U, N_0, \odot)$ is complete if every Cauchy sequence is convergent in U

(iv) $\{b_j\}$ is RFC contractive if $\exists\ \alpha(0,1)$ so that

$$N_0(b_j, b_{j+1}, t) \leq \alpha\left(N_0(b_{j-1}, b_j, t)\right) \text{ for t } \gg \theta,\ \text{j} \geq 1. \quad (1)$$

Lemma 6

Let $(U, N_0, \odot)$is a RFCMS and a sequence

$b_j \to b_1 \in \mathrm{U} \iff N_0(b_j, b_1, t) \to 0$ as $j \to \infty$ for each $\mathrm{t} \gg \theta$.

Definition 7

Let $(U, N_0, \odot)$is a RFCMS. The RFCM N_0 is triangular if

$$N_0(b_1, b_3, t) \le N_0(b_1, b_2, t) + N_0(b_2, b_3, t),\ \forall\, b_1, b_2, b_3 \in U,\ t \gg \theta, \quad (2)$$

Definition 8

Let $(U, N_0, \odot)$is a RFCMS and $A: U \to U$. Then, A is called RFC contractive if there is $\alpha \in (0,1)$ so that

$$N_0(Ab_1, Ab_2, t) \le \alpha\big(N_0(b_1, b_2, t)\big),\ \forall\, b_1, b_2 \in U \text{ and } t \gg \theta, \quad (3)$$

Definition 9

Let $(\lambda_j, \lambda_2) \in U^{\,2}$. Then, it is said to be coupled FP of a mapping $A:\ U^{\,2} \to U$ if

$$A(b_2, \mathrm{b}_3) = b_2,$$

$$A(b_3, \mathrm{b}_2) = b_3. \quad (4)$$

As a follow-up to our original work, we now prove a few special pair FP theorems in RFCMS with examples. Additionally, we offer a Lebesgue integral contractive type application.

Main results

Now, the first main result is presented.

Theorem 10.

Let $A: U^{\,2} \to U$ is a mapping on complete RFCMS $(U, N_0, \odot)$ in which N_0 is triangular and satisfies the following inequality

$$N_0(A(a,b), A(\xi,\zeta), t) \le l\big(N_0(a,\xi,t)\big) + m\big(M(A(a,b), A(\xi,\zeta), t)\big) \quad (5)$$

where

$$M(A(a,b), A(\xi,\zeta), t) = \begin{Bmatrix} N_0(a, A(a,b), t) + N_0(\xi, A(\xi,\zeta), t) \\ + N_0(a, A(\xi,\zeta), t) + N_0(\xi, A(a,b), t) \end{Bmatrix} \quad (6)$$

$\forall\ a, b, \xi, \zeta \in U,\ t \ge \theta,\ l \in 0,1)$, and $m \ge 0$ with $(l + 4m) < 1$.

Then, A has a unique coupled FP in U.

Proof:

Any $a_0, b_0 \in U$; we define sequences $\{a_j\}$ and $\{b_j\}$ in U such that

$$A(a_j, b_j) = a_{j+i},\ A(b_j, a_j) = b_{j+i}, \text{ for } j \ge 0. \quad (7)$$

From (5) for $t \ge \theta$, one gets

$$N_0(a_j, a_{j+1}, t) = N_0\big(A(a_{j-1}, b_{j-1}), A(a_j, b_j), t\big)$$

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$$\leq l\left(N_0\left(a_{j-1},a_j,t\right)\right)+m\left(M\left(A\left(a_{j-1},b_{j-1}\right),A\left(a_j,b_j\right),t\right)\right), \quad (8)$$

where

$$M\left(A\left(a_{j-1},b_{j-1}\right),A\left(a_j,b_j\right),t\right)=\begin{pmatrix}N_0\left(a_{j-1},A\left(a_{j-1},b_{j-1}\right),t\right)+N_0\left(a_j,\ A\left(a_j,b_j\right),t\right)\\ +N_0\left(a_{j-1},\ A\left(a_j,b_j\right),t\right)\end{pmatrix} \quad (9)$$

$$=\left(N_0\left(a_{j-1},a_j,t\right),N_0\left(a_j,a_{j+1},t\right),N_0\left(a_{j-1},\ a_{j+1},t\right)\right)$$

$$\leq 2\left(N_0\left(a_{j-1},a_j,t\right)+N_0\left(a_j,a_{j+1},t\right)\right)$$

From (8) and (9), for $t \gg \vartheta$,

$$N_0\left(a_j,a_{j+1},t\right)\leq l\left(N_0\left(a_{j-1},a_j,t\right)\right)+2m\left(N_0\left(a_{j-1},a_j,t\right)+N_0\left(a_j,a_{j+1},t\right)\right) \quad (10)$$

After simplification, one obtains

$$N_0\left(a_j,a_{j+1},t\right)\leq\lambda\left(N_0\left(a_{j-1},a_j,t\right)\right) \text{ for } t\gg\vartheta, \quad (11)$$

where $\lambda=\left(\frac{l+2m}{(1-2m)}\right)<1$.

Similarly,

$$N_0\left(a_{j-1},a_j,t\right)\leq\lambda\left(N_0\left(a_{j-2},a_{j-1},t\right)\right) \text{ for } t\gg\vartheta. \quad (12)$$

From (11) and (12), by induction, for $t \gg \vartheta$,

$$N_0\left(a_j,a_{j+1},t\right)\leq\lambda\left(N_0\left(a_{j-1},a_j,t\right)\right)\leq\lambda^2\left(N_0\left(a_{j-1},a_j,t\right)\right)$$

$$\leq\cdots\leq\lambda^2\left(N_0\left(a_0,a_1,t\right)\right)\to 0, \text{ as } j\to\infty. \quad (13)$$

The above shows that $\{a_j\}$ be a RFC-C; therefore,

$$\lim_{j\to\infty}N_0\left(a_j,a_{j+1},t\right)=0, \text{ for } t\gg\vartheta. \quad (14)$$

Now for $i > j$ and for $t \gg \vartheta$, then

$$N_0\left(a_j+a_{j+1},t\right)\leq\left(N_0\left(a_j+a_{j+1},t\right)+N_0\left(a_{j+1}+a_{j+2},t\right)+\cdots+\lambda^2\left(N_0\left(a_{i-1}+a_i,t\right)\right)\right)$$

$$\leq\lambda^j\left(N_0\left(a_0+a_1,t\right)\right)+\lambda^{j+1}\left(N_0\left(a_0+a_1,t\right)\right)+\cdots+\lambda^{j-1}\left(N_0\left(a_0+a_1,t\right)\right)$$

$$=\left(\lambda^j+\lambda^{j+1}+\cdots+\lambda^{j-1}\right)\left(\left(N_0\left(a_0+a_1,t\right)\right)\right) \text{ as } j\to\infty.$$

$$=\frac{\lambda^j}{1-\lambda}\left(\left(N_0\left(a_0+a_1,t\right)\right)\right) \text{ as } j\to\infty. \quad (15)$$

Hence, the sequence $\{a_j\}$ is Cauchy. Now for the sequence $\{b_j\}$ and from (5), for $t \gg \vartheta$, there is

$$N_0\left(b_j,b_{j+1},t\right)=N_0\left(A\left(b_{j-1},b_{j-1}\right),A\left(b_j,a_j\right),t\right)$$

$$\leq l\left(N_0\left(b_{j-1},b_j,t\right)\right)+m\left(M\left(A\left(b_{j-1},a_{j-1}\right),\left(b_j,a_j\right),t\right)\right), \quad (16)$$

where

$$M\left(A\left(b_{j-1},a_{j-1}\right),\left(b_j,a_j\right),t\right)=\begin{pmatrix}N_0\left(b_{j-1},A\left(b_{j-1},a_{j-1}\right),t\right)+N_0\left(b_j,\ A\left(b_j,a_j\right),t\right)\\ +N_0\left(b_{j-1},\ A\left(b_j,a_j\right),t\right)\end{pmatrix}$$

$$=\left(N_0\left(b_{j-1},b_j,t\right)+N_0\left(b_j,b_{j+1},t\right)+N_0\left(b_{j-1},b_{j+1},t\right)\right)$$

$$= 2\left(N_0(b_{j-1}, b_j, t) + N_0(b_j, b_{j+1}, t)\right) \quad (17)$$

Now, from (16) and (17), for $t \gg \vartheta$,

$$N_0(b_j, b_{j+1}, t) \le l\left(N_0(b_{j-1}, b_j, t)\right) + 2m\left(N_0(b_{j-1}, b_j, t) + N_0(b_j, b_{j+1}, t)\right) \quad (18)$$

one gets, after simplification,

$N_0(b_j, b_{j+1}, t) \le \lambda\left(N_0(b_{j-1}, b_j, t)\right)$, for $t \gg \vartheta$, (19)

where $\lambda = \frac{(l+2m)}{(1-2m)} < 1$.

Similarly,

$N_0(b_{j-1}, b_j, t) \le \lambda\left(N_0(b_{j-2}, b_{j-1}, t)\right)$, for $t \gg \vartheta$. (20)

Now, from (19) and (20) and by induction, for $t \gg \vartheta$,

$$N_0(b_j, b_{j+1}, t) \le \lambda\left(N_0(b_{j-1}, b_j, t)\right) \le \lambda^2\left(N_0(b_{j-2}, b_{j-1}, t)\right)$$
$$\le \cdots \le \lambda^j\left(N_0(b_0, b_1, t)\right) \quad (21)$$

It shows that the sequence $\{b_j\}$ is a RFC-C; therefore,

$\lim_{j\to\infty} N_0(b_j, b_{j+1}, t) = 0$, for $t \gg \vartheta$. (22)

Now, for $i > j$ and for $t \gg \vartheta$, there is

$$N_0(b_j, b_{j+1}, t) \le \left(N_0(b_j, b_{j+1}, t)\right) + \left(N_0(b_{j+1}, b_{j+2}, t)\right) + \cdots + \lambda^2\left(N_0(b_{i-1}, b_i, t)\right)$$
$$\le \lambda^j\left(N_0(b_0, b_1, t)\right) + \lambda^{j+1}\left(N_0(b_0, b_1, t)\right) + \cdots + \lambda^{j-1}\left(N_0(b_0, b_1, t)\right)$$
$$= \left(\lambda^j + \lambda^{j+1} + \cdots + \lambda^{j-1}\right)\left(N_0(b_0, b_1, t)\right) \text{ as } j \to \infty.$$
$$= \frac{\lambda^j}{1-\lambda}\left(N_0(b_0, b_1, t)\right) \text{ as } j \to \infty. \quad (23)$$

Hence, the sequence $\{b_j\}$ is Cauchy. Since A is complete and $\{a_j\}$, $\{b_j\}$ are Cauchy sequences in A, so $\exists\, a, b \in A$ such that $a_j \to a$ and $b_j \to b$ as $j \to \infty$ or this can be written as $lim_{j\to\infty}\ a_j = a$ and $lim_{j\to\infty}\ b_j = b$. Therefore,

$lim_{j\to\infty}\ N_0(a_j, a, t) = 0$, $lim_{j\to\infty}\ N_0(b_j, b, t) = 0$, for $t \gg \vartheta$. (24)

Hence,

$lim_{j\to\infty}\ a_{j+1} = lim_{j\to\infty} A(a_j, b_j) = A(lim_{j\to\infty}\ a_j, lim_{j\to\infty}\ b_j) \Rightarrow A(a, b) = a$. (25)

Similarly,

$lim_{j\to\infty}\ b_{j+1} = lim_{j\to\infty} A(b_j, a_j) = A(lim_{j\to\infty}\ b_j, lim_{j\to\infty}\ a_j) \Rightarrow A(b, a) = b$.(26)

Regarding its uniqueness, suppose (a_1, b_1) and (b_1, a_1) are another coupled FP pairs in U^2 such that $A(a_1, b_1) = a_1$ and $A(b_1, a_1) = b_1$. Now, from (5), for $t \gg \vartheta$, there exists

$$N_0(a, a_1, t) = N_0(A(a, b), A(a_1, b_1), t)$$
$$\le l\left(N_0(a, a_1, t)\right) + m\left(M(A(a, b), (a_1, b_1), t)\right), \quad (27)$$

where

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$$M(A(a,b),(a_1,b_1),t) = \begin{pmatrix} N_0(a,A(a,b),t) + N_0(a_1,A(a_1,b_1),t) \\ +N_0(a,A(a_1,b_1),t) + N_0(a_1,A(a,b),t) \end{pmatrix}$$
$$= \left(N_0(a,a,t) + N_0(a_1,a_1,t) + N_0(a,a_1,t) + N_0(a_1,a,t)\right)$$
$$= 2\left(N_0(a,a_1,t)\right) . \quad (28)$$

Now, from (27) and for $t \gg \vartheta$,

$$N_0(a,a_1,t) \le l\left(N_0(a,a_1,t)\right) + 2m\left(N_0(a,a_1,t)\right) = (l+2m)\left(N_0(a,a_1,t)\right)$$
$$= (l+2m) \le (l+2m)^2\left(N_0(a,a_1,t)\right)$$
$$\le \cdots \le (l+2m)^j\left(N_0(a,a_1,t)\right) \to 0, \text{ as } j \to \infty, \quad (29)$$

where $(l+2m) < 1$.

Hence, there exists $N_0(a,a_1,t) = 0$ for $a = a_1$ and $t \gg \vartheta$.

Similarly, again from (4), $t \gg \vartheta$, there is

$$N_0(b,b_1,t) = N_0(A(b,a),A(b_1,a_1),t)$$
$$\le l\left(N_0(b,b_1,t)\right) + m\left(M(A(b,a),(b_1,a_1),t)\right), \quad (30)$$

where

$$M(A(b,a),(b_1,a_1),t) = \begin{pmatrix} N_0(b,A(b,a),t) + N_0(b_1,A(b_1,a_1),t) \\ +N_0(b,A(b_1,a_1),t) + N_0(b_1,A(b,a),t) \end{pmatrix}$$
$$= \left(N_0(b,b,t) + N_0(b_1,b_1,t) + N_0(b,b_1,t) + N_0(b_1,b,t)\right)$$
$$= 2\left(N_0(b,b_1,t)\right) \quad (31)$$

Now, from (30) and for $t \gg \vartheta$,

$$N_0(b,\mathrm{ba}_1,t) \le l\left(N_0(b,\mathrm{ba}_1,t)\right) + 2m\left(N_0(b,b_1,t)\right) = (l+2m)\left(N_0(b,b_1,t)\right)$$
$$= (l+2m) \le (l+2m)^2\left(N_0(b,b_1,t)\right)$$
$$\le \cdots \le (l+2m)^j\left(N_0(b,b_1,t)\right) \to 0, \text{ as } j \to \infty. \quad (32)$$

Hence, there exists $N_0(b,b_1,t) = 0 \; for \; b = b_1$ and $t \gg \vartheta$.

Corollary 11

Let $A : U^2 \to U$ be a mapping on complete RFCMS $(U, N_0, \odot)$ in which N_0 is triangular and satisfies

$$N_0(A(a,b),A(\xi,\zeta),t) \le \begin{Bmatrix} l\left(N_0(a,\xi,t)\right) \\ +m[N_0(a,A(a,b),t) + N_0(\xi,A(\xi,\zeta),t)] \end{Bmatrix} \quad (33)$$

$\forall\ a,b,\xi,\zeta \in U, t \ge \theta,\ l \in 0,1)$, and $m \ge 0$ with $(l+2m) < 1$. Then, A has a unique coupled FP in U.

Corollary 12

Let $A : U^2 \to U$ be a mapping on complete RFCMS $(U, N_0, \odot)$ in which N_0 is triangular and satisfies

$$N_0(A(a,b),A(\xi,\zeta),t) \le \begin{Bmatrix} l\left(N_0(a,\xi,t)\right) \\ +m[N_0(a,A(\xi,\zeta),t) + N_0(\xi,A(a,b),t)] \end{Bmatrix} \quad (34)$$

$\forall\ a,b,\xi,\zeta \in U,\ t \geq \theta,\ l \in 0,1)$, and $m \geq 0$ with $(l+2m) < 1$. Then, A has a unique coupled FP in U.

Example 13

$A=(0,\infty)$, $\odot$is a t-conorm, and $A: U^{\,2} \times (0,\infty) \to [0,1]$is defined as

$$N_0(a,b,t)=\frac{d(a,b)}{t+d(a,b)}, d(a,b)=|a-b|, \quad (35)$$

$\forall\ a,b \in U$and $t \geq \theta$. Then, it is easy to verify that N_0 is triangular and $(U, N_0, \odot)$is a complete RFCM-space. We define

$$A(g,h)=\begin{cases}\frac{a-b}{12}, if a,b \in [0,1] \\ \frac{2a+2b-2}{3}, if a,b \in [1,\infty).\end{cases} \quad (36)$$

Now from (5), for $t \geq \theta$, one obtains

$$N_0(A(a,b),A(\xi,\zeta),t)=\begin{pmatrix}\frac{1}{12}\left(N_0(a,\xi,t)\right) \\ +\frac{1}{12}\left(M(A(a,b),A(\xi,\zeta),t)\right)\end{pmatrix}, \text{for } t \geq \theta. \quad (37)$$

It is easy to verify that conditions of Theorem 10 are satisfied with $l=m=\frac{1}{12}$. Then, A has unique coupled FP for $a=2$ and $b=2$.

$$A(a,b)=A(2,2)=\frac{2(2)+2(2)-2}{3}=2 \Rightarrow A(2,2)=2. \quad (38)$$

Theorem 14

Let $A: U^{\,2} \to U$ be a mapping on complete RFCMS $(U, N_0, \odot)$ in which N_0 is triangular and satisfies the inequality

$$N_0(A(a,b),A(\xi,\ \zeta),t) \leq \begin{Bmatrix} l\left(N_0(a,\xi,t)\right) \\ +m\left(N_0(a,A(a,b),t)+N_0(\xi,A(\xi,\ \zeta),t)\right) \\ +n\left(N_0(\xi,A(a,b),t) \odot N_0(\xi,A(\xi,\ \zeta),t)\right)\end{Bmatrix} \quad (39)$$

$\forall\ a,b,\xi,\zeta \in U,\ t \geq \theta,\ l \in 0,1)$, and $m,n \geq 0$ with $(l+2m,n) < 1$. Then, A has a unique coupled FP in U.

Corollary 15

Let $A: U^{\,2} \to U$ be a mapping on complete RFCMS $(U, N_0, \odot)$ in which N_0 is triangular and satisfies the inequality

$$N_0(A(a,b),A(\xi,\ \zeta),t) \leq \begin{Bmatrix} l\left(N_0(a,\xi,t)\right) \\ +n\left(N_0(\xi,A(a,b),t) \odot N_0(\xi,A(\xi,\ \zeta),t)\right)\end{Bmatrix} \quad (40)$$

$\forall\ a,b,\xi,\zeta \in U,\ t \geq \theta,\ l \in 0,1)$, and $m,n \geq 0$ with $(l+2m,n) < 1$. Then, A has a unique coupled FP in U.

Example 16

$A=(0,\infty)$, $\odot$ is a t-conorm, and $A: U^{\,2} \times (0,\infty) \to [0,1]$ is defined as

$N_0(a,b,t) = \frac{d(a,b)}{t+d(a,b)}, d(a,b) = |a-b|,$ (41)

$\forall\ a,b \in U$and $t \geq \theta$. Then, it is easy to verify that N_0 is triangular and $(U, N_0, \odot)$is a complete RFCM-space. We define

$$A(g,h) = \begin{cases} \frac{a-b}{8}, if a,b \in [0,1] \\ \frac{2a+2b-3}{3}, if a,b \in [1,\infty). \end{cases} \quad (42)$$

Now from (39), for $t \geq \theta$, one obtains, for $t \geq \theta$,

$$N_0(A(a,b), A(\xi,\zeta), t) = \begin{pmatrix} \frac{1}{12}\left(N_0(a,\xi,t)\right) \\ +\frac{1}{8}\left(N_0(a, A(a,b), t) + N_0(\xi, A(\xi,\zeta), t)\right) \end{pmatrix}. \quad (43)$$

It is easy to verify that conditions of Theorem 13 are satisfied with $l = m = \frac{1}{8}$ and $n = 0$.

Then, A has unique coupled FP for $a = 2$ and $b = 2$.

$A(a,b) = A(3,3) = \frac{2(3)+2(3)-2}{3} = 2 \Rightarrow A(3,3) = 3.$ (44)

Application

In this section, we present an application of Lebesgue integral (LI) mapping to support our main work. In 2002, Branciari (Branciari, 2002) proved the following result on a complete metric space for a unique FP.

Theorem 17

Let (U,d) be a complete metric space, $\alpha \in (0,1)$, and $A: U \to U$ a mapping such that $\forall\ a,b \in U$,

$$\int_0^{d(Aa,Ab)} \varphi(s)\,ds \leq \alpha \int_0^{d(a,b)} \varphi(s)\,ds\,, \quad (45)$$

where $\varphi: (0,\infty) \to (0,\infty)$ is a Lebesgue integrable mapping which is summable (i.e., with finite integral on each compact subset of $(0,\infty)$ and for each $\tau > 0$,

$$\int_0^{d(a,b)} \varphi(s)\,ds\,. \quad (46)$$

Then, A has a unique FP $u \in U$ such that for any $a \in U$, $\lim_{j\to\infty} A_a^j = u$. Now, we are in the position to use the above concept and to prove a unique coupled FP-theorem in FCM-spaces.

Theorem 18

Let $A : U^2 \to U$ be a mapping on complete RFCMS $(U, N_0, \odot)$ in which N_0 is triangular and satisfies the inequality

$$\int_0 \varphi(s)\,ds \le l \int_0 \varphi(s)\,ds + m \int_0 \varphi(s)\,ds$$

$$M(A(a,b), A(\xi,\zeta), t) = \begin{Bmatrix} N_0(a, A(a,b), t) + N_0(\xi, A(\xi,\zeta), t) \\ +N_0(a, A(\xi,\zeta), t) + N_0(\xi, A(a,b), t) \end{Bmatrix} \tag{47}$$

$\forall\ a, b, \xi, \zeta \in U$, $t \ge \theta$, $l \in 0,1)$, and $m \ge 0$ with $(l + 4m) < 1$ and where $\varphi: 0,\infty) \to 0,\infty)$ is a Lebesgue integrable mapping which is summable (i.e., with finite integral on each compact subset of $0,\infty)$ and for each $\tau > 0$,

$$\int_0^{d(a,b)} \varphi(s)\,ds \tag{48}$$

Then, A has a unique coupled FP in U.

Conclusion

This article introduced the idea of coupled FP-results in RFCMS and used "the triangular property of RFCMS" to demonstrate certain special coupled FPT under the revised contractive type requirements. Some examples are provided that supported our conclusions as well. Furthermore, a Lebesgue integral mapping application is provided to enhance our primary findings. With the aid of this novel idea, it is possible to demonstrate more modified and universal contractive type coupled FP results with various integral contractive type of conditions and applications throughout the whole RFCMS.

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Nuevo enfoque de la integral de Llebesgue en espacios métricos de conos difusos revisados mediante teoremas de punto fijo acoplados únicos

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CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: Este artículo presenta el concepto de contracción revisada del cono difuso utilizando el concepto de conorma triangular y las condiciones contractivas revisadas del cono difuso.

Métodos: Este artículo estableció nuevos teoremas de punto fijo acoplados únicos (teoremas FP) del tipo de contracción difusa revisada (RFC-C) en espacios métricos de cono difuso revisados (RFCMS) mediante el uso de la propiedad triangular de RFCMS.

Resultados: Los resultados obtenidos en puntos fijos en espacios métricos de cono difuso revisados generalizan algunos resultados conocidos en la literatura y presentan ejemplos ilustrativos para respaldar el trabajo principal.

Conclusión: Las condiciones contractivas de RFC generalizan algunos tipos de contracción importantes y examinan la existencia de un punto fijo en espacios métricos de cono difuso revisados. Además, se aplica el mapeo de tipo integral de Lebesgue para obtener el resultado de existencia de un punto fijo acoplado único en RFCMS para validar el trabajo principal.

Palabras claves: métrica difusa revisada, cono difuso revisado, punto fijo.

Новый подход к интегралу Лебега в пересмотренных нечетких конусообразных метрических пространствах с помощью единой связанной неподвижной точки

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РУБРИКА ГРНТИ: 27.25.17 Метрическая теория функций,
27.39.15 Линейные пространства, снабженные топологией, порядком и другими структурами
ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: В данной статье представлена концепция пересмотренного сокращения нечеткого конуса с использованием концепции треугольника и пересмотренных условий сокращения нечеткого конуса.

Методы: В статье представлены новые пересмотренные теоремы об единой связанной неподвижной точке с типом сжатия нечеткого конуса (RFC-C) (теоремы FP) в пересмотренных метрических пространствах нечеткого конуса (RFCMS), используя свойство треугольности RFCMS.

Результаты: Полученные результаты по неподвижным точкам в пересмотренных нечетких конусообразных метрических пространствах обобщают некоторые известные результаты в литературе и представляют иллюстративные примеры в поддержку основной работы.

Выводы: Условия сжатия RFC обобщают некоторые важные типы сжатия и исследуют существование неподвижной точки в пересмотренных нечетких конусообразных метрических пространствах. Помимо того, отображение интегрального типа Лебега применяется для получения результата существования единой связанной фиксированной точки в RFCMS для подтверждения основной работы.

Ключевые слова: пересмотренная нечеткая метрика, пересмотренный нечеткий конус, неподвижная точка.

Нови приступ Лебеговом интегралу у прерађеним фази конусним метричким просторима помоћу теорема јединствене спрегнуте непокретне тачке

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ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Овај рад уводи појам прерађене фазне конусне контракције помоћу концепта троугаоне конорме и прерађеног фазног конусног контрактивног услова.

Методе: Представљене су нове теореме јединствене спрегнуте непокретне тачке типа RFC-C (revised fuzzy cone contraction – преређене фази конусне контракције) у прерађеним фази конусним метричким просторима (RFCMS – revised fuzzy cone metric spaces) коришћењем својства троугла које поседују RFCMS.

Резултати: Добијени резултати на непокретним тачкама у прерађеним фази конусним метричким просторима генерализују неке познате резултате из литературе и представљају илустративне примере који подржавају основу овог рада.

Закључак: Контрактивни услови RFC генерализују неке важне типове контракција и испитују постојање непокретне тачке у прерађеним фази конусним метричким просторима. Примењено је и пресликавање типа Лебеговог интеграла за добијање резултата

јединствене спрегнуте непокретне тачке у RFCMS за валидацију овог рада.

Кључне речи: прерађена фази метрика, прерађен фази конус, непокретна тачка.

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