Relating Sombor and Euler indices

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DOI: https://doi.org/10.5937/vojtehg72-48818

FIELD: mathematics (mathematics subject classification: primary 05c07, secondary 05c09)
ARTICLE TYPE: original scientific paper

Abstract:

Introduction/purpose: The Euler-Sombor index (EU) is a new vertex-degree-based graph invariant, obtained by geometric consideration. It is closely related to the Sombor index (SO). The actual form of this relation is established.

Methods: Combinatorial graph theory is applied.

Results: The inequalities between EU and SO are established.

Conclusion: The paper contributes to the theory of Sombor-index-like graph invariants.

Keywords: degree(of vertex), Sombor index, Euler-Sombor index.

Introduction

Vertex-degree-based (VDB) graph invariants are much studied in the current mathematical and applied-mathematical literature; see for instance the recent papers (Das et al, 2021; Hu et al, 2022; Liu, 2023a; Monsalve & Rada, 2021; Rada et al, 2022; Yuan, 2024). A few years ago, it was discovered that some of these graph invariants have a geometric interpretation (Gutman, 2021). Eventually, this triggered a whole series of geometry-based research studies on VDB invariants (Ali et al, 2024; Gutman, 2022; Gutman et al, 2024; Imran et al, 2022; Liu, 2023b; Tang et al, 2024). The first geometry-motivated VDB invariant is the Sombor index (Gutman, 2021), defined as

$$SO = SO(G) = \sum_{uv \in E(G)} \sqrt{{d_u}^2 + {d_v}^2}\ . \tag{1}$$

Gutman, I., Relating Sombor and Euler indices, pp.1-12

Although relatively new, the Sombor index has been a subject of numerous mathematical studies; see the review (Liu et al, 2022), the most recent papers (Attarzadeh & Behtoei, 2024; Chen & Zhu, 2024; Selenge & Horoldagva, 2024; Shetty & Bhat, 2024), and the references cited therein. The Sombor index found also noteworthy applications, especially in chemistry (Hayat et al, 2024; Rauf & Ahmad, 2024; Redžepović, 2021).

In some recent studies, a similarly-looking quantity has been encountered (Ali et al, 2024, Gutman et al, 2024, Tang et al, 2024), namely

$$EU = EU(G) = \sum_{uv \in E(G)} \sqrt{{d_u}^2 + {d_v}^2 + d_u\, d_v}\ . \qquad (2)$$

For the reasons explained below, it can be named the “Euler-Sombor index“.

In this paper, we use the following notation and terminology. By *G* we denote a simple graph with *n* vertices and *m* edges. Let *E*(*G*) be its edge sets, and then $|E(G)| = m$. The edge of the graph *G*, connecting the vertices *u* and *v* is denoted by *uv*. The degree d_u of a vertex *u* is the number of the first neighbors of this vertex.

For additional details of graph theory, see (Harary, 1969; Bondy & Murty, 1976).

A geometric approach to VDB invariants

The general form of a VDB graph invariant is $\sum_{uv \in E(G)} f(d_u, d_v)$, where *f* is a pertinently chosen function with the property *f(x,y)=f(y,x)*. In (Gutman, 2021), it was recognized that the vertex-degree pair (d_u, d_v) can be interpreted as a point in a 2-dimensional coordinate system, representing the edge *uv*, called the *degree-point* of the edge *uv* (point *A* in Figure 1). If so, then (d_v, d_u) would be the *dual degree point*, pertaining to the same edge *uv* (point *B* in Figure 1).

The (Eucldean) distance between the degree-point (d_u, d_v) and the origin *O* is $\sqrt{{d_u}^2 + {d_v}^2}$, which then directly leads to the concept of the Sombor index, Eq. (1).

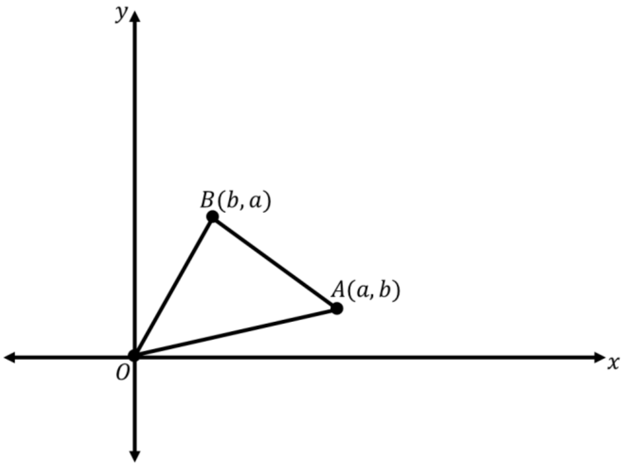

Figure 1 – A geometric representation of the edge uv of a graph G. Here d_u=a and d_v=b. The distance between the origin O and either the degree-point A or the dual degree-point B leads to the Sombor index, Eq. (1). The distance between the points A and B pertains to the Albertson irregularity index, see (Gutman, 2021).

Recently, in (Gutman et al, 2024), a geometric model was proposed, in which the degree-point and the dual degree-point play equivalent roles: these are set to be the two foci of an ellipse passing through the origin (see Figure 2).

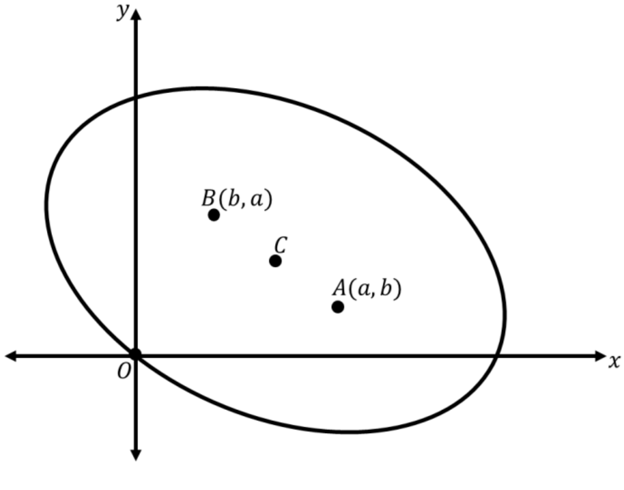

Figure 2 – Ellipse whose foci are the degree-point A and the dual degree-point B of the edge uv of a graph G.The point C is the center of the ellipse.

In (Gutman et al, 2024), it was shown that the lengths of the semi-major and the semi-minor axes of the ellipse in Figure 2 are

$$r_1 = \sqrt{a^2 + b^2} = \sqrt{d_u^{\,2} + d_v^{\,2}} \quad \text{and} \quad r_2 = a + b = d_u + d_v \,. \tag{3}$$

Using formulas (3), the area of the ellipse, equal to $\pi\sqrt{r_1 r_2}$, can easily be calculated and related to a VDB graph invariant. On the other hand, the calculation of the perimeter of the ellipse is a difficult task and (because of its importance in astronomy) a large number of various approximations have been proposed; for details, see (Gutman et al, 2024). The approximate formula for the perimeter of an ellipse, proposed by Leonhard Euler (Euler, 1773), is $\pi\sqrt{2(r_1^{\,2} + r_2^{\,2})}$. When relations (3) are substituted into this formula, and the multiplier abandoned (since it is irrelevant for the present considerations), we arrive at the expression

$$\sqrt{d_u^{\,2} + d_v^{\,2} + d_u\, d_v} \,.$$

This expression directly leads to the VDB graph invariant (2). Because of its origin, the name *Euler-Sombor index* for it would be appropriate.

Evidently, there is a close algebraic analogy between the Sombor index, Eq. (1), and the Euler-Sombor index, Eq. (2). In what follows, we determine the actual form of the relation between these two VDB graph invariants.

Estimating *SO* by means of *EU*

From now on, in order to avoid trivialities, we restrict the consideration to connected graphs. The results obtained could then be directly extended to disconnected graphs, taking into account that, for a graph *G* consisting of disconnected components G_1 and G_2,

$$SO(G) = SO(G_1) + SO(G_2) \quad \text{and} \quad EU(G) = EU(G_1) + EU(G_2)\,.$$

Theorem 1. Let *G* be a connected graph. Then

$$\sqrt{\frac{2}{3}}\, EU(G) \le SO(G) < EU(G)\,. \tag{4}$$

The equality on the left-hand side is attained if and only if the graph *G* is regular.

Proof. It suffices to verify that the relations

$$\sqrt{\frac{2}{3}}\sqrt{x^2+y^2+xy} \le \sqrt{x^2+y^2} \le \sqrt{x^2+y^2+xy} \qquad (5)$$

hold for all $x \ge 0$ and $y \ge 0$.

The right-hand side inequality in (5) is obvious. The equality in it occurs if and only if either x=0 or y=0 (or both), which in the case of vertex degrees of connected graphs cannot happen.

In order to obtain the left-hand side inequality in (5), we seek λ satisfying

$$\lambda\sqrt{x^2+y^2+xy} \le \sqrt{x^2+y^2}\ . \qquad (6)$$

From (6), we get

$$\lambda^2 xy \le (1-\lambda^2)x^2 + (1-\lambda^2)y^2$$

$$\lambda^2 xy - 2(1-\lambda^2)xy \le (1-\lambda^2)x^2 + (1-\lambda^2)y^2 - 2(1-\lambda^2)xy$$

$$\left[\lambda^2 - 2(1-\lambda^2)\right]xy \le (1-\lambda^2)(x-y)^2 .$$

Assuming that $1-\lambda^2>0$, we conclude that it must be

$$\lambda^2 - 2(1-\lambda^2) \ge 0 \quad \text{i.e.,} \quad \lambda \ge \sqrt{2/3}\ .$$

Then the best choice for relation (5) is the smallest value of λ, i.e., $\lambda = \sqrt{2/3}$.

From the above consideration, it is seen that the equality will hold if and only if x=y. Applying this to graphs, the equality will hold if the end-vertices of the edge *uv* have equal degrees. If this must be valid for all edges of the (connected) graph *G*, then all vertices of *G* must have equal degrees, i.e., then *G* must be regular.

This completes the proof of Theorem 1. ■

Inequalities (4) provide lower and upper bounds for the Sombor index in terms of the Euler-Sombor index. Of course, these relations can be inverted, so that *EU* is estimated by means of *SO*:

$$SO(G) < EU(G) \leq \sqrt{\frac{3}{2}}\, SO(G)\,.$$

Improving the inequality *SO* < *EU*

In this section, if the end-vertices of an edge uv have the degrees d_u=i and d_v=j (or vice versa), we say that the edge uv is of the (i,j)-type.

The right-hand side inequality (4) in Theorem 1 is strict. Therefore, it would be of interest to modify it so as to get equality for some graphs. In order to achieve this goal, we first establish the following:

Lemma 1. For $x, y \geq 0$, the function

$$F(x,y) = \sqrt{x^2 + y^2 + xy} - \sqrt{x^2 + y^2}$$

is monotonically increasing. More precisely, in the domain $D = \{(x,y) \in R^2 \mid x \geq 0, y \geq 0\}$, for any $(x_1, y_1), (x_2, y_2) \in D$, if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $F(x_1, y_1) \geq F(x_2, y_2)$ holds.

Proof. We need to show that the right-hand side of

$$\frac{\partial F(x,y)}{\partial x} = \frac{2x+y}{2\sqrt{x^2+y^2+xy}} - \frac{2x}{2\sqrt{x^2+y^2}} \tag{7}$$

is positive-valued for all $x, y > 0$. We start with an evident relation

$$x^2y^2 + y^4 + 4xy^3 > 0\,.$$

Using it, we have

$$4x^4 + 4x^2y^2 + 4x^3y + \left[x^2y^2 + y^4 + 4xy^3\right] > 4x^4 + 4x^2y^2 + 4x^3y$$

$$(4x^4 + y^2 + 4xy)(x^2 + y^2) > 4x^2(x^2 + y^2 + xy)$$

$$(2x+y)^2(x^2+y^2) > (2x)^2(x^2+y^2+xy)$$

$$\frac{(2x+y)^2}{x^2+y^2+xy} > \frac{(2x)^2}{x^2+y^2}$$

$$\frac{2x+y}{\sqrt{x^2+y^2+xy}} > \frac{2x}{\sqrt{x^2+y^2}}$$

from which it immediately follows that the right-hand side expression in Eq. (7) is greater than zero.

Because of F(x,y)=F(y,x), we also have $\partial F(x,y)/\partial y > 0.$ ■

In view of Lemma 1, we need to find the minumum value of F(x,y) when x and y are the degrees of adjacent vertices of some graph. Evidently, this would happen if x=1 and y=1, i.e., for an edge of the (1,1)-type, resulting in

$$F(1,1) \leq \sqrt{x^2+y^2+xy} - \sqrt{x^2+y^2}$$

i.e.,

$$\sqrt{x^2+y^2} \leq \sqrt{x^2+y^2+xy} - (\sqrt{3}-\sqrt{2}) .$$

Recall that in the above inequalities, it is assumed that *x* and *y* pertain to the degrees of vertices of graphs.

Taking into account Eqs. (1) and (2), by summation over all edges of the undelying graph, we arrive at:

Theorem 2. Let *G* be a connected graph with *n* vertices and *m* edges. Then

$$SO(G) \leq EU(G) - (\sqrt{3}-\sqrt{2})m .$$

The equality holds if and only if all edges of the graph *G* are of the (1,1)-type, which at connected graphs can happen only if *n*=2, *m*=1, i.e., if *G* is the two-vertex path.

In a fully analogous manner, we obtain the following theorems, which hold not for all connected graphs, but for those satisfying some structural requirements.

Theorem 3. Let *G* be a connected graph with *n*≥3 vertices and *m* edges. Then

$$SO(G) \leq EU(G) - (\sqrt{7}-\sqrt{5})m .$$

The equality holds if and only if all the edges of the graph G are of the (1,2)-type, which only can happen if n=3, m=2, i.e., if G is the three-vertex path.

Proof. If a connected graph has 3 or more vertices, then none of its edges can be of the (1,1)-type. Then the next-smallest value of F(x,y) is F(1,2), pertaining to an edge of the (1,2)-type. Graphs possessing (1,2)-edges exist for n≥3, but only the 3-vertex path has all its edges of the (1,2)-type. ■

Theorem 4. Let *G* be a connected graph with *n*≥3 vertices, *m* edges, and without vertices of degree 1. Then

$$SO(G) \leq EU(G) - (\sqrt{12} - \sqrt{8})m$$

The equality holds if and only if all edges of the graph G are of the (2,2)-type, which is the case with the n-vertex cycles, n≥3.

Proof. If pendent vertices (those of degree one) do not exit in the underlying graph, then the minimum possible value of F(x,y) is F(2,2), pertaining to an edge of the (2,2)-type. The connected graphs in which all edges are of the (2,2)-type are the cycles, and these exist for all n≥3. Therefore, the claim of Theorem 4 is applicable to all graphs with 3 or more vertices. ■

Theorem 5. Let *G* be a connected graph with *n* vertices, *m* edges, and let δ be its smallest vertex degree (δ≥1). Then

$$SO(G) \leq EU(G) - (\sqrt{3} - \sqrt{2})\delta m$$

The equality holds if and only if all edges of the graph G are of the (δ,δ)-type, i.e., if G is regular of the degree δ. If δ is even, then graphs of this kind exist for all n≥ δ+1. If δ is odd, then graphs of this kind exist for all even-valued n, n≥ δ+1.

Proof. By the same argument as in the previous proofs, the minimum possible value of of F(x,y) is F(δ,δ). The equality requires that all edges be of the (δ,δ)-type. If so, then all vertices must be of the degree δ. Thus, the graphs for which the equality holds must be δ-regular. In the last part of the statement of Theorem 5, the well-known conditions for the number of vertices of δ-regular graphs are repeated (Harary, 1969; Bondy & Murty, 1976). ■

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Relacionando los índices de Sombor y Euler

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CAMPO: matemáticas (clasificación de materias de matemáticas: primaria 05c07, secundaria 05c09)
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: El índice de Euler-Sombor (EU) es un nuevo gráfico invariante basado en grados de vértice, obtenido mediante consideración geométrica. Está estrechamente relacionado con el índice de Sombor (SO). Se establece la forma real de esta relación.

Métodos: Se aplica la teoría combinatoria de grafos.

Resultados: Se establecen las desigualdades entre UE y SO.

Conclusión: El artículo contribuye a la teoría de las invariantes gráficas similares al índice de Sombor.

Palabras claves: grado (de vértice), índice de Sombor, índice de Euler-Sombor.

Соотношение между индексами Сомбора и Эйлера

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РУБРИКА ГРНТИ: 27.29.19 Краевые задачи и задачи на собственные значения для обыкновенных дифференциальных уравнений и систем уравнений
ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: Сомборский индекс Эйлера является новым инвариантом графа, основанным на степени вершины, полученным путем геометрического анализа. И он соотносится с индексом Сомбора. В данной статье установлено математическое соотношение между этими двумя инвариантами графа.

Методы: В данной статье применяется комбинаторная теория графов.

Результаты: Верхняя и нижняя границы индекса Сомбора были определены в зависимости от индекса Эйлера-Сомбора и *наоборот. Затем эти границы были откорректированы с учетом структурных особенностей графов.*

Выводы: Данное исследование вносит вклад в теорию инвариантов графа сомборского вида.

Ключевые слова: степень (вершины), индекс Сомбора, индекс Эйлера-Сомбора.

Веза између Сомборског и Ојлеровог индекса

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КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Ојлер-сомборски индекс је нова, на степенима чворова заснована графовска инваријанта, добијена геометријским разматрањима. Сродан је Сомборском индексу. У раду су утврђене математичке везе између ове две графовске инваријанте.

Методе: Примењена је комбинаторна теорија графова.

Резултати: Одређене су горње и доње границе за Сомборски индекс у зависности од Ојлер-сомборског индекса, и обратно. Ове границе су затим побољшане, узимајући у обзир структурне карактеристике графова.

Закључак: Рад доприности теорији графовских инваријанти сомборског типа.

Кључне речи: степен (чвора), Сомборски индекс, Ојлер-сомборски индекс.

Paper received on: 20.01.2024.
Manuscript corrections submitted on: 01.03.2024.
Paper accepted for publishing on: 03.03.2024.