ORIGINAL SCIENTIFIC PAPERS

Confining a non-negative solution between a lower and upper solution for a sixth-degree boundary value problem

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DOI: https://doi.org/10.5937/vojtehg72-48996

FIELD: mathematics
ARTICLE TYPE: original scientific paper

Abstract:

Introduction/purpose: The aim of the paper is to prove the existence of solutions for a special case of the sixth-order boundary value problem.

Methods: The Leray-Schauder fixed point theorem is used in order to determine lower and upper bound solutions.

Results: Lower and upper bound solutions have been found.

Conclusions: The sixth-order boundary value problem admits solutions.

Key words: Leray-Schauder nonlinear alternative, Green's function, fixed point theorem, lower and upper solutions, boundary value problem.

Introduction

Many modern mathematical studies are based on modeling phenomena in various applied branches, and simulating the results. Often, it is important to evaluate the obtained results at least with approximate solutions. One of these cases concerns BVPs related to sixth-degree differential equations derived mostly from physical phenomena and applied mathematics.

Some authors have addressed the problem of obtaining positive solutions to a sixth-degree BVP by employing multiple methods, such as the spectral operator theory, the general bifurcation theorem, the minimization theorem, and the fixed point theory in the cone, see references (Möller & Zinsou, 2013; Tersian & Chaparova, 2002; Li, 2012; Agarwal et al., 2013; Ji et al., 2012; Kovács & Guedda, 2014; Zhang & An, 2010; Mirzaei, 2016; Yang, 2019). Not a long time ago, the authors of this paper analyzed sixth-order BVPs with different boundary conditions at two points (Bekri & Benaicha, 2018, 2020). In 2018, see reference (Bekri & Benaicha, 2018), the same authors formed the conditions for finding the existence of solutions for a sixth-degree BVP:

$$\begin{cases} -\xi^{(6)}(\varepsilon) + \hbar\left(\varepsilon, \xi(\varepsilon), \xi''(\varepsilon)\right) = 0, \quad 0 \leq \varepsilon \leq 1, \\ \xi(0) = \omega'(0) = \xi''(0) = 0, \\ \xi'''(1) = \xi^{(4)}(1) = \xi^{(5)}(1) = 0, \end{cases}$$

where $\hbar \in C([0,1] \times \mathbb{R}^2)$. Suitable operators are employed in the proof of the nonlinear Leray-Schauder alternative. Also, in 2020, (Bekri & Benaicha, 2020), created all the necessary factors to prove the existence of non-negative solutions for a BVP of the sixth-order

$$\begin{cases} -\xi^{(6)}(\varepsilon) + z(\varepsilon)\hbar\left(\varepsilon, \xi(\varepsilon), \xi'(\varepsilon), \xi''(\varepsilon), \xi'''(\varepsilon), \xi^{(4)}(\varepsilon), \xi^{(5)}(\varepsilon)\right) = 0, \\ 0 \leq \varepsilon \leq 1, \\ \xi(0) = \xi'(1) = \xi''(0) = 0, \\ \xi'''(1) = \xi^{(4)}(0) = \xi^{(5)}(1) = 0, \end{cases}$$

where $\hbar \in C([0,1] \times [0,+\infty) \times [0,+\infty) \times (-\infty,0] \times (-\infty,0] \times [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty))$. The authors focused on the fixed point theorem of Leray-Schauder and on a nonlinear version of the theorems of Leray-Schauder type. For more examples, see (Chabanea et al., 2022; Fabiano & Parvaneh, 2021; Samei et al., 2021b; Fabiano et al., 2020; Samei et al., 2021a; Boutiara et al., 2021; Santra et al., 2023; Houas & Samei, 2023; Hammad et al., 2022; Amiri & Samei, 2022; Boutiara et al., 2023).

This work demonstrates the existence of a solution for sixth-degree six-point BVPs

$$\begin{cases} -\xi^{(6)}(\varepsilon) + \hbar\left(\varepsilon, \xi(\varepsilon), \xi^{(5)}(\varepsilon)\right) = 0, \quad 0 < \varepsilon < 1, \\ \xi(0) = \xi^{'}(0) = \xi^{''}(0) = 0, \\ \xi^{'''}(1) = \xi^{(4)}(1) = \xi^{(5)}(0) = 0, \end{cases} \tag{1}$$

where $\hbar \in C([0,1] \times \mathbb{R}^2)$, and

$$\begin{cases} -\xi^{(6)}(\varepsilon) + \hbar\left(\varepsilon, \xi(\varepsilon), \xi^{'}(\varepsilon), \xi^{(5)}(\varepsilon)\right) = 0, \quad 0 < \varepsilon < 1, \\ \xi(0) = \xi^{'}(0) = \xi^{''}(0) = 0, \\ \xi^{'''}(1) = \xi^{(4)}(1) = \xi^{(5)}(0) = 0, \end{cases} \tag{2}$$

where $\hbar \in C([0,1] \times \mathbb{R}^3)$.

This article is structured as follows. The section on the basic concepts reviews the definition and three lemmas to realize the analysis. Then, the part on the main results presents some desired results because there are solutions for two nonlinear sixth-order BVPs. The methodology used here is based on the Schauder fixed point theorem and the lower and upper solution method by creating the aforementioned estimates. The fourth, final part, Application, offers a few similar examples to demonstrate the findings.

Basic concepts

Within this section, a BVP (1) with the condition $\hbar \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$, and $A = C^6[0,1]$ is put forward. Some lemmas that define the role of the nonlinear term are summarized and lower or upper solutions are generated.

Definition 1. *Assume $\zeta, \eta \in C^2[0,1] \cap A$ lower and upper solution of the BVP (1), respectively, if for $0 < \varepsilon < 1$,*

$$-\zeta^{(6)}(\varepsilon) + \hbar(\varepsilon, \zeta(\varepsilon), \zeta^{(5)}(\varepsilon)) \geq 0,$$

$$\zeta(0)=\zeta^{'}(0)=\zeta^{''}(0)=\zeta^{'''}(1)=\zeta^{(4)}(1)=0, \qquad \zeta^{(5)}(0)\geq 0,$$
$$-\eta^{(6)}(\varepsilon)+\hbar(\varepsilon,\eta(\varepsilon),\eta^{(5)}(\varepsilon))\leq 0,$$
$$\eta(0)=\eta^{'}(0)=\eta^{''}(0)=\eta^{'''}(1)=\eta^{(4)}(1)=0, \qquad \eta^{(5)}(0)\leq 0.$$

LEMMA 1. *Let it be $k\in C([0,1])$. So the five-point BVP*

$$\begin{cases} \xi^{(5)}(\varepsilon)=k(\varepsilon), & 0<\varepsilon<1, \\ \xi(0)=\xi^{'}(0)=\xi^{''}(0)=0, \quad \xi^{'''}(1)=\xi^{(4)}(1)=0, & \end{cases} \tag{3}$$

is simulated by the integral equation

$$\xi(\varepsilon)=\int_0^1 \chi(\varepsilon,\varrho)k(\varrho)\,\mathrm{d}\varrho,$$

where $\chi:[0,1]\times[0,1]\to[0,+\infty)$ is the Green's function defined by

$$\chi(\varepsilon,\varrho)=\frac{1}{24}\begin{cases} \varepsilon^3(4\varrho-\varepsilon), & 0\leq\varepsilon\leq\varrho\leq 1, \\ \varrho^2(\varrho^2+6\varepsilon^2-4\varepsilon\varrho), & 0\leq\varrho\leq\varepsilon\leq 1. \end{cases} \tag{4}$$

Define the integral operator $P:A\to A$ by

$$P\xi(\varepsilon)=\tfrac{1}{24}\int_0^{\varepsilon}\varrho^2\left(\varrho^2+6\varepsilon^2-4\varepsilon\varrho\right)k(\varrho)\,\mathrm{d}\varrho+\tfrac{1}{24}\int_{\varepsilon}^1\varepsilon^3(4\varrho-\varepsilon)k(\varrho)\,\mathrm{d}\varrho.$$

According to Lemma 1, if the operator P admits a fixed point in A, it implies that the BVP (1) has a solution. Thus, one found a fixed point of P in A. Using the theorem of Ascoli-Arzelà, one checks that P is an operator which is completely continuous. Therefore, a nonlinear alternative of Leray-Schauder is given.

LEMMA 2. *(Deimling, 1985) Let A be a Banach space of and Ψ be a subset that is open and bounded of A, $0\in\Psi$. Let $P:\overline{\Psi}\to A$ be a completely continuous operator. Therefore, either*

(i) $\exists\,\xi\in\partial\Psi$ *and $\nu>1$ in which $P(\xi)=\nu\xi$, or*

(ii) $\exists\,\xi^*$ *a fixed point such that $\xi^*\in\overline{\Psi}$.*

LEMMA 3. *By $\chi(\varepsilon,\varrho)$ given by (4), the following holds*

$$(J1)\quad \frac{\chi(\varepsilon,\varrho)}{\chi(\varrho,\varrho)}\leq 1 \quad for \quad \varepsilon,\varrho\in(0,1),$$

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$$(J2)\quad \frac{\chi(\varepsilon,\varrho)}{\chi(\varrho,\varrho)} \geq V > 0 \quad for \quad \varepsilon,\varrho \in (0,1),$$

$$(J3)\ \ \varsigma = \int_0^1 \chi(\varrho,\varrho)\varrho d\varrho, \ \ such\ that\ \ \varsigma > 0.$$

Main results

This section introduces some concepts in the form of lemmas indicating some bounds on the nonlinear term and the construction of lower or upper solutions. Consider that $\hbar \in C([0,1]\times\mathbb{R}^2)$, and $\hbar \in C([0,1]\times\mathbb{R}^3)$.

LEMMA 4. *Let there be a real $\varpi \geq 0$ in which*

$$\hbar(\varepsilon,\mu,\nu) \leq \varpi, \quad for \quad 0 \leq \varepsilon \leq 1, \quad \varsigma\varpi V \leq \mu \leq \varsigma\varpi, \quad -\varpi \leq \nu \leq 0,$$

so the BVP (1) *admits an upper solution.*

Proof. We put $\Sigma(\varepsilon) = \xi^{(5)}(\varepsilon)$, so problem (1) is parallel to the equivalent problem

$$\begin{cases} \Sigma'(\varepsilon) = \hbar\left(\varepsilon, (\vartheta\Sigma)(\varepsilon), \Sigma(\varepsilon)\right), & 0 < \varepsilon < 1, \\ \Sigma(0) = 0, \end{cases} \tag{5}$$

where $(\vartheta\Sigma)(\varepsilon) = \int_0^1 \chi(\varepsilon,\varrho)\Sigma(\varrho)\,\mathrm{d}\varrho$ and $\chi(\varepsilon,\varrho)$ is given by (4). It is obvious that the bound on $\hbar$ ensures that $\Xi(\varepsilon) = \varpi\varepsilon$ realizes the inequality

$$\begin{cases} \Xi'(\varepsilon) - \hbar\left(\varepsilon, (\vartheta\Xi)(\varepsilon), \vartheta(\varepsilon)\right) \geq 0, & 0 < \varepsilon < 1, \\ \Xi(0) \geq 0. \end{cases} \tag{6}$$

From this, it holds that $\eta(\varepsilon) = (\vartheta\Xi)(\varepsilon)$ is an upper solution of the BVP (1). □

LEMMA 5. *Let there be a real number $\kappa \leq 0$ such that*

$$\hbar(\varepsilon,\mu,\nu) \geq \kappa, \quad for \quad 0 \leq \varepsilon \leq 1, \quad \varsigma\kappa \leq \mu \leq \varsigma\kappa V, \quad 0 \leq \nu \leq -\kappa,$$

so the BVP (1) *has a lower solution.*

Proof. Suppose that $\Gamma(\varepsilon) = \kappa\varepsilon$. Then, it is obvious that the bound on $\hbar$ ensures that $\Gamma(\varepsilon) = \kappa\varepsilon$ provides

$$\begin{cases} \Gamma^{'}(\varepsilon) - \hbar\,(\varepsilon, (\vartheta\Gamma)(\varepsilon), \Gamma(\varepsilon)) \leq 0, & 0 < \varepsilon < 1, \\ \Gamma(0) \leq 0, \end{cases} \tag{7}$$

this inequality proves that $\zeta(\varepsilon) = (\vartheta\Gamma)(\varepsilon)$ is a lower solution of the BVP (1). □

COROLLARY 1. *The upper solution $\eta(\varepsilon)$ and the lower solution $\zeta(\varepsilon)$ of the BVP (1) can be written in a simple manner:*

$$\eta(\varepsilon) = \frac{\varpi}{720}\varepsilon^6 - \frac{\varpi}{48}\varepsilon^4 + \frac{\varpi}{18}\varepsilon^3,$$

$$\zeta(\varepsilon) = \frac{\kappa}{720}\varepsilon^6 - \frac{\kappa}{48}\varepsilon^4 + \frac{\kappa}{18}\varepsilon^3.$$

THEOREM 1. *Let there be two real numbers $\varpi \geq 0 \geq \kappa, \quad \varpi \geq |\kappa|$ such that*

$$\hbar(\varepsilon, \mu, \nu) \leq \varpi, \qquad 0 \leq \varepsilon \leq 1, \quad \varsigma\varpi V \leq \mu \leq \varsigma\varpi, \quad -\varpi \leq \nu \leq 0, \tag{8}$$

$$\hbar(\varepsilon, \mu, \nu) \geq \kappa, \qquad 0 \leq \varepsilon \leq 1, \quad \varsigma\kappa \leq \mu \leq \varsigma\kappa V, \quad 0 \leq \nu \leq -\kappa. \tag{9}$$

If $\hbar(\varepsilon, \mu, \nu)$ is nondecreasing in μ, so the BVP (1) admits a solution $\xi(\varepsilon)$ for which

$$\zeta(\varepsilon) \leq \xi(\varepsilon) \leq \eta(\varepsilon), \quad \zeta^{(5)}(\varepsilon) \leq \xi^{(5)}(\varepsilon) \leq \eta^{(5)}(\varepsilon), \qquad \varepsilon \in [0, 1],$$

where

$$\zeta(\varepsilon) = \kappa \int_0^1 \chi(\varepsilon, \varrho)\varrho \,\mathrm{d}\varrho, \quad \eta(\varepsilon) = \varpi \int_0^1 \chi(\varepsilon, \varrho)\varrho \,\mathrm{d}\varrho, \qquad \varsigma = \tfrac{1}{48}.$$

Proof. According to formulas (8) and (9), Lemmas 4 and 5 together produce the result that the BVP (1) accepts an upper solution $\eta(\varepsilon)$ and a lower solution $\zeta(\varepsilon)$. Define $\Gamma = \zeta^{(5)}$, $\Xi = \eta^{(5)}$, therefore $\Gamma^{'}(\varepsilon) = \kappa \leq \varpi = \Xi^{'}$. In another way, $\varpi \geq 0 \geq \kappa$ implies $\zeta^{(5)}(\varepsilon) \leq \eta^{(5)}(\varepsilon)$. This implies that $\chi(\varepsilon, \varrho) \geq 0$ and, depending on the boundary condition, $\zeta(\varepsilon) \leq \eta(\varepsilon)$, for $0 \leq \varepsilon \leq 1$. Now, introduce the following operators Δ, Θ, where $\Delta : \mathrm{dom}\,\Delta = \{\Sigma \in C^1(0,1) \cap C[0,1] \to C[0,1] : \Sigma(0) = 0\}$ is a derivative operator such that $(\Delta\Sigma)(\varepsilon) = \Sigma^{'}(\varepsilon)$, $\varepsilon \in (0, 1)$, and $\Theta : C[0,1] \to C[0,1]$ is a continuous operator for which

$$\Delta\Sigma = \Theta\Sigma, \qquad \Sigma \in \mathrm{dom}(\Delta), \tag{10}$$

where Θ is given by

$$(\Theta\Sigma)(\varepsilon) = \hbar\left(\varepsilon, \vartheta\left(\min\left\{\Xi, \max\{\Sigma, \Gamma\}\right\}\right), \min\left\{\Xi, \max\{\Sigma, \Gamma\}\right\}\right). \tag{11}$$

This shows that if Σ^* is a solution of (10), then $\Gamma(\varepsilon) \leq \Sigma^*(\varepsilon) \leq \Xi(\varepsilon)$, $\varepsilon \in [0,1]$. Therefore, $\Sigma^*(\varepsilon)$ is a solution of (7). In addition $\xi^*(\varepsilon) = (\vartheta\Sigma^*)(\varepsilon)$ is a solution of the BVP (1) realizing $\zeta(\varepsilon) \leq \xi^*(\varepsilon) \leq \eta(\varepsilon)$. If $\Gamma(\varepsilon) > \Sigma^*(\varepsilon)$, there exists $\varepsilon_1 \in [0,1]$ such that $\Gamma(\varepsilon_1) > \Sigma^*(\varepsilon_1)$. Also, $\Sigma^*(0) = 0 = \zeta^{(5)}(0) = \Gamma(0) = 0$, as Σ^* and Γ are continuous, so we have an interval $(\varepsilon_2, \varepsilon_3)$ in which $\Gamma(\varepsilon_2) = \Sigma^*(\varepsilon_2)$ and $\Gamma(\varepsilon) > \Sigma^*(\varepsilon)$, for $\varepsilon \in (\varepsilon_2, \varepsilon_3)$. Then, for $\varepsilon \in [\varepsilon_2, \varepsilon_3]$,

$$\begin{aligned}(\Theta\Sigma^*)(\varepsilon) &= \hbar\left(\varepsilon, \vartheta\left(\min\left\{\Xi, \max\{\Sigma^*, \Gamma\}\right\}\right), \min\left\{\Xi, \max\{\Sigma^*, \Gamma\}\right\}\right)\\ &= \hbar\left(\varepsilon, \vartheta\left(\min\left\{\Xi, \max\{\Sigma^*, \Gamma\}\right\}\right), \Gamma\right).\end{aligned}$$

There exists $\Upsilon(\varepsilon) = \Sigma^*(\varepsilon) - \Gamma(\varepsilon)$, $\hbar$ and ϑ are monotonic, and one has

$$\begin{aligned}\Gamma'(\varepsilon) = \zeta^{(6)}(\varepsilon) \leq \hbar\left(\varepsilon, \zeta(\varepsilon), \zeta^{(5)}(\varepsilon)\right) = \hbar\left(\varepsilon, \vartheta\Gamma(\varepsilon), \Gamma(\varepsilon)\right), \qquad \varepsilon \in [0,1],\\ \Sigma^{*'} = \hbar\left(\varepsilon, \vartheta\left(\min\left\{\Xi, \max\{\Sigma^*, \Gamma\}\right\}\right), \phi\right),\end{aligned}$$

so $\Upsilon'(\varepsilon) \geq 0$, $\varepsilon \in [\varepsilon_2, \varepsilon_3]$. Consequently, $\Upsilon(\varepsilon_2) = 0$ implies $\Upsilon(\varepsilon) \geq 0$, $\varepsilon \in [\varepsilon_2, \varepsilon_3]$. From the relation $\Gamma(\varepsilon) \leq \Sigma^*(\varepsilon)$, $\varepsilon \in [\varepsilon_2, \varepsilon_3]$. This contradicts the given hypothesis. So, $\Gamma(\varepsilon) \leq \Sigma^*(\varepsilon)$, $\varepsilon \in [0,1]$. The proof of $\Xi(\varepsilon) \geq \Sigma^*(\varepsilon)$, $\varepsilon \in [0,1]$, is analogous to $\Gamma(\varepsilon) \leq \Sigma^*(\varepsilon)$.

One will now introduce $\Omega : C[0,1] \to [0,1]$, given by

$$(\Omega\Sigma)(\varepsilon) = \int_0^{\varepsilon} (\Theta\Sigma)(\varrho)\,\mathrm{d}\varrho, \qquad \varepsilon \in [0,1].$$

It is obvious that the operator is continuous and a fixed point of Ω is a solution of the BVP (10). Let there be a set $\Psi_\varpi = \{\rho \in C[0,1] : \|\rho\| \leq \varpi\}$, and is easy to observe that $\Xi, \Gamma \in \Psi_\varpi$. Now, one demonstrates that $\Omega : \Psi_\varpi \to \Psi_\varpi$ is a completely continuous operator. By formula (9), one obtains

$$\left|(\Omega\Sigma)(\varepsilon)-(\Omega\Sigma)(\varrho)\right| = \left|\int_0^{\varepsilon}(\Theta\Sigma)(\tau)\,\mathrm{d}\tau\right| \leq \varpi\left|\varepsilon-\varrho\right|.$$

That is, $\{\Omega(\Psi_{\varpi})\}$ is bounded uniformly and equi-continuous. According to the theorem of Arzelà-Ascoli, it is affirmed that $\Omega:\Psi_{\varpi}\to\Psi_{\varpi}$ is a completely continuous operator, and the Schauder fixed point theorem ensures that Ω admits a fixed point $\Sigma^{*}\in\Psi_{\varpi}$. Hence,

$$\xi^{*}(\varepsilon)=\int_0^1\chi(\varepsilon,\varrho)\Sigma^{*}(\varrho)\,\mathrm{d}\varrho,$$

is a solution of problem (1) such that $\zeta(\varepsilon)\leq\xi(\varepsilon)\leq\eta(\varepsilon)$, $\zeta^{(5)}(\varepsilon)\leq\xi^{(5)}(\varepsilon)\leq\eta^{(5)}(\varepsilon)$, $\varepsilon\in[0,1]$. □

Now, let us consider problem (2). Suppose that $\hbar\in C([0,1]\times\mathbb{R}^3)$ is a continuous function. It is possible to obtain lemmas analogue to Lemmas 4 and 5. This is evident by observing that the equation of problem (2) is parallel to

$$\begin{cases}\Sigma^{'}(\varepsilon)=\hbar(\varepsilon,(\vartheta\Sigma)(\varepsilon),(\Pi\Sigma)(\varepsilon),\Sigma(\varepsilon)), & 0<\varepsilon<1,\\ \Sigma(0)=0,\end{cases}$$

where

$$(\vartheta\Sigma)(\varepsilon)=\int_0^1\chi(\varepsilon,\varrho)\Sigma(\varrho)\,\mathrm{d}\varrho,\qquad(\Pi\Sigma)(\varepsilon)=\int_0^1\Sigma(\varrho)\,\mathrm{d}\varrho.$$

LEMMA 6. *Let there be a real number $\varpi\geq0$ such that*

$$\hbar(\varepsilon,\mu,\nu,\tau)\leq\varpi,\ \textit{for}\ 0\leq\varepsilon\leq1,\quad\frac{\mu}{\varsigma},2\nu\in[0,\varpi],\quad\tau\in[-\varpi,0].$$

So the BVP (2) accepts an upper solution.

LEMMA 7. *Let there be a real number $\kappa\leq0$ such that*

$$\hbar(\varepsilon,\mu,\nu,\tau)\geq\kappa,\ \textit{for}\ 0\leq\varepsilon\leq1,\quad\mu\in[\varsigma\kappa,0],\quad2\nu,\tau\in[0,-\kappa].$$

So the BVP (2) admits a lower solution.

In an analogous fashion, we can write the same proof for Lemmas 6 and 7 in the same way, as formulated in Lemmas 4 and 5. The similarity in Theorem 1 allows obtaining the following results of problem (2).

THEOREM 2. *Let there be two real numbers $\varpi \geq 0 \geq \kappa$, $\varpi \geq |\kappa|$ such that*

$$\hbar(\varepsilon,\mu,\nu,\tau) \leq \varpi, \qquad 0 \leq \varepsilon \leq 1,\ \tfrac{\mu}{\varsigma},\ 2\nu \in [0,\varpi],\ \tau \in [-\varpi, 0], \tag{12}$$

$$\hbar(\varepsilon,\mu,\nu,\tau) \geq \kappa, \qquad 0 \leq \varepsilon \leq 1,\ \mu \in [\varsigma\kappa, 0],\ 2\nu, \tau \in [0, -\kappa]. \tag{13}$$

If $\hbar(\varepsilon,\mu,\nu,\tau)$ is nondecreasing in each μ and ν, then problem (2) admits a solution $\xi(\varepsilon)$ such that

$$\zeta(\varepsilon) \leq \xi(\varepsilon) \leq \eta(\varepsilon), \quad \zeta^{(5)}(\varepsilon) \leq \xi^{(5)}(\varepsilon) \leq \eta^{(5)}(\varepsilon), \qquad \varepsilon \in [0,1],$$

where

$$\zeta(\varepsilon) = \kappa \int_0^1 \chi(\varepsilon,\varrho)\varrho\,\mathrm{d}\varrho, \quad \eta(\varepsilon) = \varpi \int_0^1 \chi(\varepsilon,\varrho)\varrho\,\mathrm{d}\varrho, \qquad \varsigma = \tfrac{1}{48}.$$

Application

Some examples are given in order to illustrate the results obtained above.

EXAMPLE 1. Extrapolate the following application of the BVP

$$\begin{cases} -\xi^{(6)} + \frac{1}{48}\left[\varepsilon + \cos\xi(\varepsilon) + \xi^{(5)}(\varepsilon)\right] = 0, \quad 0 < \varepsilon < 1, \\ \xi(0) = \xi'(0) = \xi''(0) = 0, \\ \xi'''(1) = \xi^{(4)}(1) = \xi^{(5)}(0) = 0. \end{cases} \tag{14}$$

There is $\hbar(\varepsilon,\mu,\nu) = \frac{1}{48}\left[\varepsilon + \cos\mu + \nu\right]$. It is not difficult to verify that the values of $\varpi = 24$, $\kappa = 0$ meet conditions (8) and (9), respectively; hence,

$$\zeta(\varepsilon) = 0, \qquad \eta(\varepsilon) = 24\int_0^1 \chi(\varepsilon,\varrho)\varrho\,\mathrm{d}\varrho,$$

represented by the lower and upper solutions of (14). According to Theorem 1, problem (14) admits a positive solution ξ^* such that

$$0 \leq \xi^*(\varepsilon) \leq 24\int_0^1 \chi(\varepsilon,\varrho)\varrho\,\mathrm{d}\varrho.$$

Figure (1) shows the search for a result of the comparison of the numerical solution to (14) and its estimate given by the Green's function.

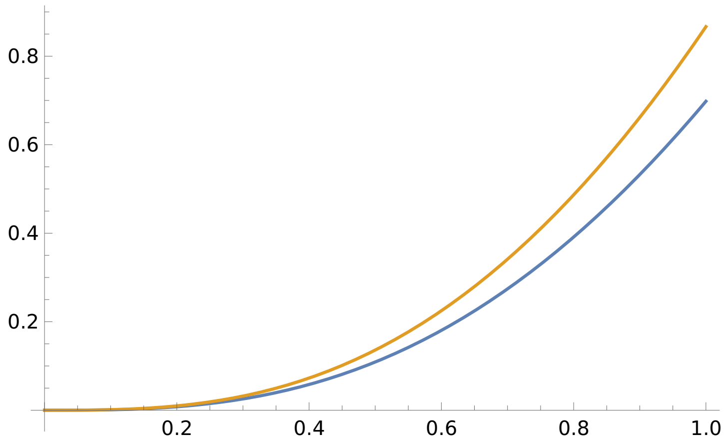

Figure 1 – The lower plot is the numerical solution of (14), multiplied by a large coefficient 4×10^3 for comparison purposes, while the upper plot is the estimate of the above, given by the Green's function. Observe how the two curves present the same behavior as functions of ε.

EXAMPLE 2. Extrapolate the following application of the BVP

$$\begin{cases} -\xi^{(6)} + \frac{1}{50}\left[\varepsilon + \sin\xi\left(\varepsilon\right) + \cos\left(\xi^{'}(\varepsilon)\right)^4 + \xi^{(5)}(\varepsilon)\right] = 0, \quad 0 < \varepsilon < 1, \\ \xi(0) = \xi^{'}(0) = \xi^{''}(0) = 0, \\ \xi^{'''}(1) = \xi^{(4)}(1) = \xi^{(5)}(0) = 0. \end{cases} \tag{15}$$

There is $\hbar(\varepsilon, \mu, \nu, \tau) = \frac{1}{50}[\varepsilon + \sin\mu + \cos(\nu)^4 + \tau]$. It is not difficult to verify that the values $\varpi = 25$, $\kappa = 0$ meet conditions (12) and (13), respectively; hence,

$$\zeta(\varepsilon) = 0, \qquad \eta(\varepsilon) = 25\int_0^1 \chi(\varepsilon, \varrho)\varrho \, \mathrm{d}\varrho,$$

are the lower and upper solutions of (15). According to Theorem 2, problem (15) admits a positive solution ξ^* such that

$$0 \leq \xi^*(\varepsilon) \leq 25\int_0^1 \chi(\varepsilon, \varrho)\varrho \, \mathrm{d}\varrho.$$

Figure (2) shows the desired result of the comparison of the numerical solution to (15) and its estimate given by the Green's function.

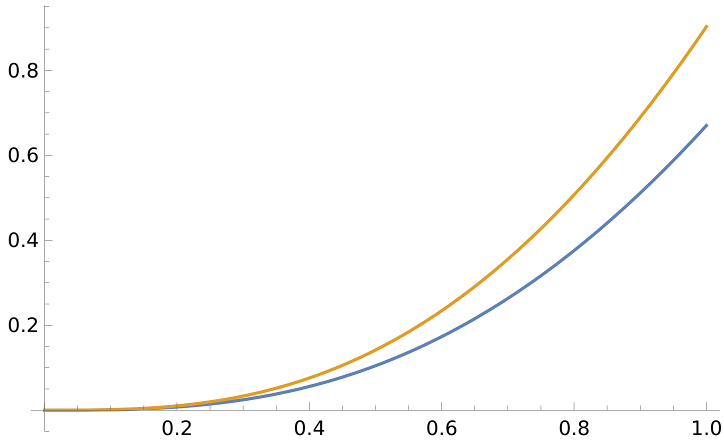

Figure 2 – The lower plot is the numerical solution of (15)*, multiplied by a large coefficient* 4×10^3 *for comparison purposes, while the upper plot is the estimate of the above, given by the Green's function. Observe how the two curves present the same behavior as functions of* ε*.*

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Confinar una solución no negativa entre una solución superior e inferior para un problema de valor límite de sexto grado

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CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: El objetivo del artículo es demostrar la existencia de soluciones para un caso especial del problema del valor límite de sexto orden.

Métodos: El teorema del punto fijo de Leray-Schauder se utiliza para determinar las soluciones de límite superior.

Resultados: Se han encontrado soluciones de límite inferior y superior.

Conclusión: El problema del valor límite de sexto orden admite soluciones.

Palabras claves: alternativa no lineal de Leray-Schauder, función de Green, teorema del punto fijo, soluciones inferior y superior, problema de valor límite.

Ограничение неотрицательного решения между нижним и верхним решениями в краевых задачах шестого порядка

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РУБРИКА ГРНТИ: 27.25.17 Метрическая теория функций,
27.29.15 Общая теория обыкновенных дифференциальных уравнений и систем уравнений,
27.39.15 Линейные пространства, снабженные топологией, порядком и другими структурами
ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: Доказательство существования решения частного случая краевой задачи шестого порядка.

Методы: Теорема Лере-Шаудера о неподвижной точке используется в определении нижней и верхней границы решения задачи.

Результаты: В ходе исследования были найдены решения нижней и верхней границы.

Выводы: Краевая задача шестого порядка допускает решения.

Ключевые слова: нелинейная альтернатива Лере-Шаудера, функция Грина, теорема о неподвижной точке, нижние и верхние решения, краевая задача.

Ограничавање ненегативног решења између доњег и горњег решења за шести степен проблема граничних вредности

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ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Доказивање постојања решења за посебан случај граничног проблема шестог реда.

Методе: Лерај-Шаудерова теорема о фиксној тачки користи се да би се одредиле ниже и горње границе решења.

Резултати: Пронађена су решења доње и горње границе.

Закључак: Гранични проблем шестог реда дозвољава решења.

Кључне речи: Лерај-Шаудерова нелинеарна алтернатива, Гринова функција, теорема фиксне тачке, доња и горња решења, проблем граничних вредности.

Paper received on: 30.01.2024.
Manuscript corrections submitted on: 07.06.2024.
Paper accepted for publishing on: 08.06.2024.