Exploring multivalued probabilistic ψ-contractions with orbits in b-Menger spaces

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Abstract:

Introduction/purpose: The paper presents a novel approach to certain well-established fixed point theorems for multivalued probabilistic contractions in b-Menger spaces, leveraging the boundedness of the orbits. The aim was to generalize and enhance the results previously derived by Fang and Hadžić.

Methods: The boundedness of orbits in b-Menger spaces is used to establish their approach for multivalued probabilistic contractions.

Results: The findings of the study not only generalized the existing fixed point theorems but also enhanced them significantly. The effectiveness of the approach in extending the results originally proposed by Fang and Hadžić was showcased. Moreover, the applicability of the coincidence fixed point theorem in fuzzy b-metric spaces was demonstrated.

Conclusions: The study presented a novel perspective on fixed point theorems in multivalued probabilistic contractions within b-Menger spaces. By leveraging boundedness and introducing a coincidence fixed point *theorem for fuzzy b-metric spaces, the work contributed to the advancement in this field.*

Key words: fixed point, b-Menger spaces, multivalued ψ-contraction, fuzzy b-metric space.

Introduction

In 1981, Vul'pe et al. (Berinde & Păcurar, 2022) introduced the concept of b-metric space as a generalization of metric spaces, a framework later utilized by Bakhtin and Czerwik (Bakhtin, 1989; Czerwik, 1993) to establish the well-known Banach fixed point theorem in these spaces (Banach, 1922). The significance of the fixed point theory resonates across various branches of pure and applied mathematics due to its broad range of applications.

The concept of probabilistic metric spaces, introduced by K. Menger in 1942 (Menger, 2003), constitutes a crucial extension of metric spaces. The exploration of the fixed point theory in Menger spaces, concerning both multivalued and single-valued contractions, has become an integral part of probabilistic analysis, attracting the attention of numerous mathematicians (Achtoun et al., 2023; Mbarki & Oubrahim, 2017; Huang et al., 2023; Mihet, 2005; Patle et al., 2019). Recently, Mbarki and Oubrahim (Mbarki & Oubrahim, 2017) introduced the b-metric version of probabilistic metric spaces, termed a b-Menger space, which stands as the most general concept among those mentioned earlier. Notably, numerous fixed point results have been derived within this type of space (Mbarki & Oubrahim, 2017; Mihet, 2005). A parallel idea emerged in the realm of fuzzy metric spaces, where Nădăban (Nădăban, 2016) introduced the concept of a fuzzy b-metric space, generalizing the notion put forth by Kramosil and Michálek (Kramosil & Michálek, 1975).

The concept of multivalued contractions in metric spaces was pioneered by Nadler (Nadler Jr, 1969), and Hadžić (Hadžic, 1989) later extended this notion to multivalued ψ-contractions in probabilistic metric spaces. She established a fixed point theorem employing the concept of probabilistic function of non-compactness. Building on this foundation, Fang (Fang, 1992) presented a generalization of Hadžić's results by substituting the condition of a continuous t-norm with a t-norm of H-type.

This paper contributes a new fixed point theorem for multivalued mappings satisfying ψ-contractive conditions in b-Menger spaces, leveraging the concept of bounded orbits. As an application, these results are extended to establish a corresponding fixed point theorem in fuzzy b-metric spaces. These authors' findings not only improve upon the work of Hadžić (Hadžic, 1989) and Fang (Fang, 1992) but also generalize their results.

The structure of this article unfolds as follows: Section 2 provides essential concepts and lemmas in b-Menger spaces. Section 3 establishes the existence of fixed points for multivalued ψ-contractions in b-Menger spaces, employing two distinct approaches and offering illustrative examples. Finally, Section 4 identifies a coincidence fixed point for multivalued ψ-contraction mappings in fuzzy b-metric spaces.

Preliminaries

To start with, here are some basic definitions and facts from b-Menger spaces.

Definition 1. *Let Δ^+ be the class of all distance distribution mappings $\gamma : [0, +\infty] \to [0, 1]$ such that:*

1. *γ is left continuous on $[0, +\infty]$,*
2. *γ is non-decreasing,*
3. *$\gamma(0) = 0$ and $\gamma(+\infty) = 1$.*

The subset $\mathcal{D}^+ \subset \Delta^+$ is the set $\mathcal{D}^+ = \left\{\gamma \in \Delta^+ : \lim_{\alpha \to +\infty} \gamma(\alpha) = 1\right\}$.

As a specific element of $\mathcal{D}^+$ is ϵ_a defined as:

$$\epsilon_a(\alpha) = \begin{cases} 0 & \text{if} \quad \alpha \in (-\infty, a], \\ 1 & \text{if} \quad \alpha \in (a, +\infty). \end{cases}$$

Definition 2. *(Schweizer & Sklar, 1983) A triangular norm (briefly t-norm) is a binary operation $\daleth$ on $[0, 1]$ such that for all $u, v, w \in [0, 1]$ the following conditions are verified:*

1. *$\daleth(\alpha, \beta) = \daleth(\beta, \alpha)$,*
2. *$\daleth(\alpha, \daleth(\beta, \gamma)) = \daleth(\daleth(\alpha, \beta), \gamma)$,*
3. *$\daleth(\alpha, \beta) < \daleth(\alpha, \gamma) \quad$ for $\beta < \gamma$,*
4. *$\daleth(\alpha, 1) = \daleth(1, \alpha) = \alpha$.*

Example 1. Here the most basic t-norms are cited:

1. The minimum t-norm $\daleth_M(\alpha,\beta) = min(\alpha,\beta)$.
2. The Lukasiewicz t-norm $\daleth_L(\alpha,\beta) = max(\alpha+\beta-1,0)$.
3. The product t-norm $\daleth_P(\alpha,\beta) = \alpha\cdot\beta$

Definition 3. *(Pap et al., 1996) A t-norm $\daleth$ is said of H-type if the family $(\daleth^n(\alpha))_{n\in\mathbb{N}}$ is equi-continuous at the point $\alpha = 1$, it means that :*

$$for\,all\,\epsilon\in(0,1),\,there\,exists\,\lambda\in(0,1):\ t>1-\lambda\ implies\ \daleth^n(t)>1-\epsilon\ for\,all\,n\geq 1,$$

where for all $\alpha\in[0,1]$ and $n\in\mathbb{N}$ there exists:

$$\daleth^n(\alpha) = \begin{cases} 1 & \text{if} \quad n=0, \\ \daleth(\daleth^{n-1}(\alpha),\alpha) & \text{otherwise}. \end{cases}$$

A simple example of H-type t-norm is $\daleth_M$, unlike $\daleth_L$ is not of H-type. The readers are referred to (Pap et al., 1996) for more details.

Definition 4. *(Mbarki & Oubrahim, 2017) A quadruple $(\Gamma,\digamma,\daleth,s)$ where Γ is a nonempty set, $\digamma$ is a function from $\Gamma\times\Gamma$ into Δ^+, $\daleth$ is a t-norm and $s\geq 1$ is a real number, is called a b-Menger space if the following requirements are verified for all $\rho,\sigma,\vartheta\in\Gamma$ and $t,v>0$:*

1. $\digamma_{\rho,\rho} = \epsilon_0$,
2. $\digamma_{\rho,\sigma} \neq \epsilon_0 \ \ if\ \rho\neq\sigma$,
3. $\digamma_{\rho,\sigma} = \digamma_{\sigma,\rho}$,
4. $\digamma_{\rho,\sigma}(s(t+v)) \geq \daleth(\digamma_{\rho,\vartheta}(t),\digamma_{\vartheta,\sigma}(v))$.

Note that a Menger space is a b-Menger space with $s=1$.
In the topology created by the family of (ϵ,λ)-neighborhoods:

$$\mathcal{N} = \{\mathcal{N}_p(\epsilon,\lambda):\ p\in\Gamma,\ \epsilon>0\ \ and\ \ \lambda>0\},$$

where:

$$\mathcal{N}_p(\epsilon,\lambda) = \{q\in\Gamma:\ \digamma_{p,q}(\epsilon)>1-\lambda\}.$$

The space $(\Gamma,\digamma,\daleth,s)$ is a Hausdorff topological space if the t-norm $\daleth$ is continuous, as demonstrated by Mbarki and Oubrahim (Mbarki & Oubrahim, 2017)

Definition 5. *A sequence $\{\omega_n\}$ in a b-Menger space $(\Gamma,\digamma,\daleth,s)$ is said to be:*

1. *Convergent to $\omega \in \Gamma$ if for any given $\epsilon > 0$ and $\lambda > 0$ there exist $N \in \mathbb{N}$ satisfying $F_{\omega_n,\omega}(\lambda) > 1-\epsilon$ whenever $n \geq N$.*
2. *Strong Cauchy sequence if for any $\epsilon > 0$ and $\lambda > 0$ there exist $N \in \mathbb{N}$ satisfying $F_{\omega_n,\omega_m}(\lambda) > 1-\epsilon$ whenever $n, m \geq N$.*

A b-Menger space $(\Gamma, F, \daleth, s)$ is complete if each Cauchy sequence in Γ is convergent to some point in Γ.

In the following, it is assumed for the b-Menger space $(\Gamma, F, \daleth, s)$ that $\daleth$ is a continuous t-norm, and the class of all nonempty closed subsets of Γ is denoted by $C(\Gamma)$, where for all $U, V \in C(\Gamma)$ and $\omega \in \Gamma$ the functions $F_{\omega,U}(.)$ and $F_{U,V}(.)$ are defined as follows:

$$F_{\omega,U}(t) = \sup_{v\in U} F_{\omega,v}(t) \quad for\ all\ t \in \mathbb{R},$$

and

$$F_{U,V}(t) = \inf_{\omega\in U} \sup_{v\in V} F_{\omega,v}(t) \quad for\ all\ t \in \mathbb{R}.$$

The first result is the following:

Lemma 1. *Let $(\Gamma, F, \daleth, s)$ be a b-Menger space, then for all $U \in C(\Gamma)$ and $\omega, v \in \Gamma$ there is*

$$F_{\omega,U}(t) = 1\ for\ all\ t > 0\ if\ and\ only\ if\ \omega \in U.$$

Proof. If $F_{\omega,U}(t) = 1$ for all $t > 0$, then for any $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $\omega_0 \in U$ such that $F_{\omega,\omega_0}(\epsilon) > 1-\lambda$. As $U \in C(\Gamma)$ then $\omega \in U$.
On the other hand, if $\omega \in U$, then

$$\begin{aligned} F_{\omega,U}(t) &= \sup_{v\in U} F_{\omega,v}(t) \\ &\geq F_{\omega,\omega}(t) = 1 \quad for\ all\ t > 0. \end{aligned}$$

Hence $F_{\omega,U}(t) = 1 \quad for\ all\ t > 0.$ □

Definition 6. *Let $(\Gamma, F, \daleth, s)$ be a b-Menger space and U a nonempty set of Γ. The function $\mathcal{D}_U$ defined on $[0, +\infty]$ by*

$$\mathcal{D}_U(\omega) = \begin{cases} \lim_{t\to\omega^-} \theta_U(t) & \textit{if}\quad 0 \leq \omega < +\infty, \\ 1 & \textit{if}\quad \omega = +\infty, \end{cases}$$

where

$$\theta_U(t) = \inf \{\mathcal{F}_{a,b}(t) \mid a, b \in U\},$$

is called the probabilistic diameter of U.

It is clear that $\mathcal{D}_U \in \Delta^+$ for any $U \subset \Gamma$, and for all $p, q \in U$. Also U is bounded if $\mathcal{D}_U$ is into $\mathcal{D}^+$.

Lemma 2. *Let $(\Gamma, \mathcal{F}, \daleth, s)$ be a b-Menger space and U a nonempty set of Γ, the probabilistic diameter has the following proprieties:*

1. *For all $a, b \in U$, there exists $F_{a,b} \geq \mathcal{D}_U$.*
2. *$\mathcal{D}_U = \epsilon_0$ if and only if U is a singleton.*
3. *If $U \subset V$, then $\mathcal{D}_U \geq \mathcal{D}_V$.*

Main result

Throughout this section, a point $z \in \Gamma$ is said to be a fixed point of $f : \Gamma \to C(\Gamma)$ if $z \in fz$. If for $\omega_0 \in \Gamma$, there exists a sequence $\{\omega_i\} \subset \Gamma$ such that $\omega_i \in f\omega_{i-1}$, then $\wp_f(\omega_0) = \{\omega_0, \omega_1, \omega_2, ...\}$ is called an orbit of f starting at ω_0.
Let χ denote the family of all function $\psi : [0, +\infty) \to [0, +\infty)$ satisyfing

$$0 < \psi(t) < t \quad and \quad \lim_{n \to +\infty} \psi^n(t) = 0 \quad for\ all\ t > 0.$$

The definition of multivalued probabilistic ψ-contraction is first introduced in a b-Menger space.

Definition 7. *Let $(\Gamma, \mathcal{F}, \daleth, s)$ be a b-Menger space and $\psi : [0, +\infty) \to [0, +\infty)$.A a mapping $f : \Gamma \to C(\Gamma)$ is called a multi-valued probabilistic ψ-contraction if for every $\omega, \upsilon \in \Gamma$ and every $\rho \in f\omega$ there exists $\sigma \in f\upsilon$ such that*

$$\mathcal{F}_{\rho,\sigma}(\psi(t)) \geq \mathcal{F}_{\omega,\upsilon}(st) \quad for\ all\ t > 0. \tag{1}$$

Remark 1. Note that if f is a multi-valued ψ-contraction, there exists

$$\mathcal{F}_{f\omega,f\upsilon}(\psi(t)) \geq \mathcal{F}_{\omega,\upsilon}(st) \quad for\ all\ \omega, \upsilon \in \Gamma,\ and\ t > 0. \tag{2}$$

Before stating the main result, one will use later the following lemma.

Lemma 3. *Let $\{\omega_n\}$ be a bounded sequence in a b-Menger space $(\Gamma, \digamma, \daleth, s)$where $Ran\digamma \subset \mathcal{D}^+$. If there exists a function $\psi \in \chi$ such that*

$$\digamma_{\omega_n,\omega_m}(\psi(t)) \geq \digamma_{\omega_{n-1},\omega_{m-1}}(st) \quad for\, all\, n, m > 0\, such\, that\, m > n\, and\, for\, all\, t > 0. \tag{3}$$

Then $\{\omega_n\}$ is a Cauchy sequence.

Proof. Let $\{\omega_n\}$ be a bounded sequence in Γ that satisfying the condition (3). Then one obtains

$$\begin{aligned}
\digamma_{\omega_n,\omega_m}(\psi^n(t)) &\geq \digamma_{\omega_{n-1},\omega_{m-1}}(s\psi^{n-1}(t)) \\
&\geq \digamma_{\omega_{n-1},\omega_{m-1}}(\psi^{n-1}(t)) \\
&\geq \digamma_{\omega_{n-2},\omega_{m-2}}(s\psi^{n-2}(t)) \\
&\geq \digamma_{\omega_{n-2},\omega_{m-2}}(\psi^{n-2}(t)) \\
&\vdots \\
&\geq \digamma_{\omega_0,\omega_{m-n}}(st) \\
&\geq \digamma_{\omega_0,\omega_{m-n}}(t) \\
&\geq \mathcal{D}_{\wp(\omega)}(t).
\end{aligned}$$

On the other hand, let $\epsilon > 0$ and $\delta \in (0, 1)$ be given, since $\mathcal{D}_{\wp(\omega)}(t) \to 1$ as $t \to +\infty$, then there exist $t_0 > 0$ such that

$$\mathcal{D}_{\wp(\omega)}(t_0) > 1 - \delta.$$

From that $\psi^n(t_0) \to 0$ as $n \to +\infty$, then there exist $n_0 \in \mathbb{N}$ satisfying

$$\psi^n(t_0) < \epsilon \quad whenever \quad n \geq n_0.$$

By using the monotonicity of $\digamma$, one obtains

$$\begin{aligned}
\digamma_{\omega_n,\omega_m}(\epsilon) &\geq \digamma_{\omega_n,\omega_m}(\psi^n(t_0)) \\
&\geq \mathcal{D}_{\wp(\omega)}(t_0) \\
&\geq 1 - \delta.
\end{aligned}$$

Therefore, $\{\omega_n\}$ is a Cauchy sequence. □

Theorem 1. *Let $(\Gamma, \digamma, \daleth, s)$ be a complete b-Menger space where $Ran\digamma \subset \mathcal{D}^+$ and $f : \Gamma \to C(\Gamma)$ is a multivalued probabilistic ψ-contraction mapping with $\psi \in \chi$. If all the orbits $\wp_{f(\omega)}$ for some $\omega \in \Gamma$ are bounded, then there exists $z \in \Gamma$ satisfying $z \in fz$.*

Proof. Let $\omega_0 \in \Gamma$ and $\omega_1 \in f\omega_0$ such that the orbit $\wp_{f(\omega)}$ starting at ω_0 is bounded. Then there exists $\omega_2 \in f\omega_1$ thus by (1) one obtains

$$F_{\omega_1,\omega_2}(\psi(t)) \geq F_{\omega_0,\omega_1}(st) \quad for\ all\ t > 0.$$

Inductively, one constructs a sequence $\{\omega_n\}$ satisfying the following conditions:

$$\omega_{n+1} \in f\omega_n \ \ and \ \ F_{\omega_n,\omega_{n+1}}(\psi(t)) \geq F_{\omega_{n-1},\omega_n}(st) \ \ for\ all\ n \in \mathbb{N},\ and\ t > 0.$$

It will shown first that for each $t > 0$,

$$F_{\omega_n,\omega_{n+m}}(\psi(t)) \geq F_{\omega_{n-1},\omega_{n+m-1}}(st) \quad for\ all\ m > 0. \tag{4}$$

It is obvious that (4) is true for $m = 1$.
It is claimed that (4) holds for $m > 0$.
Since $\omega_n \in f\omega_{n-1}$ and $\omega_{n+m+1} \in f\omega_{n+m}$, using Remark 1, one gets

$$\begin{aligned} F_{\omega_n,\omega_{n+m+1}}(\psi(t)) &\geq F_{f\omega_{n-1},\omega_{n+m+1}}(\psi(t)) \\ &\geq F_{f\omega_{n-1},f\omega_{n+m}}(\psi(t)) \\ &\geq F_{\omega_{n-1},\omega_{n+m}}(st). \end{aligned}$$

So, by induction, it is proved that (7) holds for all $m > 0$.
Therefore, from Lemma 3, it follows that $\{\omega_n\}$ is a Cauchy sequence.
As $(\Gamma, F, \urcorner, s)$ is complete, then $\{\omega_n\}$ converge to some $z \in \Gamma$.
It will be demonstrated that z is a fixed point within f. For that, let $t > 0$, then, from (1), one has

$$\begin{aligned} F_{\omega_n,fz}(\psi(t)) &\geq F_{\omega_{n-1},z}(st) \\ &\geq F_{\omega_{n-1},z}(t). \end{aligned}$$

Since that $\psi \in \chi$, it follows

$$F_{\omega_n,fz}(t) \geq F_{\omega_{n-1},z}(t),$$

by letting $n \to +\infty$ one gets

$$F_{z,fz}(t) \geq 1 \quad for\ each\ t > 0,$$

which implies by Lemma 1 that $z \in fz$. Hence, z is a fixed point of f.

□

Example 2. Let $\Gamma = [0, +\infty)$. Define $\digamma : \Gamma \times \Gamma \to \Delta^+$ by

$$\digamma_{\omega,\upsilon}(t) = \epsilon_0(t - |\omega - \upsilon|^2).$$

It is claimed that $(\Gamma, \digamma, \daleth, 2)$ is a complete b-Menger space. Let us consider the mapping $f : \Gamma \to C(\Gamma)$ given by $f(\omega) = [0, \frac{\omega}{2}]$.
It will be proven that for all $\omega, \upsilon \in \Gamma$ and $\rho \in f\omega$ there exists $\sigma \in f\upsilon$ such that

$$\digamma_{p,q}(\psi(t)) \geq \digamma_{\omega,\upsilon}(2t).$$

With $\psi(t) = \frac{3}{4}t$. Let $\omega, \upsilon \in \Gamma$ and $\rho \in f\omega$ one has

Case 1 If $\rho < \frac{\upsilon}{2}$ then $\rho \in f\upsilon$. so there exists $\sigma \in [0, \frac{\upsilon}{2}]$ such that

$$|\rho - \sigma| \leq \left|\frac{\omega}{2} - \frac{\upsilon}{2}\right|.$$

Case 2 If $\rho \geq \frac{\upsilon}{2}$, since $0 \leq \rho \leq \frac{\omega}{2}$, one gets

$$0 \leq \rho - \frac{\upsilon}{2} \leq \frac{\omega}{2} - \frac{\upsilon}{2}$$

Then if one takse $\sigma = \frac{\upsilon}{2}$, one obtains

$$|\rho - \sigma| \leq \left|\frac{\omega}{2} - \frac{\upsilon}{2}\right|.$$

Therefore, from case 1 and case 2, one obtains that

$$\begin{aligned}
\digamma_{\rho,\sigma}(\psi(t)) &= \epsilon_0(\frac{3}{4}t - |\rho - \sigma|^2) \\
&\geq \epsilon_0(\frac{3}{4}t - \left|\frac{\omega}{2} - \frac{\upsilon}{2}\right|^2) \\
&= \epsilon_0(\frac{3}{4}t - \frac{1}{4}|\omega - \upsilon|^2) \\
&\geq \epsilon_0(3t - |\omega - \upsilon|^2) \\
&\geq \digamma_{\omega,\upsilon}(2t).
\end{aligned}$$

Since $\wp_f(0)$ is bounded, then all the conditions of this Theorem are satisfied. Hence f have a fixed point which is 0.

Remark 2. One should mark down that the condition propriety about the boundedness of the orbits $\wp_f(\omega)$ is an obligatory condition to prove the existence of a fixed point as the next example shows.

Example 3. (Sherwood, 1971) Define the distribution function as:

$$\mathcal{K}(t) = \begin{cases} 0 & \text{if} \quad t \leq 4, \\ 1 - \frac{1}{a} & \text{Si} \quad 2^a < t < 2^{a+1}, \quad a > 1. \end{cases}$$

Consider $\Gamma = \{1, 2, 3, ..., a, ...\}$. Define $F : \Gamma \times \Gamma \longrightarrow \mathcal{D}^+$ as follows :

$$F_{a,a+b}(t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ \urcorner_L^b \left(\mathcal{K}(2^a t), \mathcal{K}(2^{a+1} t), ..., \mathcal{K}(2^{a+b} t)\right) & \text{if} \quad t > 0. \end{cases}$$

Then, one obtains that $(\Gamma, F, \urcorner_L, 1)$ is a complete b-Menger space, and since every single-valued mapping is a multi-valued mapping, one puts $f(a) = a + 1$, which is ψ-contractive with $\psi(t) = \frac{1}{2}t$. However, f have no fixed point, since there exists any $n \in \Gamma$ such that $\wp_f(\omega)$ is bounded.

Lemma 4. *Every Cauchy sequence in a b-Menger space* $(\Gamma, F, \urcorner, s)$ *such that* $RanF \subset \mathcal{D}^+$ *is bounded.*

Proof. Let $\{\omega_n\}$ be a Cauchy sequence. Taking $\epsilon > 0$, then for $t > 0$ there exists a positive integer $N \in \mathbb{N}$ such that

$$F_{\omega_n,\omega_m}(t) > 1 - \epsilon \quad whenever\ n, m \geq N. \tag{5}$$

Since $RanF \subset \mathcal{D}^+$ there exists $t_0 > t$ such that

$$F_{\omega_n,\omega_m}(t_0) > 1 - \epsilon \quad whenever\ n, m < N. \tag{6}$$

Then, from (5) and (6), one obtains

$$F_{\omega_n,\omega_m}(t_0) > 1 - \epsilon \quad whenever\ n, m \in \mathbb{N}.$$

Hence,

$$\theta_{\wp(\omega)}(t_0) > 1 - \epsilon,$$

which implies also that for all $t_1 \geq t_0$ one gets

$$\theta_{\wp(\omega)}(t_1) \geq \theta_{\wp(\omega)}(t_1) > 1 - \epsilon.$$

Since that $\epsilon > 0$ is arbitrary, there is $t_1 > 0$ such that

$$\mathcal{D}_{\wp(\omega)}(t_1) > 1 - \epsilon.$$

Thus,

$$\mathcal{D}_{\wp(\omega)}(t_1) \to 1 \quad as\ t_1 \to +\infty.$$

The proof is completed. □

Lemma 5. *Let* $(\Gamma, \digamma, \daleth, s)$ *be a complete b-Menger space where* $Ran\digamma \subset \mathcal{D}^+$ *and* $f : \Gamma \to C(\Gamma)$ *is a multivalued* ψ*-probabilistic contraction mapping on* Γ *with* $\psi \in \chi$*. If the t-norm* $\daleth$ *is of H-type, then for all* $\omega \in \Gamma$*, the orbit* $\wp_f(\omega)$ *is bounded.*

Proof. Let $\omega \in \Gamma$ and $\{\omega_n\}$ be a sequence of the orbit $\wp_f(\omega)$ starting at ω. From Lemma 4, it suffices to show that $\{\omega_n\}$ is a Cauchy sequence.
Since f is ψ-probabilistic, then there exists $\psi \in \chi$ such that

$$\digamma_{\omega_n,\omega_{n+1}}(\psi(t)) \geq \digamma_{\omega_{n-1},\omega_n}(st) \quad for\ all\ n \in \mathbb{N},\ and\ t > 0.$$

Then, by induction, it is shown, as in the proof of the Theorem, that

$$\digamma_{\omega_n,\omega_{n+k}}(\psi(t)) \geq \digamma_{\omega_{n-1},\omega_{n+k-1}}(st) \quad for\ all\ k > 0. \tag{7}$$

Next, for $k = 1$, one obtains that

$$\begin{aligned}\digamma_{\omega_n,\omega_{n+1}}(\psi^n(t)) &\geq \digamma_{\omega_0,\omega_1}(st)\\ &\geq \digamma_{\omega_0,\omega_1}(t) \quad for\ all\ n \in \mathbb{N}\ and t > 0.\end{aligned}$$

Since $\lim_{t\to+\infty} \digamma_{\omega_0,\omega_1}(t) = 1.$
Then, for any $\epsilon \in (0, 1]$, there exists $t_0 > 0$ such that

$$\digamma_{\omega_0,\omega_1}(t_0) > 1 - \epsilon.$$

As $\psi \in \chi$, then there exists $t_1 \geq t_0$ such that

$$\lim_{n\to+\infty} \psi^n(t_1) = 0.$$

So for any $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\psi^n(t_1) < t \quad for\ all\ n \geq n_0.$$

By the monotonicity of F, one has for all $n \geq n_0$,

$$\begin{aligned}\digamma_{\omega_n,\omega_{n+1}}(t) &\geq \digamma_{\omega_n,\omega_{n+1}}(\psi^n(t_1))\\ &\geq \digamma_{\omega_0,\omega_1}(t_1)\\ &> \digamma_{\omega_0,\omega_1}(t_0)\\ &> 1 - \epsilon.\end{aligned}$$

Hence

$$\lim_{n\to+\infty} F_{\omega_n,\omega_{n+1}}(t) = 1 \quad for\ all\ t > 0. \tag{8}$$

Now, one proves by induction that for any $k > 2$,

$$F_{\omega_n,\omega_{n+k}}(st) \geq \daleth^{k-1}\left(F_{\omega_n,\omega_{n+1}}(t-\psi(t))\right). \tag{9}$$

Inequality (9) is satisfied for $k = 3$.
Now, suppose that (9) holds for $k > 2$.
Using (7), the monotonicity of $\daleth$ and the induction hypothesis, one obtains

$$\begin{aligned} F_{\omega_n,\omega_{n+k+1}}(st) &= F_{\omega_n,\omega_{n+k+1}}\left(s(t-\psi(t)) + s\psi(t)\right) \\ &\geq \daleth(F_{\omega_n,\omega_{n+1}}(t-\psi(t)), F_{\omega_{n+1},\omega_{n+1+k}}(\psi(t)) \\ &\geq \daleth(F_{\omega_n,\omega_{n+1}}(t-\psi(t)), F_{\omega_n,\omega_{n+k}}(st)) \\ &\geq \daleth(F_{\omega_n,\omega_{n+1}}(t-\psi(t)), \daleth^{k-1}(F_{\omega_n,\omega_{n+1}}(t-\psi(t))) \\ &= \daleth^{k}(F_{\omega_n,\omega_{n+1}}(t-\psi(t))). \end{aligned}$$

Hence, (9) is proved for all $k > 2$.
Now, let $\epsilon \in (0,1)$ be given. From that $\daleth$ is of H-type, there exists $\delta > 0$ such that

$$\daleth^{n}(t) > 1-\epsilon \quad for\ all\ t \in (1-\delta, 1]\ and\ n \in \mathbb{N}. \tag{10}$$

As $\frac{t-\psi(t)}{s} > 0$, then by (8), one has that

$$\lim_{n\to+\infty} F_{\omega_n,\omega_{n+1}}\left(\frac{t-\psi(t)}{s}\right) = 1.$$

So, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ one obtains

$$F_{\omega_n,\omega_{n+1}}\left(\frac{t-\psi(t)}{s}\right) > 1-\delta.$$

Finally, from (9) and (10), one gets

$$F_{\omega_n,\omega_{n+k}}(t) \geq \daleth^{k-1}\left(F_{\omega_n,\omega_{n+1}}\left(\frac{t-\psi(t)}{s}\right)\right) > 1-\epsilon \quad for\ all\ n \geq N,\ and\ k > 1.$$

Therefore, $\{\omega_n\}$ is a Cauchy sequence, which implies from Lemma 5 that $\wp_f(\omega)$ is bounded. □

As a direct consequence of Theorem 1, Lemma 4, and Lemma 5, one obtains the following result

Corollary 1. *Let* $(\Gamma, \digamma, \daleth, s)$ *be a complete b-Menger space with* $\daleth$ *is of H-type and* $f : \Gamma \to C(\Gamma)$ *is a probabilistic* ψ*-contraction mapping where* $\psi \in \chi$*. Then there exists* $z \in \Gamma$ *satisfying* $z \in fz$*.*

Next, here is an example to illustrate corollary 1.

Example 4. Let $\Gamma = [0, +\infty)$. Define $\digamma : \Gamma \times \Gamma \to \Delta^+$ as follows

$$\digamma_{\omega,\upsilon}(t) = \epsilon_0(t - |\omega - \upsilon|^2).$$

It is easy to check that $(\Gamma, \digamma, \daleth_M, 2)$ is a complete b-Menger space with $\daleth_M$ is of H-type. And one considers the function $f : \Gamma \to C(\Gamma)$ given by $f(\omega) = \{1, \frac{\omega}{2}, \frac{\omega}{3}\}$.

Then, for any $\omega, y \in \Gamma$ and $\rho \in f\omega$, there are the following cases:

Case 1 If $\rho = 1 \in f\omega$, then one chooses $\rho = \sigma$.

Case 2 If $\rho = \frac{\omega}{2} \in f\omega$, then one chooses $\sigma = \frac{\upsilon}{2}$.

Case 3 If $\rho = \frac{\omega}{3} \in f\omega$, then one chooses $\sigma = \frac{\upsilon}{3}$.

Now, it is necessary to show that the ψ-contraction is satisfied with $\varphi(t) = \frac{1}{2}t$.

For case 1, there is

$$\digamma_{\rho,\sigma}(\frac{1}{2}t) = \epsilon_0(\frac{1}{2}t) = \epsilon_0(2t),$$

hence

$$\begin{aligned}\digamma_{\rho,\sigma}(\frac{1}{2}t) &\geq \epsilon_0(2t - |\omega - \upsilon|^2)\\ &= \digamma_{\omega,\upsilon}(2t).\end{aligned}$$

Case 2 gives

$$\digamma_{\rho,\sigma}(\frac{1}{2}t) = \epsilon_0(\frac{1}{2}t - \left|\frac{\omega}{2} - \frac{\upsilon}{2}\right|^2) = \epsilon_0(2t - |\omega - \upsilon|^2) = \digamma_{\omega,\upsilon}(2t).$$

Finally, for case 3, one obtains

$$\digamma_{\rho,\sigma}(\frac{1}{2}t) = \epsilon_0(\frac{1}{2}t - \left|\frac{\omega}{3} - \frac{\upsilon}{3}\right|^2) = \epsilon_0(\frac{9}{2}t - |\omega - \upsilon|^2),$$

hence

$$\digamma_{\rho,\sigma}(\frac{1}{2}t) \geq \epsilon_0(2t - |\omega - \upsilon|^2)$$

$$= F_{\omega,\upsilon}(2t).$$

Thus, all the conditions of the above Corollary are satisfied, which implies that f admits a fixed point.

Coincidence point theorems in a fuzzy b-metric space

Prior to announcing the coincidence, recall first the Definition of a fuzzy b-metric space.

Definition 8. *A quadruple* $(\Gamma, \mathcal{Z}, \urcorner, s)$ *where* Γ *is an arbitrary nonempty set,* Γ *is a continuous t-norm,* $\mathcal{Z}$ *is a fuzzy set on* $\Gamma \times \Gamma \times (0, +\infty)$ *and* $s \geq 1$ *is a real number, is called a fuzzy b-metric space if the following conditions are verified:*

1. $\mathcal{Z}(\alpha, \beta, 0) = 0$*,*
2. $\mathcal{Z}(\alpha, \beta, t) = 1$ *for all* $t > 0$ *if and only if* $\alpha = \beta$*,*
3. $\mathcal{Z}(\alpha, \beta, t) = \mathcal{Z}(\beta, \alpha, t)$*,*
4. $\mathcal{Z}(\alpha, \gamma, s(t + \upsilon)) \geq \urcorner(\mathcal{Z}(\alpha, \beta, t), \mathcal{Z}(\beta, \gamma, \upsilon))$*,*
5. $\mathcal{Z}(\alpha, \beta, .) : [0, +\infty) \longrightarrow [0, 1]$ *is left-continuous and nondecreasing for all* $\alpha, \beta, \gamma \in \Gamma$ *and* $t, \upsilon > 0$*.*

When $s = 1$ *then* $(\Gamma, \mathcal{Z}, \urcorner, s)$ *is a fuzzy metric space in the form of Kramosil and Michalek (Kramosil & Michálek, 1975).*

The definition of a multivalued ψ-contraction in a fuzzy b-metric version is given as the following.

Definition 9. *Let* $(\Gamma, \mathcal{Z}, \urcorner, s)$ *be a fuzzy b-metric space and* $\psi : [0, +\infty) \to [0, +\infty)$*. A mapping* $f : \Gamma \to C(\Gamma)$ *is called a multivalued fuzzy* ψ*-contraction if for every* $\omega, \upsilon \in \Gamma$ *and every* $\rho \in f\omega$ *there exists* $\sigma \in f\upsilon$ *such that*

$$\mathcal{Z}(\rho, \sigma, \psi(t)) \geq \mathcal{Z}(\omega, \upsilon, st) \quad for\ all\ t > 0.$$

Theorem 2. *Let* $(\Gamma, \mathcal{Z}, \urcorner, s)$ *be a complete fuzzy b-metric space where* $\lim_{t \to +\infty} \mathcal{Z}(\omega, \upsilon, t) = 1$ *for all* $\omega, \upsilon \in \Gamma$ *and* $f : \Gamma \to C(\Gamma)$ *is a multivalued fuzzy* ψ*-contraction mapping with* $\psi \in \chi$*. If all the orbits* $\wp_{f(\omega)}$ *for some* $\omega \in \Gamma$ *are bounded, then there exists* $z \in \Gamma$ *satisfying* $z \in fz$*.*

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Proof. From that $\mathcal{Z}(\omega, \upsilon, .)$ is left-continuous and nondecreasing mapping for all $\omega, \upsilon \in \Gamma$, then by taking $F_{\omega,\upsilon}(t) = F(\omega, \upsilon, t)$ for all $t > 0$ and since the condition of $F_{\rho,\sigma}(+\infty) = 1$ has not been used in the proof of theorem 1, it implies that this result holds. □

Similarly, from Corollary 1, one obtains

Corollary 2. *Let* $(\Gamma, \mathcal{Z}, \daleth, s)$ *be a complete fuzzy b-metric space with* $\daleth$ *is of H-type and* $f : \Gamma \to C(\Gamma)$ *is a fuzzy* ψ*-contraction mapping where* $\psi \in \chi$*. Then there exists* $z \in \Gamma$ *satisfying* $z \in fz$*.*

Conclusion

In summary, the novel approach applied in this study has led to significant advancements, generalizing and enhancing the results originally proposed by Fang (Fang, 1992) and Hadžić (Hadžic, 1989). These achievements mark a notable contribution to the fixed point theory literature, particularly in the context of multivalued maps within probabilistic metric spaces. Additionally, the authors introduced and defined the concept of multivalued ψ-contraction in a b-Menger space, extending it to encompass fuzzy b-metric spaces. Moreover, this exploration uncovered a meaningful connection between the boundedness of orbits and the H-type t-norms, providing valuable insights into the interplay between these concepts. As a consequential outcome, coincidence point theorems applicable to fuzzy b-metric spaces are derived, adding a new dimension to the understanding of these spaces and their applications in the context of multivalued mappings. This comprehensive study not only broadens the theoretical foundations but also opens avenues for further research and exploration in the rich and diverse field of the fixed point theory.

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Explorando la probabilidad multivaluada ψ-contracciones con órbitas en espacios b-Menger

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CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: El artículo presenta un enfoque novedoso para ciertos teoremas bien establecidos de punto fijo para contracciones probabilísticas multivaluadas en espacios de b-Menger, aprovechando la acotación de las órbitas. El objetivo era generalizar y mejorar los resultados obtenidos anteriormente por Fang y Hadžić.

Métodos: La delimitación de las órbitas en los espacios b-Menger se utiliza para establecer su enfoque para contracciones probabilísticas multivaluadas.

Resultados: Los hallazgos del estudio no sólo generalizaron los teoremas del punto fijo existentes, sino que también los mejoraron significativamente. Se demostró la eficacia del enfoque para ampliar los resultados propuestos originalmente por Fang y Adžić. Además, se demostró la aplicabilidad del teorema del punto fijo de coincidencia en espacios b-métricos difusos.

Conclusión: El estudio presentó una perspectiva novedosa sobre los teoremas del punto fijo en contracciones probabilísticas multivaluadas dentro de espacios de b-Menger. Al aprovechar la acotación e introducir un teorema de punto fijo de coincidencia para espacios b-métricos difusos, el trabajo contribuyó al avance en este campo.

Palabras claves: punto fijo, espacios b-Menger, contracción ψ-multivaluada, espacio b-métrico difuso.

Исследование многозначной вероятности ψ-сжатия с орбитами в b-пространстве Менгера

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РУБРИКА ГРНТИ: 27.25.17 Метрическая теория функций,
27.39.15 Линейные пространства, снабженные топологией, порядком и другими структурами
ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: В данной статье представлен новый подход к некоторым общепризнанным теоремам о неподвижной точке для многозначных вероятностных сокращений в b-менгеровских пространствах с использованием ограниченности орбит. Целью данной статьи было обобщить и улучшить предыдущие результаты, полученные Фангом и Хаджичем.

Методы: Ограниченность орбит в b-пространствах Менгера используется для определения их подхода к многозначным вероятностным сжатиям.

Результаты: Результаты исследования не только обобщили существующие теоремы о неподвижной точке, но и существенно их усовершенствовали. Была продемонстрирована эффективность подхода в расширении результатов, первоначально предложенного Фангом и

Хаджичем. Кроме того, была продемонстрирована применимость теоремы о неподвижных точках и точках совпадения в нечетких b-метрических пространствах.

Выводы: В исследовании представлен новый взгляд на теоремы о неподвижных точках в многозначных вероятностных сжатиях в b-пространствах Менгера. Используя ограниченность и вводя теорему о совпадении неподвижных точек для нечетких b-метрических пространств, данная статья вносит большой вклад в изучение данной области.

Ключевые слова: неподвижная точка, b-пространства Менгера, многозначное ψ-сжатие, нечеткое b-метрическое пространство.

Истраживање вишевредносне вероватноће ψ-контракције са орбитама у b-Менгеровим просторима

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КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Рад представља нови приступ одређеним добро утврђеним теоремама о фиксној тачки за вишезначне вероватносне контракције у b-Менгеровим просторима, користећи ограниченост орбите. Циљ је био да се генерализују и побољшају претходни резултати које су извели Фанг и Хаџић.

Методе: Коришћене су ограничености орбита у b-Менгеровим просторима које успостављају свој приступ за вишезначне вероватносне контракције.

Резултати: Налази студије нису само генерализовали постојеће теореме о фиксној тачки већ су их и значајно побољшали. Представљена је и ефективност приступа у проширењу резултата који су првобитно предложили Фанг *и Хаџић. Такође, демонстрирана је примењивост теореме коинциденције о фиксној тачки у расплинутим b-метричким просторима.*

Закључак: Студија је представила нову перспективу теореме фиксне тачке у вишезначним вероватносним контракцијама унутар b-Менгерових простора. Коришћење ограничености и увођење фиксне случајности теорема тачке за расплинуте b-метричке просторе представља допринос унапређењу ове области.

Кључне речи: фиксна тачка, b-Менгерови простори, вишезначно пресликавање ψ-контракција, расплинута b-метрика.

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