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On some fixed point results for expansive mappings in S-metric spaces

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Abstract:

Introduction/purpose: The aim of this paper is to establish some existence results of a fixed point for a class of expansive mappings defined on a complete S-metric space.
Methods: An iteration scheme was used.
Results: Some existing results of mappings satisfying contractive conditions are expanded to expansive ones, providing a new condition expressed in one variable under which the existence of a fixed point holds.
Conclusions: This work provides new tools to fixed point theory together with their applications.
Key words: fixed point, S-metric space, expansive mapping.

Introduction and preliminaries

One of the most important instruments to treat nonlinear problems with the aid of functional analytic methods is the fixed point approach. Metric fixed point theory provides essential tools for solving problems emerging from various areas of mathematical analysis, such as variational and linear inequalities, equilibrium problems, complementarity problems, optimization and approximation theory, as well as problems of proving the existence of a solution of integral and differential equations. The first important result of fixed points in metric spaces was the well known contraction mapping theorem, established by S. Banach in his dissertation and published for the first time in 1922. Later, some generalized metric spaces have been studied to obtain new fixed point theorems.

In 2006, Z. Mustafa and B. Sims (Mustafa & Sims, 2006) introduced the notion of G-metric spaces as a generalization of ordinary metric spaces, and analysed the topological structure of G-metric spaces. In this first paper, the authors developed some fixed point results for various classes of mappings in the setting of a G-metric space. For this and more details, the reader can see (Abbas et al., 2016; An et al., 2015; Dosenovic et al., 2018; Vujaković et al., 2023). In 2012, S. Sedghi, N. Shobe and A. Aliouche (Sedghi et al., 2012) introduced S-metric spaces as a generalization of G-metric spaces and metric spaces, and proved several fixed point results in the setting of S-metric spaces. For other results, see (Mojaradiafra, 2016; Mojaradiafra & Sabbaghan, 2021; Sedghi & Dung, 2014). In 1984, (Wang et al., 1984), introduced the concept of expanding mappings and proved some fixed point theorems in metric spaces. For more details on expanding mappings and related results, we refer the reader to (Mohanta, 2012; Mojaradiafra et al., 2020; Mustafa et al., 2010). This paper establishes some existence results of a fixed point for a class of expansive mappings in S-metric spaces. Some existing results from G-metric and S-metric spaces are extended. The contribution of this paper is in providing new tools to fixed point theory together with their applications. Let us recall some basic definitions and properties of S-metric spaces.

Definition 1. *(Sedghi et al., 2012) Let X be a non-empty set. An S-metric on X is a function*
$S : X \times X \times X \to [0, +\infty[$ that satisfies the following conditions for all $x, y, z, a \in X$:

(S1) $S(x, y, z) = 0$ *if and only if* $x = y = z$,

(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Example 1.
1. Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.
2. Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.
3. Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.

DEFINITION 2. *(Bakhtin, 1989) Let X be a nonempty set. A b-metric on X is a function $d : X \times X \to [0, +\infty[$ if there exists a real number $b \geq 1$ such that the following conditions hold for all $x, y, z \in X$.*

B1 $d(x, y) = 0$ *if and only if* $x = y$,

B2 $d(x, y) = d(y, x)$,

B3 $d(x, z) \leq b[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space.

We will prove that every S-metric space (X, S) will define a b-metric space (X, d).

PROPOSITION 1. *(Sedghi & Dung, 2014) Let (X, S) an S-metric space and let*

$$d(x, y) = S(x, x, y)$$

for all $x, y \in X$. Then

1. *d is a b-metric on X,*
2. *$x_n \to x$ in (X, S) if and only if $x_n \to x$ in (X, d),*
3. *(x_n) is a Cauchy sequence in (X, S) if and only if (x_n) is a Cauchy sequence in (X, d).*

DEFINITION 3. *(Sedghi et al., 2012) Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, we define the open ball $B_S(x, r)$ and the closed ball $B_S[x, r]$ with the center x and the radius r as follows*

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\} \quad (1)$$
$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}. \quad (2)$$

The topology induced by the S-metric is the topology generated by the base of all open balls in X.

DEFINITION 4. *(Sedghi et al., 2012) Let (X, S) be an S-metric space.*

1. *A sequence (x_n) converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to +\infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one gets $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.*
2. *A sequence (x_n) is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to +\infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ one gets $S(x_n, x_n, x_m) < \varepsilon$.*

3. *The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.*

LEMMA 1. *(Sedghi et al., 2012) Let (X, S) be an S-metric space. If the sequence (x_n) in x converges to x, then x is unique.*

The next three lemmas are well known, see for example (Sedghi et al., 2012).

LEMMA 2. *In an S-metric space, there exists*

$$S(x, x, y) = S(y, y, x),$$

for all $x, y \in X$.

LEMMA 3. *Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ then $S(x_n, x_n, y_n) \to S(x, x, y)$.*

LEMMA 4. *(Mojaradiafra et al., 2020) Any S-metric space is Hausdorff.*

DEFINITION 5. *Let (X, S_1) and (Y, S_2) be S-metric spaces. A map $f : X \to Y$ is called continuous at $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*
$S_1(x, x, y) < \delta$ implies that $S_2(f(x), f(x), f(y)) < \varepsilon$, or

$$f(B_{S_1}(x, \delta)) \subset B_{S_2}(f(x), \varepsilon).$$

The next result is also known, see (Sedghi et al., 2012).

LEMMA 5. *Let (X, S) be an S-metric space. The map $f : X \to X$ is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

DEFINITION 6. *Let (X, S) be an S-metric space. A map $T : X \to X$ is said to be a contraction if there exists a constant $0 \leq k < 1$ such that*

$$S(Tx, Tx, Ty) \leq kS(x, x, y), \text{ for all } x, y \in X.$$

THEOREM 1. *Let (X, S) be a complete S-metric space and $T : X \to X$ be a contraction. Then T has a unique fixed point.*

DEFINITION 7. *Let* (X, S) *be an* S*-metric space and* T *be a self-map on* X*. Then* T *is called an expansive map if there exists a constant* $a > 1$ *such that for all* $x, y \in X$*, one gets*

$$S(Tx, Tx, Ty) \geq aS(x, x, y).$$

The constant a *is called the expansion coefficient.*

REMARK 1. Expansive map on an S-metric space does not need to be continuous.

THEOREM 2. *(Mojaradiafra et al., 2020) Let* (X, S) *be a complete S-metric space, and let* $T : X \to X$ *be an onto continuous mapping satisfying*

$$S(Tx, Tx, T^2x) \geq aS(x, x, Tx) \tag{3}$$

for all $x \in X$*, where* $a > 1$*. Then* T *has a fixed point in* X*.*

EXAMPLE 2. (Mojaradiafra et al., 2020) Let
$T : (\mathbb{R}, S) \to (\mathbb{R}, S)$ be defined by

$$T(x) = \begin{cases} 4x \text{ for } x \leq 2, \\ 4x + 3 \text{ for } x > 2, \end{cases}$$

where $S(x, y, z) = \max\{|x - z|, |y - z|\}$. Then $(\mathbb{R}, S)$ is a complete S-metric space and T is an expansive map with the coefficient $a = 2$.

Main results

Our first main result is as follows:

THEOREM 3. *Let* (X, S) *be a complete* S*-metric space. Let* $T : X \to X$ *be an onto mapping such that,*

$$S(Tx, Ty, Tz) \geq aN(x, y, z) \tag{4}$$

for all x, y, z*, where* $a > 1$ *and*

$$N(x, y, z) = \min \left\{ \begin{array}{c} S(x, y, Tx), S(Tx, Ty, y), S(Tx, Ty, x), S(Tx, Tx, y), \\ S(z, z, x), S(Tz, Tz, T^2x), S(T^2y, T^2x, Tz), S(Tx, Ty, z), \\ S(x, y, z), S(Tx, Tx, x), S(Ty, Ty, y), S(T^2x, T^2y, Tz), \\ S(Tx, Tx, z), S(Ty, Ty, x), S(Tz, Tz, T^2y) \end{array} \right\}. \tag{5}$$

Then T admits a fixed point.

Proof. Let $x_0 \in X$, since T is onto then there exists an element x_1 satisfying $x_1 \in T^{-1}(x_0)$. Continuing in this way, one gets a sequence (x_n), where $x_n \in T^{-1}(x_{n-1})$. If $x_n = x_{n-1}$ for some n, then x_n is a fixed point of T. Assume $x_n \neq x_{n-1}$ for every $n \in \mathbb{N}$, then from (5) one obtains

$$S(x_{n-1}, x_{n-1}, x_{n-2}) = S(Tx_n, Tx_n, Tx_{n-1}) \geq aN(x_n, x_n, x_{n-1}),$$

where

$$N(x_n, x_n, x_{n-1}) = \tag{6}$$
$$= \min \left\{ \begin{array}{c} S(x_n, x_n, Tx_n), S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, x_n), \\ S(Tx_n, Tx_n, x_n), S(x_{n-1}, x_{n-1}, x_n), S(Tx_{n-1}, Tx_{n-1}, T^2x_n), \\ S(T^2x_n, T^2x_n, Tx_{n-1}), S(Tx_n, Tx_n, x_{n-1}), S(x_n, x_n, x_{n-1}), \\ S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, x_n), S(T^2x_n, T^2x_n, Tx_{n-1}), \\ S(Tx_n, Tx_n, x_{n-1}), S(Tx_n, Tx_n, x_n), S(Tx_{n-1}, Tx_{n-1}, T^2x_n) \end{array} \right\},$$

so

$$N(x_n, x_n, x_{n-1}) = \tag{7}$$
$$= \min \left\{ \begin{array}{c} S(x_n, x_n, x_{n-1}), S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), \\ S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_{n-2}, x_{n-2}, x_{n-2}), \\ S(x_{n-2}, x_{n-2}, x_{n-2}), S(x_{n-1}, x_{n-1}, x_{n-1}), S(x_n, x_n, x_{n-1}), \\ S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_{n-2}, x_{n-2}, x_{n-2}), \\ S(x_{n-1}, x_{n-1}, x_{n-1}), S(x_{n-1}, x_{n-1}, x_n), S(x_{n-2}, x_{n-2}, x_{n-2}) \end{array} \right\},$$

then

$$S(x_{n-1}, x_{n-1}, x_{n-2}) \geq aS(x_n, x_n, x_{n-1}),$$

this implies that

$$S(x_n, x_n, x_{n-1}) \leq \frac{1}{a} S(x_{n-1}, x_{n-1}, x_{n-2}). \tag{8}$$

Set $k = \frac{1}{a}$, then $k < 1$. By induction one obtains

$$\begin{aligned} S(x_n, x_n, x_{n-1}) &\leq kS(x_{n-1}, x_{n-1}, x_{n-2}) \\ &\leq k^2 S(x_{n-2}, x_{n-2}, x_{n-3}) \end{aligned} \tag{9}$$

$$\begin{aligned} &\leq \quad ... \\ &\leq \quad k^{n-1}S(x_1,x_1,x_0). \end{aligned}$$

Then, by $(S2)$ and Lemma 2, we get for all $n,m \in \mathbb{N}$; $n < m$,

$$\begin{aligned} S(x_m,x_m,x_n) &\leq 2S(x_m,x_m,x_{m-1}) + S(x_n,x_n,x_{m-1}) \\ &= 2S(x_m,x_m,x_{m-1}) + S(x_{m-1},x_{m-1},x_n) \\ &\leq 2S(x_m,x_m,x_{m-1}) + 2S(x_{m-1},x_{m-1},x_{m-2}) \\ &+ S(x_n,x_n,x_{m-2}) \\ &= 2S(x_m,x_m,x_{m-1}) + 2S(x_{m-1},x_{m-1},x_{m-2}) \\ &+ S(x_{m-2},x_{m-2},x_n) \\ &... \\ &\leq 2\sum_{i=n+2}^{m} S(x_i,x_i,x_{i-1}) + S(x_{n+1},x_{n+1},x_n), \end{aligned} \tag{10}$$

by (10), we obtain

$$\begin{aligned} S(x_m,x_m,x_n) &\leq 2\sum_{i=n+2}^{m} S(x_i,x_i,x_{i-1}) + S(x_{n+1},x_{n+1},x_n) \\ &\leq 2\sum_{i=n+2}^{m} k^{i-1}S(x_1,x_1,x_0) + k^nS(x_1,x_1,x_0) \\ &\leq 2k^{n+1}\frac{1-k^{m-n-1}}{1-k}S(x_1,x_1,x_0) + k^nS(x_1,x_1,x_0) \\ &\leq 2\frac{k^{n+1}}{1-k}S(x_1,x_1,x_0) + k^nS(x_1,x_1,x_0), \end{aligned} \tag{11}$$

so, $S(x_m,x_m,x_n) \to 0$ as $n,m \to +\infty$ and (x_n) is a Cauchy sequence. Since (X,S) is complete, then there exists $u \in X$ such that (x_n) is convergent to u.

We need to show that $Tu = u$, let $y \in T^{-1}(u)$, for n such that $x_n \neq u$, we get

$S(x_n,x_n,u) = S(Tx_{n+1},Tx_{n+1},Ty) \geq aN(x_{n+1},x_{n+1},y)$, where

$$N(x_{n+1},x_{n+1},y) = \tag{12}$$

$$= \min \left\{ \begin{array}{l} S(x_{n+1}, x_{n+1}, x_n), S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), \\ S(x_n, x_n, x_{n+1}), S(y, y, x_{n+1}), S(Ty, Ty, x_{n-1}), \\ S(x_{n-1}, x_{n-1}, Ty), S(x_n, x_n, y), S(x_{n+1}, x_{n+1}, y), \\ S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), S(x_{n-1}, x_{n-1}, Ty), \\ S(x_n, x_n, y), S(x_n, x_n, x_{n+1}), S(Ty, Ty, x_{n-1}) \end{array} \right\},$$

taking the limit as $n \to +\infty$ we obtain $S(u, u, y) \leq 0$, which implies that $Ty = u = y$; hence u is a fixed point of T.

□

Our second result is given by

THEOREM 4. *Let (X, S) be a complete S-metric space and let $T : X \to X$ be an onto S-continuous mapping. Assume that there exist nonnegative reals a, b, c, d, e, f, g with $b < 1$, and $a + b + c + d + e + f + g > 1$ such that*

$$\begin{aligned} & S^4(Tx, Ty, Tz) \geq \\ & \geq aS^4(x, y, z) + bS^4(T^2x, T^2y, z) + cS^4(Tx, Ty, x) + \\ & + dS^2(Tz, T^2y, Tx)S^2(y, x, Tx) + eS^2(z, Tx, y)S^2(z, Tx, y) + \\ & + fS^2(z, Tx, T^2y)S^2(Tx, Ty, y) + gS^2(T^2x, T^2x, y)S^2(y, x, Tx), \end{aligned} \tag{13}$$

for all $x \in X$, then T has a fixed point in X.

Proof. Let $x_0 \in X$, since T is onto there exists $x_1 \in T^{-1}(x_0)$. Continuing in this way, we get a sequence (x_n), where $x_n \in T^{-1}(x_{n-1})$.
If, $x_n = x_{n-1}$ for some n, then we obtain x_n as a fixed point of T. Hence, without loss of generality, we may assume that $x_n \neq x_{n-1}$ for every $n \in \mathbb{N}$.
By (13), we get

$$\begin{aligned} S^4(x_{n-1}, x_{n-1}, x_{n-2}) &= S^4(Tx_n, Tx_n, Tx_{n-1}) \\ &\geq aS^4(x_n, x_n, x_{n-1}) + bS^4(x_{n-2}, x_{n-2}, x_{n-1}) \\ &+ +cS^4(x_{n-1}, x_{n-1}, x_n) \\ &+ dS^2(x_{n-2}, x_{n-2}, x_{n-1})S^2(x_n, x_n, x_{n-1}) \\ &+ eS^2(x_{n-1}, x_{n-1}, x_n)S^2(x_{n-1}, x_{n-1}, x_n) \\ &+ fS^2(x_{n-1}, x_{n-1}, x_{n-2})S^2(x_{n-1}, x_{n-1}, x_n) \\ &+ gS^2(x_{n-2}, x_{n-2}, x_n)S^2(x_n, x_n, x_{n-1}), \end{aligned}$$

hence, by Lemma 2 we obtain

$$(a + c)S^4(x_n, x_n, x_{n-1}) + \tag{14}$$

$$+ (d+e+f+g)S^2(x_n,x_n,x_{n-1})S^2(x_{n-1},x_{n-1},x_{n-2}) - \\ - (1-b)S^4(x_{n-1},x_{n-1},x_{n-2}) \leq 0,$$

which is equivalent to

$$(a+c)r^4 + (d+e+f+g)r^2 - (1-b) \leq 0, \tag{15}$$

where

$$r = \frac{S(x_n,x_n,x_{n-1})}{S(x_{n-1},x_{n-1},x_{n-2})}.$$

Let $f : [0,+\infty[\to \mathbb{R}$ be the function given by

$$f(r) = (a+c)r^4 + (d+e+f+g)r^2 - (1-b) = 0,$$

then, using assumptions, we get $f(0) = b-1 < 0$ and $f(1) = a+b+c+d+e+f+g-1 > 0$.
One can deduce that, there exists $k \in]0,1[$ for which inequality (15) holds whenever $r \leq k$, and hence

$$\begin{aligned} S(x_n,x_n,x_{n-1}) &\leq kS(x_{n-1},x_{n-1},x_{n-2}) \\ &\leq k^2S(x_{n-2},x_{n-2},x_{n-3}) \\ &\leq \ldots \\ &\leq k^{n-1}S(x_1,x_1,x_0), \end{aligned}$$

then, by $(S2)$ and Lemma 2, we get for all $n,m \in \mathbb{N}$; $n < m$,

$$\begin{aligned} S(x_m,x_m,x_n) &\leq 2S(x_m,x_m,x_{m-1}) + S(x_n,x_n,x_{m-1}) \\ &= 2S(x_m,x_m,x_{m-1}) + S(x_{m-1},x_{m-1},x_n) \\ &\leq 2S(x_m,x_m,x_{m-1}) + 2S(x_{m-1},x_{m-1},x_{m-2}) \\ &+ S(x_n,x_n,x_{m-2}) \\ &= 2S(x_m,x_m,x_{m-1}) + 2S(x_{m-1},x_{m-1},x_{m-2}) \\ &+ S(x_{m-2},x_{m-2},x_n) \\ &\ldots \\ &\leq 2\sum_{i=n+2}^{m} S(x_i,x_i,x_{i-1}) + S(x_{n+1},x_{n+1},x_n), \end{aligned} \tag{16}$$

by (16), we obtain

$$S(x_m,x_m,x_n) \leq 2\sum_{i=n+2}^{m} S(x_i,x_i,x_{i-1}) + S(x_{n+1},x_{n+1},x_n) \tag{17}$$

$$\begin{aligned}
&\leq 2\sum_{i=n+2}^{m} k^{i-1}S(x_1,x_1,x_0)+k^nS(x_1,x_1,x_0)\\
&\leq 2k^{n+1}\frac{1-k^{m-n-1}}{1-k}S(x_1,x_1,x_0)+k^nS(x_1,x_1,x_0)\\
&\leq 2\frac{k^{n+1}}{1-k}S(x_1,x_1,x_0)+k^nS(x_1,x_1,x_0),
\end{aligned}$$

so, $S(x_m,x_m,x_n)\to 0$ as $n,m\to+\infty$ and (x_n) is a Cauchy sequence, and by the completeness of (X,S), there exists $u\in X$ such that (x_n) converges to u. By continuity of T, we get

$$T(x_n)=x_{n-1}\to Tu,$$

thus $u=Tu$. We conclude that u is a fixed point of T. □

We state our third result in the sequel.

THEOREM 5. *Let (X,S) be a complete S-metric space and let $T:X\to X$ be an onto continuous mapping satisfying*

$$\begin{aligned}
S(Tx,Ty,Tz)\geq\ & c_1S(Tx,Ty,x)+c_2S(Tz,T^2y,Tx)+c_3S(y,y,Ty)+ \\
&+c_4S(Tx,Tx,Tz)+c_5\left[\frac{S(Ty,z,T^2x)+S(z,Tx,y)+S(T^2x,T^2y,z)}{3}\right]+\\
&+c_6\left[\frac{S(z,Tx,T^2y)+S(T^2y,Tz,Tx)+S(x,y,z)}{3}\right]+\\
&+c_7\left[\frac{S(Tx,Ty,y)+S(Tz,T^2x,Ty)+S(y,y,z)}{3}\right],
\end{aligned}\tag{18}$$

where $c_1,c_2,c_3,c_4,c_5,c_6,c_7$ are non negative reals that verify

$c_1+c_2+c_3+c_4+c_5+c_6+c_7>1,$ *and* $3-3c_2-3c_4-2c_5-2c_6-c_7>0,$

then, T has a fixed point in X.

Proof. Replacing y by x and z by Tx in (18), we obtain

$$\begin{aligned}
S(Tx,Tx,T^2x)\ \geq\ & c_1S(Tx,Tx,x)+c_2S(T^2x,T^2x,Tx)\\
+\ & c_3S(x,x,Tx)+c_4S(Tx,Tx,T^2x)\\
+\ & c_5\left[\frac{S(Tx,Tx,T^2x)+S(Tx,Tx,x)+S(T^2x,T^2x,Tx)}{3}\right]
\end{aligned}\tag{19}$$

$$+\quad c_6\left[\frac{S(Tx,Tx,T^2x)+S(T^2x,T^2x,Tx)+S(x,x,Tx)}{3}\right]$$
$$+\quad c_7\left[\frac{S(Tx,Tx,x)+S(T^2x,T^2x,Tx)+S(x,x,Tx)}{3}\right],$$

without loss of generality, we may assume that $T(x) \neq T^2(x)$, then (19) entails

$$\begin{aligned} S(Tx,Tx,T^2x) \quad &\geq \quad (c_1+\frac{c_5+c_7}{3})S(Tx,Tx,x) \\ &+ \quad (c_2+\frac{c_5+c_6+c_7}{3})S(T^2x,T^2x,Tx) \\ &+ \quad (c_3+\frac{c_6+c_7}{3})S(x,x,Tx) \\ &+ \quad (c_4+\frac{c_5+c_6}{3})S(Tx,Tx,T^2x). \end{aligned} \tag{20}$$

Using Lemma 2, we obtain

$$\begin{aligned} (1-(c_2+c_4+\tfrac{c_5+c_6}{3}+\tfrac{c_5+c_6+c_7}{3}))S(Tx,Tx,T^2x) \geq \\ (c_1+c_3+\tfrac{c_5+c_7}{3}+\tfrac{c_6+c_7}{3})S(x,x,Tx), \end{aligned} \tag{21}$$

which implies that

$$\begin{aligned} (3-3c_2-3c_4-2c_5-2c_6-c_7)S(Tx,Tx,T^2x) \geq (3c_1+3c_3+ \\ +c_5+c_6+2c_7)S(x,x,Tx), \end{aligned} \tag{22}$$

hence

$$S(Tx,Tx,T^2x) \geq \frac{3c_1+3c_3+c_5+c_6+2c_7}{3-3c_2-3c_4-2c_5-2c_6-c_7}S(x,x,Tx). \tag{23}$$

Setting, $a = c_1+c_2+c_3+c_4+c_5+c_6+c_7$, then by assumption we obtain $a > 1$. So, (23) becomes condition (3); therefore, the result follows from Theorem 2. □

Now, we give an example illustrating our result in Theorem 5.

EXAMPLE 3. Let $X = \mathbb{R}$ be the set of real numbers. Define

$$S(x,y,z) = \begin{cases} 0 \text{ if } x=y=z, \\ \max\{|x|,|y|,|z|\} \text{ otherwise.} \end{cases}$$

Then (X,S) is an S-metric space.
Assume

$$Tx = \begin{cases} \sqrt{2}x \text{ if } x < 0, \\ \frac{5}{2}x \text{ if } x \geq 0. \end{cases}$$

We get

$$T^2x = \begin{cases} 2x \text{ if } x < 0, \\ \frac{25}{4} \text{ if } x \geq 0. \end{cases}$$

$$S(x, x, Tx) = |Tx| = \begin{cases} -\sqrt{2}x \text{ if } x < 0, \\ \frac{5}{2}x \text{ if } x \geq 0. \end{cases}$$

$$S(Tx, Tx, T^2x) = |T^2x| = \begin{cases} -2x \text{ if } x < 0, \\ \frac{25}{4}x \text{ if } x \geq 0. \end{cases}$$

Set: $c_1 = \frac{1}{20}, c_2 = \frac{1}{8}$, $c_3 = \frac{1}{5}$, $c_4 = \frac{1}{4}$, $c_5 = \frac{7}{40}$, $c_6 = \frac{3}{20}$ and $c_7 = \frac{3}{40}$, so conditions of Theorem 5 are verified; hence T admits a fixed point in $\mathbb{R}$. One can see that the fixed point here is zero.

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Sobre algunos resultados de punto fijo para mapeos expansivos en espacios S-métricos

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CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: El objetivo de este artículo es establecer algunos resultados de existencia de un punto fijo para una clase de mapeos expansivos definidos en un espacio S-métrico completo.

Métodos: Se utilizó un esquema de iteración.

Resultados: Algunos resultados existentes de mapeos que satisfacen condiciones contractivas se expanden a resultados expansivos, proporcionando una nueva condición expresada en una variable bajo la cual se cumple la existencia de un punto fijo.

Conclusión: Este trabajo proporciona nuevas herramientas a la teoría del punto fijo junto con sus aplicaciones.

Palabras claves: punto fijo, espacio S-métrico, mapeo expansivo.

О некоторых результатах с неподвижной точкой в расширенных отображениях

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РУБРИКА ГРНТИ: 27.25.17 Метрическая теория функций,
27.39.15 Линейные пространства, снабженные топологией, порядком и другими структурами

ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: Цель данной статьи − выявить результаты о наличии неподвижной точки в классе расширяющихся отображений, определенных в полном S-метрическом пространстве.

Методы: В исследовании использована итерационная схема.

Результаты: В данной статье развернуты некоторые из существующих результатов расширяющихся отображений, которые были дополнены новыми сжимающими усло-

виями, выраженными одной переменной, при наличии неподвижной точки.

Выводы: В данной статье представлены способы применения новых инструментов в теории неподвижных точек.

Ключевые слова: неподвижная точка, S-метрическое пространство, расширенное отображение.

О неким резултатима фиксне тачке за експанзивна пресликавања у S-метричким просторима

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Сажетак:

Увод/циљ: Циљ овог рада јесте да установи неке резултате постојања фиксне тачке за класу експанзивних пресликавања дефинисаних на комплетном S-метричком простору.

Методе: У истраживању је примењена шема итерације.

Резултати: Проширени су неки постојећи резултати за експанзивна пресликавања са новим контрактивним условима израженим са једном променљивом под којом постоји фиксна тачка.

Закључак: Ауторски допринос пружа нове алате за теорију фиксне тачке и њихову примену.

Кључне речи: фиксна тачка, S-метрички простор, експанзивно пресликавање.

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