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Existence theorems for a unified interpolative Kannan contraction with an application on nonlinear matrix equations

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Abstract:

Introduction/purpose: This paper established a new mathematical framework by uncovering the relationships between Kannan contractions and interpolative Kannan contractions. The concept of unified interpolative Kannan contractions was introduced in the framework of a relational metric space. Additionally, the study aimed to broaden the concept of alpha admissibility by incorporating specific relational metric ideas.

Methods: A detailed exploration of the properties and characteristics of Kannan contractions and interpolative Kannan contractions was conducted. The research introduced the concept of unified interpolative Kannan contractions and formulated new fixed point results for these mappings.

Result: The study successfully established fixed point results for unified interpolative Kannan contractions within the framework of relational metric spaces. Additionally, an application of these results to solve a problem concerning nonlinear matrix equations was provided, further emphasizing their significance.

Conclusion: The findings of this study significantly advanced the understanding of Kannan contractions and interpolative Kannan contractions, offering a unified framework for their analysis. The introduction of unified *interpolative Kannan contractions and the expansion of alpha admissibility have broad implications for the field of mathematics.*

Key words: unified interpolative Kannan contraction, $\mathcal{R}$-admissible, relational metric space.

Introduction

Kannan made a significant contribution to metric fixed point theory after Banach's influential fixed point theorem. While mappings satisfying the Banach contraction inequality are necessarily continuous, Kannan introduced a novel class of contractions in 1968, addressing the intriguing question of whether discontinuous mappings defined in a complete metric space and satisfying specific contractive conditions could possess a fixed point.

Kannan stated the following result.

THEOREM 1. *Let (X, ∂) be a complete metric space, and $\mathcal{S}$ be a self-map defined on X. If $\mathcal{S}$ is a Kannan contraction (KC, for brief), meaning that there exists a λ in the interval $\left[0, \frac{1}{2}\right)$ such that,*

$$\partial(\mathcal{S}\nu, \mathcal{S}\mu) \leq \lambda[\partial(\nu, \mathcal{S}\nu) + \partial(\mu, \mathcal{S}\mu)], \quad \text{for all } \nu, \mu \in X, \tag{1}$$

then, $\mathcal{S}$ possesses a unique fixed point $\gamma \in X$, and for each $\nu \in X$, the sequence of iterates $\{\mathcal{S}^n\nu\}$ converges to γ.

Kannan's fixed-point theorem represents a notable extension of Banach's remarkable work (Banach, 1922), leading to several generalizations, see (Debnath et al., 2021). Among these, a recent variant introduced by Karapinar, termed as interpolative Kannan-type contraction (or Kannan interpolative contraction), was demonstrated in Karapinar (2018). To guarantee the existence of a fixed point in a complete metric space, this contraction condition allows more flexibility in choosing the constants that control the contraction rate and can also incorporate the distance between points in the contractive condition. Additionally, it is worth mentioning that many classical and advanced contraction concepts have been recently reexamined through interpolation, see (Debnath et al., 2020; Hammad et al., 2023; Jain & Radenović, 2023; Jain et al., 2022; Karapinar, 2021; Karapinar et al., 2018a,b, 2021).

In his work, Karapinar (2018) presents an example that falls outside the scope of Kannan contractions but aligns with interpolative Kannan contractions. This highlights an additional advantage of interpolative Kannan contractions over Kannan contractions. Despite existing research on the subject, there is a notable gap in the literature concerning the converse relationship, i.e., whether Kannan contractions imply interpolative Kannan contractions.

Recent studies Nazam et al. (2023a,b) have suggested that Kannan contractions do indeed imply interpolative Kannan contractions. However, this paper diverges from this perspective and, through illustrative examples, establishes that not every Kannan contraction implies an interpolative Kannan contraction. Consequently, this paper asserts that these two classes of contractions are independent from each other. This comprehensive understanding emphasizes the significance of both contraction types, providing valuable insights into their practical applications.

Karapinar (2018) introduced the concept of an interpolative Kannan contraction as follows:

DEFINITION 1. *A self-mapping $\mathcal{S}$ defined on a metric space (X, ∂) is considered as an interpolative Kannan type contraction (IKC, for brief) if there exists a pair of constant $\alpha, \lambda \in [0, 1)$ with $\alpha \neq 0$, satisfying*

$$\partial(\mathcal{S}\nu, \mathcal{S}\mu) \leq \lambda[\partial(\nu, \mathcal{S}\nu)^{\alpha} \cdot \partial(\mu, \mathcal{S}\mu)^{1-\alpha}], \text{ for all } \nu, \mu \in X, \text{ and } \nu \neq \mathcal{S}\nu. \quad (2)$$

By employing the interpolative Kannan contraction, Karapinar (2018) established a unique fixed point theorem. Subsequently, Karapinar et al. (2018a) identified a limitation in the aforementioned result, highlighting that fixed points obtained from the contractive condition (2) may not necessarily be unique. To illustrate that not every Kannan contraction implies an interpolative Kannan contraction, these authors examine the following example.

EXAMPLE 1. Let $X = [0, 1]$ and consider the mapping $\mathcal{S} : X \to X$ defined by $\mathcal{S}\nu = \frac{\nu}{5}$. Let ∂ denote the usual metric.
Then, one can observe that, $\partial(\mathcal{S}\nu, \mathcal{S}\mu) = \frac{1}{5}|\nu - \mu|, \partial(\nu, \mathcal{S}\nu) = \frac{4\nu}{5}$, and $\partial(\mu, \mathcal{S}\mu) = \frac{4\mu}{5}$.
For $\lambda = \frac{2}{5} \in \left[0, \frac{1}{2}\right)$, one can verify that:

$$\partial(\mathcal{S}\nu, \mathcal{S}\mu) = \frac{1}{5}|\nu - \mu| \leq \frac{2}{5} \cdot \frac{4}{5}(\nu + \mu) = \lambda \cdot [\partial(\nu, \mathcal{S}\nu) + \partial(\mu, \mathcal{S}\mu)].$$

This confirms that $\mathcal{S}$ fulfills condition (1). Now, the next task is to demonstrate that $\mathcal{S}$ does not satisfy (2). Suppose if possible $\mathcal{S}$ satisfies (2),

then, two points are chosen, $\frac{1}{100}$ and $\frac{99}{100}$, from the interval $[0,1]$. Clearly, $\frac{1}{100} \neq \mathcal{S}\left(\frac{1}{100}\right)$ and $\frac{99}{100} \neq \mathcal{S}\left(\frac{99}{100}\right)$.
Case I: When $\nu = \frac{1}{100}$ and $\mu = \frac{99}{100}$, there have,

$$\frac{49}{250} = \partial(\mathcal{S}\nu, \mathcal{S}\mu) \leq \lambda\left[\partial(\nu, \mathcal{S}\nu)^{\alpha} \cdot \partial(\mu, \mathcal{S}\mu)^{1-\alpha}\right] = \frac{\lambda \cdot 99^{1-\alpha}}{125}. \tag{3}$$

Case II: When $\nu = \frac{99}{100}$ and $\mu = \frac{1}{100}$, there exists,

$$\frac{49}{250} = \partial(\mathcal{S}\nu, \mathcal{S}\mu) \leq \lambda\left[\partial(\nu, \mathcal{S}\nu)^{\alpha} \cdot \partial(\mu, \mathcal{S}\mu)^{1-\alpha}\right] = \frac{\lambda \cdot 99^{\alpha}}{125}. \tag{4}$$

Since $\mathcal{S}$ satisfies (2) for all $\nu, \mu \in X \backslash F(\mathcal{S})$, from (3) and (4), there exists a pair of constants $\lambda, \alpha \in [0,1)$ with $\alpha \neq 0$, such that

$$\frac{49}{2} \leq \lambda \cdot \min\{99^{1-\alpha}, 99^{\alpha}\}. \tag{5}$$

Now, if $\lambda = 0$, a contradiction of (5) is obtained. Therefore, $\lambda \in (0,1)$, and in such a case, there is

$$\frac{49}{\lambda} \leq 2 \cdot \min\{99^{1-\alpha}, 99^{\alpha}\}.$$

However, this again leads to a contradiction, as expressed by the following inequality

$$\inf_{\lambda \in (0,1)} \frac{49}{\lambda} > 2 \cdot \sup_{\alpha \in (0,1)} \left[\min\{99^{1-\alpha}, 99^{\alpha}\}\right].$$

Therefore, there does not exist any $\alpha \in (0,1)$ and $\lambda \in (0,1)$ for which equation (5) holds true for all $\nu, \mu \in X \backslash F(\mathcal{S})$. Thus, the initial assumption is incorrect, and $\mathcal{S}$ does not satisfy condition (2).

Therefore, based on Example 1 and Example 2.3 of Karapinar et al. (2018a), it can be inferred that conditions (1) and (2) are independent. In the current study, these authors endeavor to establish connections between these conditions by extending them to a more generalized contraction condition in a relational metric space.

It is noteworthy that in relational metric spaces, one often considers weaker properties such as $\mathcal{R}$-continuous (not necessarily continuous), $\mathcal{R}$-complete (not necessarily complete), etc. In this setting, additional flexibility is beneficial in that the contraction condition need not be applied to every element but rather to related elements only. Importantly, these contraction conditions revert to their conventional counterparts when the universal relation is taken into account.

Preliminaries

Before presenting the main results of this paper, it is important to introduce formal notations that will be used throughout. Let X be a non-empty set, with a binary relation $\mathcal{R}$. In this context, the pair $(X, \mathcal{R})$ is acknowledged as a relational set. Similarly, within the framework of a metric space (X, ∂), one designates the triplet $(X, \partial, \mathcal{R})$ which constitutes a relational metric space (RMS, for brevity). The collection of fixed points of the self-mapping $\mathcal{S}$ is indicated by $F(\mathcal{S})$, and let $X_{\mathcal{R}}$ denote the set defined by, $X_{\mathcal{R}} = \{(\nu, \mu) \in X^2 : (\nu, \mu) \in \mathcal{R}$ and $\nu, \mu \notin F(\mathcal{S})\}$. Furthermore, $X(\mathcal{S}, \mathcal{R})$ is a subset of X, containing elements ν such that $(\nu, \mathcal{S}\nu) \in \mathcal{R}$. These formalized notations ensure precision and consistency throughout the subsequent analyses and discussions.

Definition 2. (Alam & Imdad, 2015) *Let $\mathcal{S}$ be self-map on X, and $(X, \mathcal{R})$ be a relational set,*

- (i) *any two elements $\nu, \mu \in X$ are considered $\mathcal{R}$-comparative if $(\nu, \mu) \in \mathcal{R}$ or $(\mu, \nu) \in \mathcal{R}$. This relationship is symbolically represented as $[\nu, \mu] \in \mathcal{R}$,*
- (ii) *a sequence $\{\nu_k\} \subset X$ satisfies the condition $(\nu_k, \nu_{k+1}) \in \mathcal{R}$ for all $k \in \mathbb{N}_0$, is referred to as an $\mathcal{R}$-preserving sequence.*
- (iii) *$\mathcal{R}$ is designated as $\mathcal{S}$-closed when it satisfies the condition that if (ν, μ) belongs to $\mathcal{R}$, then $(\mathcal{S}\nu, \mathcal{S}\mu)$ also belongs to $\mathcal{R}$, for any $\nu, \mu \in X$.*
- (iv) *$\mathcal{R}$ is referred to as ∂-self-closed under the condition that whenever there exists a $\mathcal{R}$-preserving sequence $\{\nu_k\}$ such that $\nu_k \xrightarrow{\partial} \nu$, there can always be found a subsequence $\{\nu_{k_n}\}$ of $\{\nu_k\}$ such that $[\nu_{k_n}, \nu]$ belongs to $\mathcal{R}$ for all $n \in \mathbb{N}_0$.*

Definition 3. (Alam & Imdad, 2017) *$(X, \partial, \mathcal{R})$ is considered $\mathcal{R}$-complete if every sequence in X, which is both $\mathcal{R}$-preserving and Cauchy, converges.*

Definition 4. (Alam & Imdad, 2017) *A self-map $\mathcal{S}$ defined on X is termed $\mathcal{R}$-continuous at $\nu \in X$, if any $\mathcal{R}$-preserving sequence $\nu_k \xrightarrow{\partial} \nu$, implies $\mathcal{S}\nu_k \xrightarrow{\partial} \mathcal{S}\nu$. Furthermore, if $\mathcal{S}$ exhibits this behavior at every point in X, it is simply categorized as $\mathcal{R}$-continuous.*

DEFINITION 5. (Alam & Imdad, 2018) *Consider a self-mapping $\mathcal{S}$ defined on X. If for every $\mathcal{R}$-preserving sequence $\{\nu_n\} \subset \mathcal{S}(X)$, with a range denoted as $E = \{\nu_n : n \in \mathbb{N}\}$, $\mathcal{R}|_E$ is transitive, then $\mathcal{S}$ is designated as locally $\mathcal{S}$-transitive.*

Samet et al. (2012) introduced the concept of α-admissible mappings, which has been applied by various authors in numerous fixed-point theorems.

DEFINITION 6. (Samet et al., 2012) *Suppose $\mathcal{S}$ is a self-map on X, and $\alpha : X \times X \to \mathbb{R}^+$ is a function. Then, $\mathcal{S}$ is considered α-admissible if $\alpha(\nu, \mu) \geq 1 \Rightarrow \alpha(\mathcal{S}\nu, \mathcal{S}\mu) \geq 1$ for all $\nu, \mu \in X$.*

In the following definition, this concept is generalized by incorporating certain relational metrical notions.

DEFINITION 7. *Let $(X, \mathcal{R})$ be a relational set. A self-map $\mathcal{S}$ defined on X is termed $\mathcal{R}$-admissible if there exists a function $\vartheta : X \times X \to [0, +\infty)$, satisfying the following conditions:*

(r_1) *$\vartheta(\nu, \mu) \geq 1$ for all $(\nu, \mu) \in \mathcal{R}$,*

(r_2) *$\mathcal{R}$ is $\mathcal{S}$-closed.*

REMARK 1. From the above two definitions, it can be observed that if $\mathcal{S}$ is α-admissible, it also holds that $\mathcal{S}$ is $\mathcal{R}$-admissible when considering $\mathcal{R} = \{(\nu, \mu) \in X^2 : \vartheta(\nu, \mu) \geq 1\}$. However, it should be noted that the converse is not necessarily true, as illustrated in the following example.

EXAMPLE 2. Let $X = \{0, 1, 2, 3\}$, $\vartheta : X \times X \to \mathbb{R}^+$ by

$$\vartheta(\nu, \mu) = \begin{cases} 2, & (\nu, \mu) \in \{(0,1), (1,2), (2,3)\} \\ 1, & (\nu, \mu) \in \{(0,2), (1,1), (2,1), (2,2)\} \\ \frac{2}{\nu+5}, & \text{otherwise.} \end{cases}$$

and $\mathcal{S} : X \to X$ is defined by $\mathcal{S}0 = 0, \mathcal{S}1 = 2, \mathcal{S}2 = 1$, and $\mathcal{S}3 = 3$.
In this example, it is evident that $\vartheta(2, 3) \geq 1$, but $\vartheta(\mathcal{S}2, \mathcal{S}3) = \vartheta(1, 3) \ngeq 1$, indicating that $\mathcal{S}$ is not ϑ-admissible. Now, let us consider the binary relation $\mathcal{R}$ defined as,

$$\mathcal{R} = \{(0,1), (0,2), (1,2), (2,1), (1,1), (2,2)\}.$$

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It is straightforward to observe that $\mathcal{R}$ is $\mathcal{S}$-closed, and for all $\nu,\mu \in X$ with $(\nu,\mu) \in \mathcal{R}$, $\vartheta(\nu,\mu) \geq 1$. Therefore, $\mathcal{S}$ is $\mathcal{R}$-admissible.

Let $\psi,\phi : [0,+\infty) \to [0,+\infty)$ be two functions. Then the following conditions are considered:

(C_1) ϕ is $u.s.c.$ such that $\phi(0) = 0$,

(C_2) ψ is $l.s.c.$,

(C_3) ψ,ϕ are non-decreasing,

(C_4) $\psi(t) > \phi(t),$ for all $t > 0$,

(C_5) $\limsup\limits_{t\to c+} \phi(t) < \psi(c+),$ for all $c > 0$,

(C_6) $\limsup\limits_{t\to 0} \phi(t) \leq \liminf\limits_{t\to e+} \psi(t)$.

Main results

This section introduce a novel concept of a unified interpolative Kannan contraction condition and establish some fixed-point results for such contractions. Through an example, it will be demonstrated how the unified interpolative Kannan contraction condition extends the classical notions of contraction mappings defined in (Kannan, 1968; Karapinar, 2018; Nazam et al., 2023a).

Definition 8. *Let $(X,\partial,\mathcal{R})$ be an RMS. A self-mapping $\mathcal{S}$ defined on X is characterized as a unified interpolative Kannan contraction (UIKC, for brief) if there exist functions $\psi,\phi : [0,+\infty) \to [0,+\infty)$, and a function $\vartheta : X \times X \to [0,+\infty)$, along with a parameter $\alpha \in (0,1)$, such that*

$$\vartheta(\nu,\mu)\psi(\partial(\mathcal{S}\nu,\mathcal{S}\mu)) \leq \phi\left(\Omega\left(\partial(\nu,\mathcal{S}\nu),\partial(\mu,\mathcal{S}\mu)\right)\right), \quad \text{for all } \nu,\mu \in X_{\mathcal{R}}, \tag{6}$$

where $\Omega : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying $\Omega(\nu,\mu) \leq \max\left\{\nu,\mu,\nu^{\alpha}\mu^{1-\alpha}\right\}$.

Example 3. Let (X,d) be a metric space with $X = [0,+\infty)$ and ∂ is the usual metric, define the self-map $\mathcal{S}$ on X by,

$$\mathcal{S}\nu = \begin{cases} \frac{\nu}{5}, & \text{if } \nu \leq 1, \\ \nu^2, & \text{if } \nu > 1. \end{cases}$$

Then, it is important to note that $\mathcal{S}$ is not a Kannan contraction (Kannan, 1968). This is evident that when considering $\nu = \frac{1}{2}$ and $\mu = 2$, as there does not exist any $\lambda \in \left[0, \frac{1}{2}\right)$ that satisfies (1). Additionally, for the same values of $\nu = \frac{1}{2}$ and $\mu = 2$, there is no pair of $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ for which (2) holds. Consequently, $\mathcal{S}$ is not an interpolative Kannan contraction (Karapinar, 2018). Now, let us define the binary relation $\mathcal{R}$ on X as,

$$\mathcal{R} = \{(\nu, \mu) \in X^2 : \max\{\nu, \mu\} \leq 1\}.$$

Observing the definition of $\mathcal{R}$, it is evident that $\mathcal{R}$ is not an orthogonal relation. It is important to recall that a binary relation $\mathcal{R}$ is considered as an orthogonal relation if for any element $\nu_0 \in X$, either (for all $\mu, (\nu_0, \mu) \in \mathcal{R}$) or (for all $\mu, (\mu, \nu_0) \in \mathcal{R}$). As a consequence, the function $\mathcal{S}$ is not a (ψ, ϕ)-orthogonal interpolative Kannan-type contraction (Nazam et al., 2023a). However, it will now be demonstrated that $\mathcal{S}$ is indeed a unified interpolative Kannan contraction. Consider $\vartheta : X \times X \to [0, +\infty)$ defined by

$$\vartheta(\nu, \mu) = \begin{cases} \frac{3}{2}, & \text{if } \nu, \mu \in [0, 1], \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Observing that $\vartheta(\nu, \mu) > 1$ for all $\nu, \mu \in X$ with $(\nu, \mu) \in \mathcal{R}$, and that $(\nu, \mu) \in \mathcal{R}$ implies $(\mathcal{S}\nu, \mathcal{S}\mu) \in \mathcal{R}$, it follows that $\mathcal{S}$ is $\mathcal{R}$-admissible. Suppose there exist functions $\psi, \phi : [0, +\infty) \to [0, +\infty)$ defined by $\phi(t) = \frac{t}{6}$,

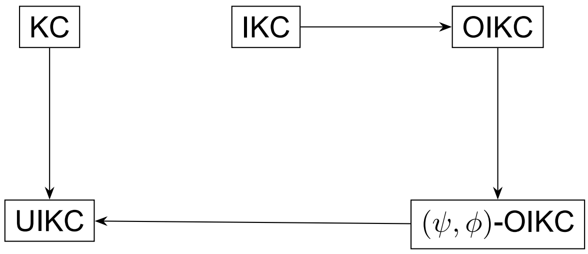
$$\text{and, } \psi(t) = \begin{cases} \frac{t}{5}, & \text{if } t \leq 1, \\ \frac{2t}{9}, & \text{if } t > 1. \end{cases}$$

The aim now is to show that $\mathcal{S}$ satisfies (6). Consider the function $\Omega : X \times X \to [0, +\infty)$ defined as $\Omega(\nu, \mu) = \frac{\nu+\mu}{2}$. For every $\nu, \mu \in X_{\mathcal{R}}$, the following inequality holds,

$$\begin{aligned} \vartheta(\nu, \mu)\psi(\partial(\mathcal{S}\nu, \mathcal{S}\mu)) &= \frac{3}{2}\psi\left(\left|\frac{\nu}{5} - \frac{\mu}{5}\right|\right) \\ &= \frac{3}{50}|\nu - \mu| \\ &\leq \frac{1}{6}\left(\Omega(\partial(\nu, \mathcal{S}\nu), \partial(\mu, \mathcal{S}\mu))\right) \\ &= \phi\left(\Omega(\partial(\nu, \mathcal{S}\nu), \partial(\mu, \mathcal{S}\mu))\right). \end{aligned}$$

Consequently, it is deduced that $\mathcal{S}$ is a unified interpolative Kannan contraction.

REMARK 2. From Example 1, Example 3, and Example 2.3 in Karapinar et al. (2018a), one arrives at the following conclusion



Now, let us proceed to establish this paper's main results concerning the unified interpolative Kannan contraction maps.

THEOREM 2. *Consider the RMS $(X, \partial, \mathcal{R})$ where $\mathcal{R}$ is a locally $\mathcal{S}$-transitive binary relation. Suppose that $\mathcal{S}$ is a unified interpolative Kannan contraction and there exist functions $\psi, \phi : [0, +\infty) \to [0, +\infty)$ satisfying conditions C_i, $(i = 1, 2, 3, 4)$. Under the following conditions:*

(D_1) $\mathcal{S}$ is $\mathcal{R}$-admissible,

(D_2) there exists $Y \subseteq X$ with $\mathcal{S}(X) \subseteq Y$, such that $(Y, \partial, \mathcal{R})$ is $\mathcal{R}$-complete,

(D_3) $X(\mathcal{S}, \mathcal{R})$ is non-empty,

(D_4) either $\mathcal{S}$ is $\mathcal{R}|_Y$-continuous or $\mathcal{R}$ is ∂-self-closed,

there exists at least one $\gamma \in X$ such that $\gamma \in F(\mathcal{S})$.

Proof. Under the assumption (D_3), suppose that $\nu_0 \in X(\mathcal{S}, \mathcal{R})$. Define the sequence $\{\nu_n\}$ of Picard iterates with the initial point ν_0, i.e. $\nu_n = \mathcal{S}^n \nu_0$ for all $n \in \mathbb{N}_0$. As $(\nu_0, \mathcal{S}\nu_0) \in \mathcal{R}$ and $\mathcal{S}$ is $\mathcal{R}$-admissible, using (r_1) it follows that $(\mathcal{S}^n \nu_0, \mathcal{S}^{n+1}\nu_0) \in \mathcal{R}$. Consequently, $(\nu_n, \nu_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$, and this yields that the sequence $\{\nu_n\}$ is $\mathcal{R}$-preserving and from (r_2) there holds $\vartheta(\nu_n, \nu_{n+1}) \geq 1$. Let $\partial_n = \partial(\nu_n, \nu_{n+1})$; applying contractive condition (1) yields that

$$\begin{aligned}
\psi\left(\partial_n\right) &\leq \vartheta(\nu_{n-1}, \nu_n)\psi\left(\partial(\mathcal{S}\nu_{n-1}, \mathcal{S}\nu_n)\right) \\
&\leq \phi\left(\Omega(\partial(\nu_{n-1}, \mathcal{S}\nu_{n-1}), \partial(\nu_n, \mathcal{S}\nu_n))\right) \\
&= \phi\left(\Omega\left(\partial_{n-1}, \partial_n\right)\right) \\
&\leq \phi\left(\max\left\{\partial_{n-1}, \partial_n, \partial_{n-1}^{\alpha} \cdot \partial_n^{1-\alpha}\right\}\right) \\
&< \psi\left(\max\left\{\partial_{n-1}, \partial_n, \partial_{n-1}^{\alpha} \cdot \partial_n^{1-\alpha}\right\}\right). \qquad (7)
\end{aligned}$$

By the monotonicity of the function ψ one obtains

$$\partial_n < \max\left\{\partial_{n-1}, \partial_n, \partial_{n-1}^{\alpha} \cdot \partial_n^{1-\alpha}\right\}. \tag{8}$$

Now suppose there exists $n \in \mathbb{N}$ for which $\partial_{n-1} \leq \partial_n$, then from (8) it yields that $\partial_n < \partial_n$, a contradiction. Therefore $\partial_n \leq \partial_{n-1}$, now it can be concluded that $\{\nu_n\}$ is a non-increasing sequence and thus a non-negative constant C exists such that, $\lim_{n\to+\infty} \partial_n = C+$. Suppose if possible $C > 0$, then from (7), it can be deduced that

$$\psi(C+) \leq \liminf \psi(\partial_n) \leq \limsup \phi(\partial_{n-1}) \leq \phi(C+),$$

but, from (C_4) there exists $\psi(\nu) > \phi(\nu)$ for all $\nu > 0$, therefore C must be 0, i.e. $\lim_{n\to+\infty} \partial_n = 0$. The next objective is to establish that the sequence $\{\nu_n\}$ is Cauchy. For the sake of contradiction, suppose it is not; then there exists a positive real number $\epsilon > 0$ along with sub-sequences $\{\nu_{n_k}\}$ and $\{\nu_{m_k}\}$ of $\{\nu_n\}$, with $n_k > m_k \geq k$, such that

$$\partial(\nu_{m_k}, \nu_{n_k}) \geq \epsilon, \quad \text{for all } k \in \mathbb{N}. \tag{9}$$

Selecting n_k as the smallest integer exceeding m_k such that (9) holds, it is deduced that

$$\partial(\nu_{m_k}, \nu_{n_k-1}) < \epsilon. \tag{10}$$

Using triangular inequality and (9), (10) one obtains that

$$\begin{aligned}\epsilon \leq \partial(\nu_{m_k}, \nu_{n_k}) &\leq \partial(\nu_{m_k}, \nu_{n_k-1}) + \partial(\nu_{n_k-1}, \nu_{n_k}) \\ &< \epsilon + \partial(\nu_{n_k-1}, \nu_{n_k}).\end{aligned}$$

on taking the limit $k \to +\infty$ and utilizing the fact that $\lim_{n\to+\infty} \partial_n = 0$, one gets

$$\lim_{k\to+\infty} \partial(\nu_{m_k}, \nu_{n_k}) = \epsilon + . \tag{11}$$

By using triangular inequality, one obtains that

$$|\partial(\nu_{m_k+1}, \nu_{n_k+1}) - \partial(\nu_{m_k}, \nu_{n_k})| \leq \partial\nu_{m_k} + \partial\nu_{n_k}.$$

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Letting limit $k \to +\infty$ in the above inequality and employing (11) yields the following:

$$\lim_{k \to +\infty} \partial(\nu_{m_k+1}, \nu_{n_k+1}) = \lim_{k \to +\infty} \partial(\nu_{m_k}, \nu_{n_k}) = \epsilon + . \tag{12}$$

Since $\{\nu_n\} \subset \mathcal{S}(X)$ and $\{\nu_n\}$ is $\mathcal{R}$-preserving, the local $\mathcal{S}$-transitivity of $\mathcal{R}$ leads to the implication that $(\nu_{m_k}, \nu_{n_k}) \in \mathcal{R}$. Thus, it can be deduced

$$\begin{aligned}
\psi(\partial(\nu_{m_{k+1}}, \nu_{n_{k+1}})) &\leq \vartheta(\nu_{m_k}, \nu_{n_k})\psi(\partial(\mathcal{S}\nu_{m_k}, \mathcal{S}\nu_{n_k})) \\
&\leq \phi\left(\Omega\left(\partial(\nu_{m_k}, \mathcal{S}\nu_{m_k}), \partial(\nu_{n_k}, \mathcal{S}\nu_{n_k})\right)\right) \\
&= \phi\left(\Omega\left(\partial_{m_k}, \partial_{n_k}\right)\right) \\
&\leq \phi\left(\max\left\{\partial_{m_k}, \partial_{n_k}, \partial_{m_k}^{\alpha} \cdot \partial_{n_k}^{1-\alpha}\right\}\right).
\end{aligned}$$

Taking the limit as $k \to +\infty$ in the aforementioned inequality leads to the conclusion that $\epsilon < 0$, a contradiction. Hence, $\{\nu_n\}$ is the $\mathcal{R}$- preserving Cauchy sequence in Y. The $\mathcal{R}$-completeness of the metric space $(Y, \partial, \mathcal{R})$ now guarantees the existence of a point $\gamma \in Y$ such that, $\lim_{n \to +\infty} \nu_n = \gamma$.

First, one assumed that $\mathcal{S}$ is $\mathcal{R}$-continuous; one can deduce that $\lim_{n \to +\infty} \nu_{n+1} = \lim_{n \to +\infty} \mathcal{S}\nu_n = \mathcal{S}\gamma$. Applying the uniqueness of the limit, one consequently establishes that $\mathcal{S}\gamma = \gamma$, indicating that $\gamma \in F(\mathcal{S})$.
Alternatively, let $\mathcal{R}|_Y$ is ∂-self-closed. The fact that $\{\nu_n\}$ is $\mathcal{R}$-preserving and $\{\nu_n\} \to \gamma$ can be utilized again. This implies the existence of a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k}, \gamma] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$. If $(\nu_{n_k}, \gamma) \in \mathcal{R}$, then since $\mathcal{S}$ is a unified interpolative Kannan contraction, there exists

$$\begin{aligned}
\psi(\partial(\mathcal{S}\nu_{n_k}, \mathcal{S}\gamma)) &\leq \vartheta(\nu_{n_k}, \gamma)\psi(\partial(\mathcal{S}\nu_{n_k}, \mathcal{S}\gamma)) \\
&\leq \phi(\Omega(\partial(\nu_{n_k}, \mathcal{S}\nu_{n_k}), \partial(\gamma, \mathcal{S}\gamma))) \\
&= \phi(\Omega(\partial_{n_k}, \partial(\gamma, \mathcal{S}\gamma))) \\
&\leq \phi\left(\max\left\{\partial_{n_k}, \partial(\gamma, \mathcal{S}\gamma), \partial_{n_k}^{\alpha} \cdot \partial(\gamma, \mathcal{S}\gamma)^{1-\alpha}\right\}\right),
\end{aligned} \tag{13}$$

on taking the limit $k \to +\infty$, in (13), one obtains

$$\psi(\partial(\gamma, \mathcal{S}\gamma)) \leq \phi(\partial(\gamma, \mathcal{S}\gamma)). \tag{14}$$

It is important to note that in equation (14), if $\partial(\gamma, \mathcal{S}\gamma) \neq 0$, it is contradictory to (C_4). Similarly, if $(\gamma, \nu_{n_k}) \in \mathcal{R}$, then by utilizing the symmetry of ∂, we once again encounter a contradiction of (C_4). Therefore, $\partial(\gamma, \mathcal{S}\gamma) = 0$, implying $\gamma \in F(\mathcal{S})$. □

THEOREM 3. *Consider the RMS* $(X, \partial, \mathcal{R})$ *where* $\mathcal{R}$ *is a locally* $\mathcal{S}$*-transitive binary relation. Suppose that* $\mathcal{S}$ *is a unified interpolative Kannan contraction and there exist functions* $\psi, \phi : [0, +\infty) \to [0, +\infty)$ *satisfying conditions* C_i*,* $(i = 3, 4, 5, 6)$ *and* D_j*,* $(j = 1, 2, 3, 4)$ *holds. Then, there exists at least one* $\gamma \in X$ *such that* $\gamma \in F(\mathcal{S})$*.*

Proof. Following the steps of the previous theorem, one can obtain an $\mathcal{R}$-preserving and non-increasing sequence $\{\nu_n\}$ such that there exists some $C \geq 0$ and ν_n converges to $C+$ as $n \to +\infty$. Suppose $C > 0$, then (7) implies that

$$\begin{aligned}\psi(C+) &\leq \limsup_{n\to+\infty} \psi(\partial_n) \\ &\leq \limsup_{n\to+\infty} \phi\left(\max\left\{\partial_{n-1}, \partial_n, \partial_{n-1}^{\alpha} \cdot \partial_n^{1-\alpha}\right\}\right) \\ &\leq \limsup_{k\to C+} \phi(k),\end{aligned}$$

a contradiction of (C_5), thus $C = 0$ i.e. $\lim_{n\to+\infty} \partial_n = 0$. Now, to establish that the sequence $\{\nu_n\}$ is Cauchy, one makes a counter assumption. Suppose it is not Cauchy, then following the steps outlined in the previous theorem, there exists a positive real number $\epsilon > 0$, along with sub-sequences $\{\nu_{n_k}\}$ and $\{\nu_{m_k}\}$ of $\{\nu_n\}$, where $n_k > m_k \geq k$, satisfying condition (12). Since $\{\nu_n\} \subset \mathcal{S}(X)$ and $\{\nu_n\}$ is $\mathcal{R}$-preserving, the local $\mathcal{S}$-transitivity of $\mathcal{R}$ leads to the implication that $(\nu_{m_k}, \nu_{n_k}) \in \mathcal{R}$. Thus, it can be deduced

$$\begin{aligned}\psi(\partial(\nu_{m_{k+1}}, \nu_{n_{k+1}})) &\leq \vartheta(\nu_{m_k}, \nu_{n_k})\psi(\partial(\mathcal{S}\nu_{m_k}, \mathcal{S}\nu_{n_k})) \\ &\leq \phi\left(\max\left\{\partial_{m_k}, \partial_{n_k}, \partial_{m_k}^{\alpha} \cdot \partial_{n_k}^{1-\alpha}\right\}\right),\end{aligned}$$

on taking the limit $k \to +\infty$ in the above equation, it implies that

$$\begin{aligned}\liminf_{a\to\epsilon+} \psi(a) &\leq \liminf_{k\to+\infty} \psi(\partial(\nu_{m_{k+1}}, \nu_{n_{k+1}})) \\ &\leq \limsup_{k\to+\infty} \phi\left(\max\left\{\partial_{m_k}, \partial_{n_k}, \partial_{m_k}^{\alpha} \cdot \partial_{n_k}^{1-\alpha}\right\}\right) \\ &\leq \limsup_{a\to 0} \phi(a).\end{aligned}$$

This results in a contradiction of (C_6), thus establishing that the $\{\nu_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence is in Y. Given that $(Y, \partial, \mathcal{R})$ is an $\mathcal{R}$-complete metric space, there exists $\gamma \in Y$ such that $\lim_{n\to+\infty} \nu_n = \gamma$. If the self-mapping $\mathcal{S}$ is $\mathcal{R}$-continuous, the desired conclusion can be derived, as demonstrated in the previous theorem.

Alternatively, let $\mathcal{R}|_Y$ be ∂-self-closed then utilizing the fact that $\{\nu_n\}$ is $\mathcal{R}$-preserving and $\{\nu_n\} \to \gamma$. This implies the existence of a sub-sequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k}, \gamma] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$. One claims that $\partial(\gamma, \mathcal{S}\gamma) = 0$. Let us assume that $\partial(\gamma, \mathcal{S}\gamma) > 0$, if $(\nu_{n_k}, \gamma) \in \mathcal{R}$, then since $\mathcal{S}$ is a unified interpolative Kannan contraction, there exists

$$\begin{aligned}\psi(\partial(\nu_{n_k+1}, \mathcal{S}\gamma)) &\leq \vartheta(\nu_{n_k}, \gamma)\psi(\partial(\mathcal{S}\nu_{n_k}, \mathcal{S}\gamma)) \\ &\leq \phi\left(\Omega\left(\partial(\nu_{n_k}, \mathcal{S}\nu_{n_k}), \partial(\gamma, \mathcal{S}\gamma)\right)\right) \\ &= \phi\left(\Omega\left(\partial_{n_k}, \partial(\gamma, \mathcal{S}\gamma)\right)\right) \\ &\leq \phi\left(\max\left\{\partial_{n_k}, \partial(\gamma, \mathcal{S}\gamma), \partial_{n_k}^{\alpha} \cdot \partial(\gamma, \mathcal{S}\gamma)^{1-\alpha}\right\}\right) \\ &< \psi\left(\max\left\{\partial_{n_k}, \partial(\gamma, \mathcal{S}\gamma), \partial_{n_k}^{\alpha} \cdot \partial(\gamma, \mathcal{S}\gamma)^{1-\alpha}\right\}\right),\end{aligned}$$

by using (C_3) and taking the limit as $k \to +\infty$, one deduces $\partial(\gamma, \mathcal{S}\gamma) < \partial(\gamma, \mathcal{S}\gamma)$, which leads to a contradiction. Furthermore, if $(\gamma, \nu_{n_k}) \in \mathcal{R}$, then by utilizing the symmetry of ∂, one encounters again a contradiction. Hence, $\partial(\gamma, \mathcal{S}\gamma) = 0$, implying $\gamma \in F(\mathcal{S})$ □

THEOREM 4. *Consider the RMS $(X, \partial, \mathcal{R})$, where $\mathcal{R}$ is a locally $\mathcal{S}$-transitive and $\mathcal{S}$-closed. Suppose the conditions $D_j, (j = 1, 2, 3, 4)$ hold and there exist the functions $\psi, \phi : [0, +\infty) \to [0, +\infty)$ satisfying the conditions C_i, $(i = 1, 2, 3, 4)$ or $(i = 3, 4, 5, 6)$, such that,*

$$\psi(\partial(\mathcal{S}\nu, \mathcal{S}\mu)) \leq \phi\left(\Omega\left(\partial(\nu, \mathcal{S}\nu), \partial(\mu, \mathcal{S}\mu)\right)\right), \quad \textit{for all } \nu, \mu \in X_{\mathcal{R}} \tag{15}$$

Then there exists at least one $\gamma \in X$ such that $\gamma \in F(\mathcal{S})$.

By considering the specific values of the functions ψ, ϕ, Ω, and ν, in Theorem 4, one can derive the following relational theoretic versions of Kannan fixed-point results and Interpolative Kannan fixed-point results respectively.

COROLLARY 1. *Let $(X, \partial, \mathcal{R})$ be an $\mathcal{R}$-complete RMS, where $\mathcal{R}$ is a locally $\mathcal{S}$-transitive and $\mathcal{S}$-closed. Suppose that the conditions $D_j, (j = 1, 2, 3)$ hold and there exists a parameter $0 \leq \lambda < \frac{1}{2}$, such that*

$$\partial(\mathcal{S}\nu, \mathcal{S}\mu) \leq \lambda\left[\partial(\nu, \mathcal{S}\nu) + \partial(\mu, \mathcal{S}\mu)\right], \quad \textit{for all } \nu, \mu \in X_{\mathcal{R}}.$$

Then there exists at least one $\gamma \in X$ such that $\gamma \in F(\mathcal{S})$.

COROLLARY 2. *Let* $(X, \partial, \mathcal{R})$ *be an* $\mathcal{R}$*-complete RMS, where* $\mathcal{R}$ *is a locally* $\mathcal{S}$*-transitive and* $\mathcal{S}$*-closed. Suppose that the conditions* $D_j, (j = 1, 2, 3)$ *hold and there exists a pair of constants* $\alpha, \lambda \in [0, 1)$ *with* $\alpha \neq 0$*, satisfying*

$$\partial(\mathcal{S}\nu, \mathcal{S}\mu) \leq \lambda \left[\partial(\nu, \mathcal{S}\nu)^{\alpha} \cdot \partial(\mu, \mathcal{S}\mu)^{1-\alpha}\right], \quad \text{for all } \nu, \mu \in X_{\mathcal{R}}.$$

Then there exists at least one $\gamma \in X$ *such that* $\gamma \in F(\mathcal{S})$.

An application

In this section, the authors have applied their research findings to derive a result concerning the existence of solutions for a nonlinear matrix equation. In this context, let the set denoted as $\mathcal{M}(n)$ encompasses all square matrices with dimensions of $n \times n$, while $\mathcal{H}(n)$, $\mathcal{P}(n)$, and $\mathcal{K}(n)$, respectively represent the sets of Hermitian matrices, positive definite positive, and semi-definite matrices. When there is a matrix $\mathcal{C}$ from $\mathcal{H}(n)$, one uses the notation $\|\mathcal{C}\|_{tr}$ to refer to its trace norm, which is the sum of all its singular values. If there are matrices $\mathcal{P}$ and $\mathcal{Q}$ from $\mathcal{H}(n)$, the notation $\mathcal{P} \succeq \mathcal{Q}$ signifies that the matrix $\mathcal{P} - \mathcal{Q}$ is an element of the set $\mathcal{K}(n)$, while $\mathcal{P} \succ \mathcal{Q}$ indicates that $\mathcal{P} - \mathcal{Q}$ belongs to the set $\mathcal{P}(n)$. The upcoming discussion relies on the significance of the following lemmas.

LEMMA 1. (Ran & Reurings, 2002) *If* $X \in \mathcal{H}(n)$ *satisfies* $X \prec \mathcal{I}_n$*, then* $\|X\| < 1$.

LEMMA 2. (Ran & Reurings, 2002) *For* $n \times n$ *matrices* $X \succeq O$ *and* $Y \succeq O$*, the following inequalities hold:*

$$0 \leq tr(XY) \leq \|X\| tr(Y).$$

Examine now the following nonlinear matrix equation,

$$X = \mathcal{A} + \sum_{i=1}^{u} \sum_{k=1}^{v} \mathcal{C}_j^{*} \Upsilon_k(X) \mathcal{C}_j \tag{16}$$

In the above equation, $\mathcal{A}$ is defined as a Hermitian and positive definite matrix. Additionally, the notation $\mathcal{C}_j^{*}$ refers to the conjugate transpose of a square matrix $\mathcal{C}_j$ of size $n \times n$. Furthermore, Υ_k represents continuous functions that preserve order, mapping from $\mathcal{H}(n)$ to $\mathcal{P}(n)$. It is noteworthy that $\Upsilon(O) = O$, where O represents a zero matrix.

THEOREM 5. *Consider the nonlinear matrix equation expressed in (16) and assume the following:*

(H_1) *there exists* $\mathcal{A} \in \mathcal{P}(n)$ *with* $\sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(\mathcal{A})\mathcal{C}_j \succ 0$*;*

(H_2) *for every* $X, Y \in \mathcal{P}(n)$*,* $X \preceq Y$ *implies*

$$\sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(X)\mathcal{C}_j \preceq \sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(Y)\mathcal{C}_j;$$

(H_3) $\sum_{j=1}^{u} \mathcal{C}_j\mathcal{C}_j^* \prec N\mathcal{I}_n$*, for some positive number* N*, and for all* $X, Y \in \mathcal{P}(n)$ *with* $X \preceq Y$*, the following inequality holds*

$\max_k(tr(\Upsilon_k(Y) - \Upsilon_k(X))) \leq$

$$\frac{1}{2Nv} \times \max \left\{ \begin{array}{l} tr\left(X - \mathcal{A} - \sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(X)\mathcal{C}_j\right), tr\left(Y - \mathcal{A} - \sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^*, \Upsilon_k(Y)\mathcal{C}_j\right), \\ \left(tr\left(X - \mathcal{A} - \sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(X)\mathcal{C}_j\right)\right)^{\frac{1}{2}} \cdot \left(tr\left(Y - \mathcal{A} - \sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^*, \Upsilon_k(Y)\mathcal{C}_j\right)\right)^{\frac{1}{2}} \end{array} \right\}.$$

Then, there exists at least one solution of the nonlinear matrix equation (16). Moreover, the iteration

$$X_r = \mathcal{A} + \sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(X_{r-1})\mathcal{C}_j, \tag{17}$$

where $X_0 \in \mathcal{P}(n)$ *satisfies* $X_0 \preceq \mathcal{A} + \sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(X_0)\mathcal{C}_j$*, Convergence towards the solution of the matrix equation, in the context of trace norm* $\|\cdot\|_{tr}$*.*

Proof. Let $\mathfrak{T} : \mathcal{P}(n) \to \mathcal{P}(n)$ be a mapping defined by

$$\mathfrak{T}(X) = \mathcal{A} + \sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(X)\mathcal{C}_j, \quad \text{for all } X \in \mathcal{P}(n).$$

Consider $\mathcal{R} = \{(X, Y) \in \mathcal{P}(n) \times \mathcal{P}(n) : X \preceq Y\}$. Consequently, the fixed point of $\mathfrak{T}$ serves as a solution to the nonlinear matrix equation (16). It is pertinent to mention that $\mathcal{R}$ is $\mathfrak{T}$-closed and $\mathfrak{T}$ is well-defined as well as $\mathcal{R}$-continuous. From condition (H_1) there is $\sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^* \Upsilon_k(X)\mathcal{C}_j \succ 0$ for some $X \in \mathcal{P}(n)$, thus $(X, \mathfrak{T}(X)) \in \mathcal{R}$ and consequently $\mathcal{P}(n)(\mathfrak{T}, \mathcal{R})$ is non-empty.

Define $\partial : \mathcal{P}(n) \times \mathcal{P}(n) \to \mathbb{R}^+$ by

$$\partial(X, Y) = \|X - Y\|_{tr}, \text{ for all } X, Y \in \mathcal{P}(n).$$

Then $(\mathcal{P}(n), \partial, \mathcal{R})$ is $\mathcal{R}$-complete RMS. Then

$$\begin{aligned}
\|\mathfrak{T}(Y) - \mathfrak{T}(X)\|_{tr} &= tr(\mathfrak{T}(Y) - \mathfrak{T}(X)) \\
&= tr\left(\sum_{j=1}^{u}\sum_{k=1}^{v} \mathcal{C}_j^*(\Upsilon_k(Y) - \Upsilon_k(X))\mathcal{C}_j\right) \\
&= \sum_{j=1}^{u}\sum_{k=1}^{v} tr(\mathcal{C}_j\mathcal{C}_j^*(\Upsilon_k(Y) - \Upsilon_k(X))) \\
&= tr\left(\left(\sum_{j=1}^{u} \mathcal{C}_j\mathcal{C}_j^*\right)\sum_{k=1}^{v}(\Upsilon_k(Y) - \Upsilon_k(X))\right) \\
&\le \left\|\sum_{j=1}^{u} \mathcal{C}_j\mathcal{C}_j^*\right\| \times v \times \max \|(\Upsilon_k(Y) - \Upsilon_k(X))\|_{tr} \\
&\le \frac{1}{2} \times \max\Big\{\|X - \mathfrak{T}X\|_{tr}, \|Y - \mathfrak{T}Y\|_{tr}, \\
&\quad \|X - \mathfrak{T}X\|_{tr}^{\frac{1}{2}} \cdot \|Y - \mathfrak{T}Y\|_{tr}^{\frac{1}{2}}\Big\} \\
&= \frac{1}{2}(\Omega(\|X - \mathfrak{T}X\|_{tr}, \|Y - \mathfrak{T}Y\|_{tr})) \qquad (18)
\end{aligned}$$

Now, when considering $\psi(\nu) = \nu$, $\phi(\nu) = \frac{\nu}{2}$, then equation (18) becomes

$$\psi(\partial(\mathfrak{T}X, \mathfrak{T}Y)) \le \phi\left(\Omega\left(\partial(X, \mathfrak{T}(X)), \partial(Y, \mathfrak{T}(Y))\right)\right).$$

Consequently, upon fulfilling all the hypotheses stated in Theorem 2, it can be deduced that there exists an element $X^* \in \mathcal{P}(n)$ for which $\mathfrak{T}(X^*) = X^*$ holds good. As a result, the matrix equation (16) is guaranteed to possess a solution within the set $\mathcal{P}(n)$. □

EXAMPLE 4. Consider the nonlinear matrix equation (16) for $u = v = 2$, and $n = 3$, with $\Upsilon_1(X) = X^{\frac{1}{4}}$, $\Upsilon_2(X) = X^{\frac{1}{5}}$, i.e.,

$$X = \mathcal{A} + \mathcal{C}_1^* X^{\frac{1}{4}}\mathcal{C}_1 + \mathcal{C}_1^* X^{\frac{1}{5}}\mathcal{C}_1 + \mathcal{C}_2^* X^{\frac{1}{4}}\mathcal{C}_2 + \mathcal{C}_2^* X^{\frac{1}{5}}\mathcal{C}_2 \qquad (19)$$

where

$$\mathcal{A} = \begin{bmatrix} 0.177855454222667 & 0.001123654123643 & 0.144563214565439 \\ 0.001123532012243 & 0.177856213654500 & 0.133214521452362 \\ 0.144562121365390 & 0.133214526352116 & 0.266521364125960 \end{bmatrix}$$

$$\mathcal{C}_1 = \begin{bmatrix} 0.222353216521933 & 0.104402312563210 & 0.077854213651530 \\ 0.277652136521619 & 0.122365475632174 & 0.066321541236599 \\ 0.144563125462493 & 0.111232145236838 & 0.244512365214147 \end{bmatrix}$$

$$\mathcal{C}_2 = \begin{bmatrix} 0.255541232145296 & 0.177563214532317 & 0.277854621452056 \\ 0.074456321236541 & 0.222351452365355 & 0.100321256321427 \\ 0.155462136521421 & 0.133652123652627 & 0.199663251400003 \end{bmatrix}$$

By taking $N = \frac{3}{5}$, the conditions specified in Theorem 5 can be validated numerically by evaluating various specific values for the matrices involved. For example, they can be tested (and verified to be true) for

$$X = \begin{bmatrix} 0.285221251452362 & 0.123815632145236 & 0.016912136521452 \\ 0.192072365214523 & 0.219152365214523 & 0.026932365214569 \\ 0.232862136541254 & 0.172062136521452 & 0.096802123652145 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.385224563214521 & 0.123811236521452 & 0.016912365214896 \\ 0.192076541236541 & 0.319150000000000 & 0.026931236541526 \\ 0.232861236541256 & 0.172061236521452 & 0.196823652145230 \end{bmatrix}$$

To ascertain the convergence of $\{X_n\}$ defined in (17), one commences with three distinct initial values.

$$U_0 = \begin{bmatrix} \frac{1}{20} & 0 & 0 \\ 0 & \frac{1}{15} & 0 \\ 0 & 0 & \frac{1}{15} \end{bmatrix}$$

$$V_0 = \begin{bmatrix} 0.500354112000372 & 0.454632123005061 & 0.398954120000949 \\ 0.022141236541532 & 0.151234561235184 & 0.104256348563137 \\ 0.054621236525374 & 0.045213625456758 & 0.103456212563418 \end{bmatrix}$$

$$W_0 = \begin{bmatrix} 0.100963214521244 & 0.066321213621732 & 0.005445632123530 \\ 0.255632122000784 & 0.210032145632300 & 0.288632512325983 \\ 0.111232152412246 & 0.080032356212332 & 0.177521363201611 \end{bmatrix}$$

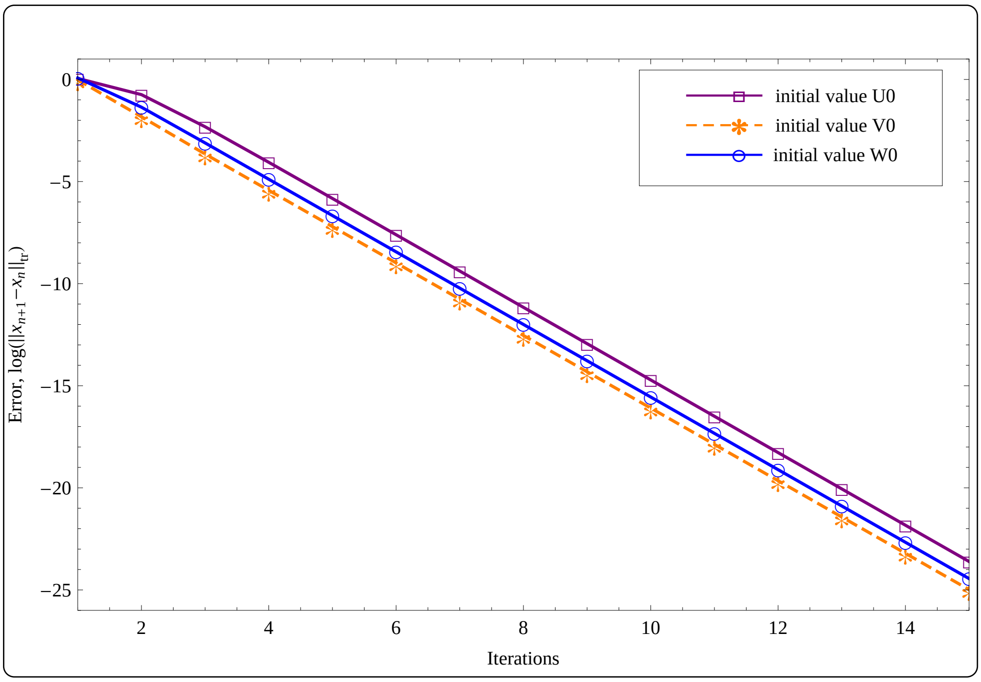

Figure - 1 Convergence behavior

After conducting 15 iterations, the subsequent approximation of the positive definite solution for the system presented in (16) is as follows:

$$\hat{U} \approx U_{15} = \begin{bmatrix} 0.683105295072446 & 0.342270947152747 & 0.548634067670387 \\ 0.342270819335426 & 0.468237110722328 & 0.450115496162419 \\ 0.548632937935167 & 0.450115485036356 & 0.678165537273724 \end{bmatrix}$$

with error 1.92601×10^{-11},

$$\hat{V} \approx V_{15} = \begin{bmatrix} 0.683105295076171 & 0.342270947155381 & 0.548634067673204 \\ 0.342270819338060 & 0.468237110724480 & 0.450115496164750 \\ 0.548632937937984 & 0.450115485038687 & 0.678165537276339 \end{bmatrix}$$

with error 1.1608×10^{-11},

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$$\hat{W} \approx W_{15} = \begin{bmatrix} 0.683105295075270 & 0.342270947154744 & 0.548634067672523 \\ 0.342270819337423 & 0.468237110723960 & 0.450115496164186 \\ 0.548632937937303 & 0.450115485038123 & 0.678165537275706 \end{bmatrix}$$

with error 5.3354×10^{-12}.
Figure 1 is a graphical illustration of the convergence phenomenon.

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Teoremas de existencia para una contracción unificada Kannan interpolativa con una aplicación en ecuaciones matriciales no lineales

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CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: Este artículo estableció un nuevo marco matemático al descubrir las relaciones entre las contracciones Kannan y las contracciones Kannan interpolativas. El concepto de contracciones Kannan interpolativas unificadas se introdujo en el marco de un espacio métrico relacional. Además, el estudio tuvo como objetivo ampliar el concepto de admisibilidad alfa incorporando ideas métricas relacionales específicas.

Métodos: Se realizó una exploración detallada de las propiedades y características de las contracciones Kannan y las contracciones Kannan interpolativas. La investigación introdujo el concepto de contracciones Kannan interpolativas unificadas y formuló nuevos resultados de punto fijo para estas asignaciones.

Resultados: El estudio estableció con éxito resultados de punto fijo para las contracciones unificadas Kannan interpolativas dentro del marco de los espacios métricos relacionales. Además, se proporcionó una aplicación de estos resultados para resolver un problema relacionado con ecuaciones matriciales no lineales, enfatizando aún más su importancia.

Conclusión: Los hallazgos de este estudio han permitido avanzar significativamente en la comprensión de las contracciones Kannan y las contracciones Kannan interpolativas, ofreciendo un marco unificado para su análisis. La introducción de contracciones unificadas Kannan interpolativas y la expansión de la admisibilidad alfa tienen amplias implicaciones para el campo de las matemáticas.

Palabras claves: contracción interpolativa unificada de Kannan, $\mathcal{R}$-admisible, espacio métrico relacional.

Теоремы существования унифицированного интерполяционного сокращения Каннана с применением нелинейных матричных уравнений

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Резюме:

Введение/цель: В данной статье были обозначены новые математические рамки, освещающие взаимосвязи между сокращениями Каннана и интерполяционными сокращениями Каннана. Концепция унифицированных интерполяционных сокращений Каннана была введена в рамках реляционного метрического пространства. Помимо того, целью исследования было развитие концепции альфа-допустимости за счет включения конкретных идей в отношении относительных показателей.

Методы: Было проведено подробное исследование свойств и характеристик сокращений Каннана и интерполяции сокращений Каннана. В ходе исследования была представлена концепция унифицированной интерполяции сокращений Каннана и сформулированы новые результаты с фиксированными точками.

Результаты: Исследование дало успешные результаты с фиксированной точкой для унифицированных интерполяционных сокращений Каннана в рамках реляционных метрических пространств. Помимо того, было представлено применение этих результатов для решения задачи, касающейся нелинейных матричных уравнений, тем самым подчеркивая их значимость.

Выводы: Результаты данного исследования значительно улучшили понимание сокращений Каннана и интерполяционных сокращений Каннана, представив единую основу для их анализа. Введение унифицированных интерполяционных сокращений Каннана и расширение допустимости альфа-допустимости широко применяются в области математики.

Ключевые слова: унифицированное интерполяционное сжатие Каннана, $\mathcal{R}$-допустимое, реляционное метрическое пространство.

Теореме постојања за јединствену интерполативну Канапову контракцију са применама код нелинеарних матричних једначина

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ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Овим радом успостављен је нови математички оквир откривањем односа између Канановe контракције и његове интерполативне контракције. Концепт обједињене интерполативне Канановe контракције уведен је у оквиру релационог метричког простора. Поред тога, студија је имала за циљ да прошири концепт алфа-прихватљивости уграђивањем специфичних релационих метричких идеја.

Методе: Детаљно истраживање својстава и карактеристика Канановe контракције и његове интерполативне контракције били су и раније разматрани. Овим истраживањем уведен је концепт унифициране интерполације Канановe контракције чиме су формулисани нови резултати фиксне тачке за њих.

Резултати: Студија је успешно потврдила резултате фиксне тачке за унифициране интерполативне Канановe контракције у оквиру релационих метричких простора. Поред тога, примена ових резултата за решавање проблема који се тиче нелинеарних матричних једначина додатно наглашава њихов значај.

Закључак: Налази ове студије значајно су унапредили недовољно разумевање Канановиx контракција и његових интерполативних контракција, нудећи јединствен оквир за њихову анализу. Увод у унифициране интерполативне Канановe контракције и проширење алфа-прихватљивости има широку примену у области математике.

Кључне речи: унифицирана интерполативна Кананова контракција, $\mathcal{R}$-допустив, релациони метрички простор.

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