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Multi-valued mappings on Pompeiu-Hausdorff fuzzy b-metric spaces and a common fixed point result

Noreddine Makran[a], *Stojan* N. Radenović[b]

[a] University Mohammed Premier, Faculty of Sciences, Department of Mathematics, Oujda, Kingdom of Morocco, e-mail: makranmakran83@gmail.com, **corresponding author**, ORCID iD: https://orcid.org/0000-0001-8131-6733

[b] University of Belgrade, Faculty of Mechanical Engineering, Belgrade, Republic of Serbia, e-mail: radens@beotel.rs, ORCID iD: https://orcid.org/0000-0001-8254-6688

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Abstract:

Introduction/purpose: The study of the theory of fuzzy sets was prompted by the presence of uncertainty as an essential part of real-world problems, leading Zadeh to address the problem of indeterminacy. The notion of fuzzy logic was introduced by Zadeh. Unlike traditional logic theory, where an element either belongs to the set or does not, in fuzzy logic, the affiliation of the element to the set is expressed as a number from the interval $[0, 1]$.

Methods: The theory of a fixed point in fuzzy metric spaces can be viewed in different ways, one of which involves the use of fuzzy logic. Fuzzy metric spaces, which are specific types of topological spaces with pleasing "geometric" characteristics, possess a number of appealing properties and are commonly used in both pure and applied sciences. Metric spaces and their various generalizations frequently occur in computer science applications. For this reason, a new space called a Pompeiu-Hausdorff fuzzy b*-metric space is constructed in this paper.*

Results: In this space, some new fixed point results are also formulated and proven. Additionally, a general common fixed point theorem for a pair of multi-valued mappings in Pompeiu-Hausdorff fuzzy b*-metric spaces is investigated. The findings obtained in fuzzy metric spaces,* *such as those discussed in the article of Shen, Y. et al. (2012. Fixed point theorems in fuzzy metric spaces. Applied Mathematics Letters, 25, pp.138-141), are generalized by the results in this paper, and additional specific findings are produced and supported by examples.*

Conclusions: The study of denotational semantics and their applications in control theory using fuzzy b-metric spaces and Pompeiu-Hausdorff fuzzy b-metric spaces will be an important next step.

Key words: fuzzy metric space, fuzzy b-metric space, t-norm, fixed point, implicit relation.

Introduction and preliminaries

In 1975, Kramosil and Michalek (Kramosil & Michalek, 1975) proposed the idea of a fuzzy distance between two elements of a nonempty set, using the concepts of a fuzzy set and a t-norm.

A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions: T is continuous, associative, and commutative, $T(a,1) = a$ for all $a \in [0,1]$ and for all $a,b,c,d \in [0,1]$ if $a \leq c$ and $b \leq d$ then $T(a,b) \leq T(c,d)$.

The study of fixed point theory in metric spaces has a variety of applications in mathematics, particularly in solving differential equations. Many authors have studied the new class of generalized metric spaces, known as a b-metric space, introduced by Bakhtin (Bakhtin, 1989). For example, see (Aghajani et al., 2014; Aydi et al., 2012; Bota et al., 2011; Czerwik, 1993; Došenović et al., 2023; George & Veeramani, 1997; Makran et al., 2023; Rakić et al., 2020).

The relationship between b-metric and fuzzy metric spaces is considered in (Hassanzadeh & Sedghi, 2018). Conversely, (Sedghi & Shobe, 2012) introduced the concept of a fuzzy b-metric space, substituting the triangle inequality with a weaker one.

In this paper, a general common fixed point theorem for a pair of multi-valued mappings in Pompeiu-Hausdorff fuzzy b-metric spaces is of interest to be proven.

DEFINITION 1. *(Sedghi & Shobe, 2012) A 3-tuple (X, M, T) is called a fuzzy $b-$metric space if X is an arbitrary nonempty set, T is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the conditions for all $x, y, z \in X,\ t, u > 0$ and a given real number $s \geq 1$:*

(b_1) $M(x,y,t) > 0,$

(b_2) $M(x,y,t)=1$ *if and only if* $x=y$,

(b_3) $M(x,y,t)=M(y,x,t)$,

(b_4) $M(x,z,s(t+u))\geq T(M(x,y,t),M(y,z,u))$,

(b_5) $M(x,y,.):(0,\infty)\to[0,1]$ *is continuous.*

REMARK 1. In this paper we will further use a fuzzy b-metric space in the sense of Definition 1 with the additional condition $\lim\limits_{t\to\infty} M(x,y,t)=1$.

Note that every fuzzy metric space is a fuzzy $b-$metric space with $s=1$. But the converse need not be true as is shown in the following example.

EXAMPLE 1. (Dahhouch et al., 2024) Let $M(x,y,t)=e^{-\frac{|x-y|^p}{t}}$, where $p>1$ is a real number, and $T(a,b)=a.b$ for all $a,b\in[0,1]$. Then (X,M,T) is a fuzzy b-metric space with $s=2^{p-1}$.

DEFINITION 2. *(Sedghi & Shobe, 2012, 2014)*
Let (X,M,T) be a fuzzy b-metric space.

(i) A sequence (x_n) converges to x if $M(x_n,x,t)\to 1$ as $n\to\infty$ for each $t>0$. In this case, we write $\lim\limits_{n\to\infty} x_n=x$.

(ii) A sequence (x_n) is called a Cauchy sequence if for all $\varepsilon\in(0,1)$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ for all $n,m\geq n_0$.

(iii) The fuzzy b-metric space (X,M,T) is said to be complete if every Cauchy sequence is convergent.

(iv) A subset $A\subset X$ is said to be closed if every sequence $x_n\in A$ such that $x_n\longrightarrow x$ we have $x\in A$.

(v) A subset $A\subset X$ is said to be compact if every sequence $x_n\in A$ has a convergent subsequence.

LEMMA 1. *(Sedghi & Shobe, 2012, 2014)*
In a fuzzy $b-$metric space (X,M,T)

(i) If a sequence (x_n) in X converges to x, then x is unique.

(ii) If a sequence (x_n) in X converges to x, then it is a Cauchy sequence.

Main results

In this section, a new space called a Pompeiu-Hausdorff fuzzy b-metric space was constructed. Several examples were presented to illustrate the properties of this space. In this framework, some new common fixed point results were formulated and proven. The following was introduced as the starting point:
Throughout this paper, $C(X)$ will denote the family of nonempty compact subsets of X, and $CL(X)$ will denote the family of nonempty closed subsets of X. For all $A, B \in C(X)$ and for all $t > 0$, we define a function on $C(X) \times C(X) \times (0, \infty)$ by

$$H_M(A, B, t) = \min\left\{\inf_{a\in A} M(a, B, t), \inf_{b\in B} M(A, b, t)\right\},$$

where $M(C, y, t) = \sup_{z\in C} M(z, y, t)$.

Then H_M is called the Pompeiu-Hausdorff fuzzy b-metric induced by the fuzzy b-metric M. The triplet $(C(X), H_M, T)$ is called a Pompeiu-Hausdorff fuzzy b-metric space.

We define also $\delta_M(A, B, t)$ by

$$\delta_M(A, B, t) = \inf\{M(a, b, t), \quad a \in A \quad b \in B\}, \quad t > 0,$$

and it follows immediately from the definition of δ_M that

$$\delta_M(A, B, t) = 1 \iff A = B = \{.\} \text{ and}$$

$$M(a, b, t) \geq \delta_M(A, B, t) \quad \forall a \in A \quad \forall b \in B, \quad t > 0.$$

Indeed: If $A = B = \{.\}$, then $\delta_M(A, B, t) = \delta_M(\{a\}, \{a\}, t) = M(a, a, t) = 1$.

Now, if $\delta_M(A, B, t) = 1$, $\forall t > 0$, then $\forall a \in A$, $b \in B$, $\quad M(a, b, t) \geq \delta_M(A, B, t) = 1$.

So, $A = B = \{a\} = \{b\} = \{.\}$.

LEMMA 2. *Let (X, M, T) be a fuzzy b-metric space with the constant s. Then H_M is a fuzzy set on $C(X) \times C(X) \times (0, \infty)$ satisfying the conditions for all $A, B, C \in C(X)$, $t, u > 0$:*

(H_1) $H_M(A, B, t) > 0$,

(H_2) $H_M(A,B,t)=1$ *if and only if* $A=B$,

(H_3) $H_M\left(A,C,s(t+u)\right)\geq T\left(H_M\left(A,B,t\right),H_M(B,C,u)\right)$,

(H_4) $H_M(A,B,.):(0,\infty)\to[0,1]$ *is continuous.*

(H_5) $\lim_{t\to\infty}H_M(A,B,t)=1$ *if and only if* $\lim_{t\to\infty}M(x,y,t)=1$.

Proof

(H_1) Let $A,B\in C(X)$, and show that $H_M(A,B,t)>0,\quad\forall t>0$, one obtains

$$\begin{aligned}H_M(A,B,t) &= \min\left\{\inf_{a\in A}M(a,B,t),\inf_{b\in B}M(A,b,t)\right\},\\ &= \min\left\{\inf_{a\in A}\sup_{b\in B}M(a,b,t),\inf_{b\in B}\sup_{a\in A}M(a,b,t)\right\}.\end{aligned}$$

Suppose that $H_M(A,B,t)=\inf_{a\in A}\sup_{b\in B}M(a,b,t)$, according to the characterization of $\inf$ and $\sup$, there exist $a_n\in A, b_n\in B$ such that $\lim_{n\to\infty}M(a_n,b_n,t)=H_M(A,B,t),\quad\forall t>0.$

Since A,B two compact, then $(a_n),(b_n)$ admit respectively two subsequences also noted $(a_n),(b_n)$ such that $a_n\to a\in A$ and $b_n\to b\in B$ $(n\to\infty)$.

By Definition 1 (b_4), one gets

$$\begin{aligned}M(a_n,b_n,t) &\geq T\left(M\left(a_n,a,\frac{t}{2s}\right),M\left(a,b_n,\frac{t}{2s}\right)\right)\\ &\geq T\left(M\left(a_n,a,\frac{t}{2s}\right),T\left(M\left(a,b,\frac{t}{4s^2}\right),M\left(b,b_n,\frac{t}{4s^2}\right)\right)\right),\end{aligned}$$

letting $n\to\infty$ one obtains

$$\begin{aligned}H_M(A,B,t) &\geq T\left(1,T\left(M\left(a,b,\frac{t}{4s^2}\right),1\right)\right)\\ &= M\left(a,b,\frac{t}{4s^2}\right)>0,\quad\forall t>0.\end{aligned}$$

Similarly, if $H_M(A,B,t)=\inf_{b\in B}\sup_{a\in A}M(a,b,t)$, then $H_M(A,B,t)>0$, $\forall t>0.$

(H_2) Let $A,B\in C(X)$, $t>0$ and show that $H_M(A,B,t)=1,\Leftrightarrow\ A=B$.

If $A=B$, there exists $H_M(A,B,t)=H_M(A,A,t)=\inf_{a\in A}M(a,A,t)$.

According to the characterization of $\inf$, there exists $a_n \in A$, such that $\lim_{n\to\infty} M(a_n, A, t) = H_M(A, A, t)$. Since A is a compact, then (a_n) admits a subsequence also noted (a_n), such that $a_n \to a \in A$. On the other hand, there is

$$M(a_n, A, t) = \sup_{x\in A} M(a_n, x, t) \geq M(a_n, a, t),$$

letting $n \to \infty$ one obtains

$$H_M(A, A, t) = \lim_{n\to\infty} M(a_n, A, t) \geq \lim_{n\to\infty} M(a_n, a, t) = 1.$$

So, $H_M(A, A, t) = 1$.

Now, if $H_M(A, B, t) = 1$, then

$$\begin{aligned}
& \inf_{a\in A} M(a, B, t) = 1 \text{ and } \inf_{b\in B} M(A, b, t) = 1 \\
\Rightarrow\quad & \forall\, a \in A,\ M(a, B, t) = 1 \text{ and } \forall\, b \in B,\ M(A, b, t) = 1, \\
\Rightarrow\quad & \forall\, a \in A,\ \sup_{b\in B} M(a, b, t) = 1 \text{ and } \forall\, b \in B,\ \sup_{a\in A} M(a, b, t) = 1, \\
\Rightarrow\quad & \forall\, a \in A,\ \exists b_n \in B : \lim_{n\to\infty} M(a, b_n, t) = 1 \\
\text{and}\quad & \forall\, b \in B,\ \exists a_n \in A : \lim_{n\to\infty} M(a_n, b, t) = 1, \\
\Rightarrow\quad & \forall\, a \in A,\ \exists b_n \in B :\ b_n \to a \in B \text{ and } \forall\, b \in B,\ \exists a_n \in A :\ a_n \to b \in A, \\
\Rightarrow\quad & A = B.
\end{aligned}$$

(H_3) Let $A, B, C \in C(X),\ t, u > 0$ and $s \geq 1$, show that

$$H_M\left(A, C, s(t+u)\right) \geq T\left(H_M\left(A, B, t\right), H_M(B, C, u)\right).$$

Suppose that $H_M\left(A, C, s(t+u)\right) = \inf_{a\in A} M\left(a, C, s(t+u)\right)$. First one proves that $\forall a \in A,\ \forall b \in B$, one has

$$M\left(a, C, s(t+u)\right) \geq T(M(a, b, t), M(b, C, u)).$$

Indeed, there is $M(b, C, u) = \sup_{c\in C} M(b, c, u)$, then there exists $c_n \in C$ such that:

$\lim_{n\to\infty} M(b, c_n, u) = M(b, C, u)$. By Definition 1 (b_4) one gets

$$M(a, c_n, s(t+u)) \geq T(M(a, b, t), M(b, c_n, u)),$$

letting $n \to \infty$ one has

$$
\begin{aligned}
M(a,C,s(t+u)) &\geq \lim_{n\to\infty} M(a,c_n,s(t+u)) \\
&\geq \lim_{n\to\infty} T(M(a,b,t),M(b,c_n,u)) \\
&= T(M(a,b,t),\lim_{n\to\infty} M(b,c_n,u)) \\
&= T(M(a,b,t),M(b,C,u)).
\end{aligned}
$$

So,

$$
M\,(a,C,s(t+u)) \geq T(M(a,b,t),M(b,C,u)). \tag{1}
$$

Now, by (1), one has

$$
\begin{aligned}
M\,(a,C,s(t+u)) &\geq T(M(a,b,t),M(b,C,u)), \\
&\geq T(M(a,b,t),\inf_{b\in B} M(b,C,u)), \quad \forall b\in B.
\end{aligned}
$$

Then

$$
\begin{aligned}
M\,(a,C,s(t+u)) &\geq T(\sup_{b\in B} M(a,b,t),\inf_{b\in B} M(b,C,u)), \\
&= T(M(a,B,t),\inf_{b\in B} M(b,C,u)), \quad \forall a\in A.
\end{aligned}
$$

This implies

$$
\begin{aligned}
H_M\,(A,C,s(t+u)) &= \inf_{a\in A} M\,(a,C,s(t+u)) \\
&\geq T(\inf_{a\in A} M(a,B,t),\inf_{b\in B} M(b,C,u)) \\
&\geq T(H_M(A,B,t),H_M(B,C,u)).
\end{aligned}
$$

Because $(\inf_{a\in A} M(a,B,t) \geq H_M(A,B,t)$ and $\inf_{b\in B} M(b,C,u) \geq H_M(B,C,u))$.

Similarly, if $H_M\,(A,C,s(t+u)) = \inf_{c\in C} M\,(A,c,s(t+u))$, then

$$
H_M\,(A,C,s(t+u)) \geq T(H_M(A,B,t),H_M(B,C,u)).
$$

(H_4) Let $A,B\in C(X)$, show that $H_M(A,B,.):(0,\infty)\to[0,1]$ is continuous, let (t_n) a sequence of elements in $(0,\infty)$ which converges to t, then one proves $\lim_{n\to\infty} H_M(A,B,t_n) = H_M(A,B,t)$.

By Definition 1 (b_5), for all $(a,b) \in A \times B$, one gets

$$\lim_{n\to\infty} M(a,b,t_n) = M(a,b,t), \text{ i.e.,}$$

$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n > n_0 :$

$$\begin{aligned}
& M(a,b,t) - \varepsilon \le M(a,b,t_n) \le M(a,b,t) + \varepsilon, \\
\Rightarrow \; & \sup_{b\in B} M(a,b,t) - \varepsilon \le \sup_{b\in B} M(a,b,t_n) \le \sup_{b\in B} M(a,b,t) + \varepsilon, \\
\Rightarrow \; & M(a,B,t) - \varepsilon \le M(a,B,t_n) \le M(a,B,t) + \varepsilon,
\end{aligned}$$

$$\Rightarrow \; \inf_{a\in A} M(a,B,t) - \varepsilon \le \inf_{a\in A} M(a,B,t_n) \le \inf_{a\in A} M(a,B,t) + \varepsilon. \tag{2}$$

Similarly, one finds

$$\inf_{b\in B} M(A,b,t) - \varepsilon \le \inf_{b\in B} M(A,b,t_n) \le \inf_{b\in B} M(A,b,t) + \varepsilon. \tag{3}$$

Thus, by (2) and (3), one can deduce

$$H_M(A,B,t) - \varepsilon \le H_M(A,B,t_n) \le H_M(A,B,t) + \varepsilon.$$

So, $\lim_{n\to\infty} H_M(A,B,t_n) = H_M(A,B,t)$.

(H_5) Let $A, B \in C(X),\ t > 0$ show that $\lim_{t\to\infty} H_M(A,B,t) = 1$ if and only if $\lim_{t\to\infty} M(x,y,t) = 1$.

If $A = \{x\}$ and $B = \{y\}$, one has

$$\begin{aligned}
\lim_{t\to\infty} H_M(A,B,t) = 1 \quad & \Rightarrow \quad \lim_{t\to\infty} H_M(\{x\},\{y\},t) = 1 \\
& \Rightarrow \quad \lim_{t\to\infty} M(x,y,t) = 1.
\end{aligned}$$

Now, if $\lim_{t\to\infty} M(x,y,t) = 1$. Suppose that $H_M(A,B,t) = \inf_{a\in A} \sup_{b\in B} M(a,b,t)$, according to the characterization of inf and sup, there exist $a_n \in A, b_n \in B$ such that $\lim_{n\to\infty} M(a_n,b_n,t) = H_M(A,B,t), \quad \forall t > 0$. Thus

$$\begin{aligned}
\lim_{t\to\infty} H_M(A,B,t) &= \lim_{t\to\infty} \left(\lim_{n\to\infty} M(a_n,b_n,t) \right) \\
&= \lim_{n\to\infty} \left(\lim_{t\to\infty} M(a_n,b_n,t) \right) = 1.
\end{aligned}$$

Similarly, if $H_M(A,B,t) = \inf_{b\in B} \sup_{a\in A} M(a,b,t)$, then $\lim_{t\to\infty} H_M(A,B,t) = 1$.

DEFINITION 3. *Let Ψ be the set of all continuous functions $\psi(t_1, t_2, t_3, t_4) : [0,1]^4 \longrightarrow \mathbb{R}$ such that:*
(ψ_0) : *ψ is nonincreasing in variable t_1 and nondecreasing in variables t_2, t_3, t_4.*
(ψ_1) : $\forall u, v \in [0,1] \quad \psi(u, v, v, u) \leq 0$ *or* $\psi(u, v, u, v) \leq 0 \Longrightarrow v < u.$
(ψ_2) : $\forall u \in [0,1] \quad \psi(u, u, 1, 1) \leq 0 \Longrightarrow u = 1.$

EXAMPLE 2. $\psi(t_1, t_2, t_3, t_4) = \frac{1}{p} ln\,(t_2) - ln\,(t_1)$, with $p > 1$.

EXAMPLE 3. $\psi(t_1, t_2, t_3, t_4) = \dfrac{1}{1+t_1} - \dfrac{t_1}{2} - \lambda \left(\dfrac{1}{1+t_2} - \dfrac{t_2}{2} \right),\ 0 < \lambda < 1.$

EXAMPLE 4. $\psi(t_1, t_2, t_3, t_4) = \beta(t_1) - \lambda\beta(t_2)$, with $\beta : (0,1] \to [0,\infty)$ and $\beta(1) = 0$ is a continuous function strictly decreasing, $0 < \lambda < 1$.

EXAMPLE 5. $\psi(t_1, t_2, t_3, t_4) = \beta(t_1) - \lambda\beta(\frac{t_2+t_3}{4} + \frac{t_4}{2})$, with $\beta : (0,1] \to [0,\infty)$ and $\beta(1) = 0$ is a continuous function strictly decreasing, $0 < \lambda < 1$.

EXAMPLE 6. $\psi(t_1, t_2, t_3, t_4) = \gamma(t_1) - k(t)\gamma(t_2)$, with $\gamma : (0,1] \to [0,\infty)$, $\gamma(1) = 0$ is a continuous function strictly decreasing and k is a function from $(0,\infty)$ into $(0,1)$.

DEFINITION 4. *A function $\Im : X \longrightarrow CL(X)$, where (X, M, T) is a fuzzy b-metric space, called closed if for all sequences (x_n) and (y_n) of elements from X and $x, y \in X$ such that $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$ and $y_n \in \Im(x_n)$ for every $n \in \mathbb{N}$, one has $y \in \Im(x)$.*

THEOREM 1. *Let (X, M, T) be a complete fuzzy b-metric space with constant s, it can be supposed that M is continuous with respect to one of its variables. Let $\Im, \Re : X \longrightarrow C(X)$ be two closed maps and $\psi \in \Psi$ such that*

$$\psi\left(H_M(\Im(x), \Re(y), t), M(x, y, t), M(\Im(x), x, t), M(\Re(y), y, t) \right) \leq 0. \quad (4)$$

Then $\Im$ and $\Re$ have a common fixed point $x \in X$.

Moreover, if x is absolutely fixed for $\Im$ and $\Re$ (which means that $\Im(x) = \{x\}$ and $\Re(x) = \{x\}$), then the fixed point is unique.

For the proof of this theorem one needs two lemmas.

LEMMA 3. *In a fuzzy b-metric space (X, M, T), if the function M is continuous with respect to one of its variable, then it is continuous with respect to the other.*

Proof

Suppose that M is continuous with respect to the first variable, and let (y_n) be a sequence of the elements of X such that (y_n) is b-convergent to $y \in X$. Then since M is symmetric one has for all $x \in X$:

$$\lim_{n\to\infty} M(x, y_n, t) = \lim_{n\to\infty} M(y_n, x, t) = M(y, x, t) = M(x, y, t), \quad t > 0.$$

LEMMA 4. *Let (X, M, T) be a fuzzy b-metric space and let $A \in C(X)$. If M is continuous with respect to one of its variables, then for all $x \in X$, there exists $y_0 \in A$ such that*

$$M(x, A, t) = \sup_{y\in A} M(x, y, t) = M(x, y_0, t), \quad t > 0.$$

Proof

One has $M(x, A, t) = \sup_{y\in A} M(x, y, t)$, so for every $n \in \mathbb{N}^*,\ t > 0$, there exists $x_n \in A$ such that

$$M(x, A, t) - \frac{1}{n} < M(x, x_n, t) \leq M(x, A, t) < M(x, A, t) + \frac{1}{n}.$$

Since A is compact, (x_n) has a subsequence, also noted (x_n), which is b-convergent to $y_0 \in A$. Therefore:

$$|M(x, x_n, t) - M(x, A, t)| < \frac{1}{n} \longrightarrow 0 \quad \text{when} \quad n \longrightarrow \infty.$$

from where $\lim_{n\to\infty} M(x, x_n, t) = M(x, A, t)$, so since M is continuous, one deduces that

$$\lim_{n\to\infty} M(x, x_n, t) = M(x, y_0, t),$$

hence from the uniqueness of the limit in a fuzzy b-metric space, one has $M(x, y_0, t) = M(x, A, t), \quad t > 0$.

Proof of the Theorem

Let $x_0 \in X$, and $x_1 \in \Im x_0$. Since $x_1 \in \Im x_0$, one has

$$H_M(\Im x_0, \Re x_1, t) = \min\left\{\inf_{x\in\Im x_0} M(x, \Re x_1, t), \inf_{y\in\Re x_1} M(\Im x_0, y, t)\right\}$$

$$\leq \inf_{x \in \Im x_0} M(x, \Re x_1, t)$$
$$\leq M(x_1, \Re x_1, t) = \sup_{y \in \Re x_1} M(x_1, y, t).$$

$\Re x_1$ is compact and M is continuous with respect to one of its variables, so, according to Lemma 4, there exists $x_2 \in \Re x_1$ such that

$$H_M(\Im x_0, \Re x_1, t) \leq M(x_1, x_2, t), \quad t > 0.$$

In the same manner, since $x_2 \in \Re x_1$, one gets

$$H_M(\Im x_2, \Re x_1, t) \leq M(\Im x_2, x_2, t) = \sup_{x \in \Im x_2} M(x, x_2, t).$$

$\Im x_2$ is compact and M is continuous with respect to one of its variables, so, according to Lemma 4, there exists $x_3 \in \Im x_2$ such that

$$H_M(\Im x_2, \Re x_1, t) \leq M(x_2, x_3, t), \quad t > 0.$$

In the same way, there exists $x_4 \in \Re x_3$ such that

$$H_M(\Im x_2, \Re x_3, t) \leq M(x_3, x_4, t), \quad t > 0.$$

By recurrence, we construct a sequence (x_n) such that $x_{2n+1} \in \Im x_{2n}$, and $x_{2n+2} \in \Re x_{2n+1}$ which satisfies:

$$H_M(\Im x_{2n}, \Re x_{2n-1}, t) \leq M(x_{2n+1}, x_{2n}, t), \tag{5}$$

and

$$H_M(\Im x_{2n}, \Re x_{2n+1}, t) \leq M(x_{2n+1}, x_{2n+2}, t). \tag{6}$$

According to (4), with $x = x_{2n}$ and $y = x_{2n-1}$ one has

$$\psi\left(\begin{array}{c} H_M(\Im x_{2n}, \Re x_{2n-1}, t), M(x_{2n}, x_{2n-1}, t), M(\Im x_{2n}, x_{2n}, t), \\ M(\Re x_{2n-1}, x_{2n-1}, t) \end{array}\right) \leq 0.$$

Since $x_{2n+1} \in \Im x_{2n}$, and $x_{2n} \in \Re x_{2n-1}$, then

$$M(\Im x_{2n}, x_{2n}, t) \geq M(x_{2n+1}, x_{2n}, t), \tag{7}$$

and

$$M(\Re x_{2n-1}, x_{2n-1}, t) \geq M(x_{2n}, x_{2n-1}, t). \tag{8}$$

By (5), (7), (8) and (ψ_0) one has

$$\psi\left(\begin{array}{c} M\left(x_{2n+1}, x_{2n}, t\right), M\left(x_{2n}, x_{2n-1}, t\right), M\left(x_{2n+1}, x_{2n}, t\right), \\ M\left(x_{2n}, x_{2n-1}, t\right) \end{array}\right) \leq 0.$$

By (ψ_1), one has

$$M\left(x_{2n}, x_{2n+1}, t\right) < M\left(x_{2n-1}, x_{2n}, t\right), \qquad n \in \mathbb{N},\ t > 0. \tag{9}$$

In the same way, by (4) with $x = x_{2n}$ and $y = x_{2n+1}$, one gets

$$\psi\left(\begin{array}{c} H_M\left(\Im x_{2n}, \Re x_{2n+1}, t\right), M\left(x_{2n}, x_{2n+1}, t\right), M\left(\Im x_{2n}, x_{2n}, t\right), \\ M\left(\Re x_{2n+1}, x_{2n+1}, t\right) \end{array}\right) \leq 0.$$

Since $x_{2n+2} \in \Re x_{2n+1}$, then

$$M\left(\Re x_{2n+1}, x_{2n+1}, t\right) \geq M\left(x_{2n+2}, x_{2n+1}, t\right). \tag{10}$$

By (6), (7), (10) and (ψ_0) one has

$$\psi\left(\begin{array}{c} M\left(x_{2n+1}, x_{2n+2}, t\right), M\left(x_{2n}, x_{2n+1}, t\right), M\left(x_{2n+1}, x_{2n}, t\right), \\ M\left(x_{2n+2}, x_{2n+1}, t\right) \end{array}\right) \leq 0.$$

By (ψ_1), one has

$$M\left(x_{2n}, x_{2n+1}, t\right) < M\left(x_{2n+1}, x_{2n+2}, t\right), \qquad n \in \mathbb{N},\ t > 0. \tag{11}$$

According to (9) and (11), one has:

$$M\left(x_{n-1}, x_n, t\right) < M\left(x_n, x_{n+1}, t\right), \qquad n \in \mathbb{N}^*,\ t > 0. \tag{12}$$

Therefore, $(M\left(x_n, x_{n+1}, t\right))$ is a strictly increasing sequence of positive real numbers in $[0, 1]$.

One puts $M_n(t) = M\left(x_n, x_{n+1}, t\right)$. Then, $(M_n(t))$ is a strictly increasing sequence.

So, there exists $M(t)$ such that $\lim\limits_{n \to \infty} M_n(t) = M(t),\ t > 0.$

Suppose that $0 < M(t) < 1$. According to (4), with $x = x_{2n}$ and $y = x_{2n+1}$ one has

$$\psi\left(\begin{array}{c} H_M\left(\Im x_{2n}, \Re x_{2n+1}, t\right), M\left(x_{2n}, x_{2n+1}, t\right), M\left(\Im x_{2n}, x_{2n}, t\right), \\ M\left(\Re x_{2n+1}, x_{2n+1}, t\right) \end{array}\right) \leq 0.$$

By (6), (7), (10) and (ψ_0), one has

$$\begin{aligned} &\psi\left(\begin{array}{c} M\left(x_{2n+1}, x_{2n+2}, t\right), M\left(x_{2n}, x_{2n+1}, t\right), M\left(x_{2n+1}, x_{2n}, t\right), \\ M\left(x_{2n+2}, x_{2n+1}, t\right) \end{array}\right) \leq 0, \\ \Rightarrow\quad &\psi\left(M_{2n+1}(t), M_{2n}(t), M_{2n}(t), M_{2n+1}(t) \right) \leq 0. \end{aligned}$$

Since ψ is continuous function, letting $n \to \infty$ one obtains

$$\psi\left(M(t), M(t), M(t), M(t) \right) \leq 0.$$

By (ψ_1), one gets $M(t) < M(t)$, a contradiction. Then $M(t) = 1$. Now, we will prove that (x_n) is a Cauchy squence. Suppose that (x_n) is not a Cauchy sequence, then for all $0 < \varepsilon < 1$, there exist two sub-sequences $(x_{n(i)})$ and $(x_{m(i)})$ such that for each $i \in \mathbb{N}$, let $n(i), m(i) \in \mathbb{N}$ satisfying $n(i) > m(i) \geq i$, such that

$$M\left(x_{2n(i)}, x_{2m(i)+1}, t\right) \leq 1 - \varepsilon. \tag{13}$$

Now, we choose $2n(i)$ corresponding to $2m(i)+1$ such that it is the smallest even integer with $n(i) > m(i)$ and satisfies Inequality (13). Hence,

$$M\left(x_{2n(i)-1}, x_{2m(i)+1}, t\right) > 1 - \varepsilon \tag{14}$$

By (13), we get

$$\begin{aligned} 1 - \varepsilon &\geq M\left(x_{2n(i)}, x_{2m(i)+1}, t\right) \\ &\geq T\left(M\left(x_{2n(i)}, x_{2n(i)-1}, \frac{t}{2s}\right), M\left(x_{2n(i)-1}, x_{2m(i)+1}, \frac{t}{2s}\right)\right). \end{aligned}$$

By (14), we have

$$1 - \varepsilon \geq M\left(x_{2n(i)}, x_{2m(i)+1}, t\right) \;\;\geq\;\; T\left(M\left(x_{2n(i)}, x_{2n(i)-1}, \frac{t}{2s}\right), 1 - \varepsilon\right)$$

$$= T\left(M_{2n(i)-1}\left(\frac{t}{2s}\right), 1-\varepsilon\right)$$

letting $i \to \infty$ we obtain

$$1-\varepsilon \geq \lim_{i\to\infty} M\left(x_{2n(i)}, x_{2m(i)+1}, t\right) \geq T\left(1, 1-\varepsilon\right) = 1-\varepsilon.$$

So

$$\lim_{i\to\infty} M\left(x_{2n(i)}, x_{2m(i)+1}, t\right) = 1-\varepsilon.$$

By (4), with $x = x_{2m(i)}$ and $y = x_{2n(i)-1}$, we get

$$\psi\begin{pmatrix} H_M\left(\Im x_{2m(i)}, \Re x_{2n(i)-1}, t\right), M\left(x_{2m(i)}, x_{2n(i)-1}, t\right), \\ M\left(\Im x_{2m(i)}, x_{2m(i)}, t\right), M\left(\Re x_{2n(i)-1}, x_{2n(i)-1}, t\right) \end{pmatrix} \leq 0.$$

Since $x_{2m(i)+1} \in \Im x_{2m(i)}$, and $x_{2n(i)} \in \Re x_{2n(i)-1}$, then

$$M\left(\Im x_{2m(i)}, x_{2m(i)}, t\right) \geq M\left(x_{2m(i)+1}, x_{2m(i)}, t\right), \tag{15}$$

and

$$M\left(\Re x_{2n(i)-1}, x_{2n(i)-1}, t\right) \geq M\left(x_{2n(i)}, x_{2n(i)-1}, t\right). \tag{16}$$

By (15), (16) and (ψ_0), we have

$$\psi\begin{pmatrix} H_M\left(\Im x_{2m(i)}, \Re x_{2n(i)-1}, t\right), M\left(x_{2m(i)}, x_{2n(i)-1}, t\right), \\ M\left(x_{2m(i)+1}, x_{2m(i)}, t\right), M\left(x_{2n(i)}, x_{2n(i)-1}, t\right) \end{pmatrix} \leq 0.$$

Since $x_{2m(i)+1} \in \Im x_{2m(i)}$ and $x_{2n(i)} \in \Re x_{2n(i)-1}$, then

$$\begin{aligned} H_M\left(\Im x_{2m(i)}, \Re x_{2n(i)-1}, t\right) &\leq M\left(x_{2m(i)+1}, \Re x_{2n(i)-1}, t\right) \\ &= T\left(M\left(x_{2m(i)+1}, \Re x_{2n(i)-1}, t\right), 1\right) \\ &= T\begin{pmatrix} M\left(x_{2m(i)+1}, \Re x_{2n(i)-1}, t\right), \\ M\left(x_{2n(i)}, \Re x_{2n(i)-1}, t\right) \end{pmatrix} \\ &\leq M\left(x_{2m(i)+1}, x_{2n(i)}, 2ts\right). \end{aligned}$$

So

$$H_M\left(\Im x_{2m(i)}, \Re x_{2n(i)-1}, t\right) \leq M\left(x_{2m(i)+1}, x_{2n(i)}, 2ts\right). \tag{17}$$

By (17) and (ψ_0), we get

$$\psi\begin{pmatrix} M\left(x_{2n(i)}, x_{2m(i)+1}, 2st\right), \\ T\left(M_{2m(i)}\left(\frac{t}{2s}\right), M\left(x_{2m(i)+1}, x_{2n(i)-1}, \frac{t}{2s}\right)\right), \\ M_{2m(i)}(t), M_{2n(i)-1}(t) \end{pmatrix} \leq 0$$

$$\Rightarrow \quad \psi \left(\begin{array}{c} M\left(x_{2n(i)}, x_{2m(i)+1}, 2st\right), T\left(M_{2m(i)}\left(\frac{t}{2s}\right), 1-\varepsilon\right), \\ M_{2m(i)}\left(t\right), M_{2n(i)-1}\left(t\right) \end{array} \right) \leq 0.$$

letting $i \to \infty$, we obtain

$$\begin{aligned} &\psi\left(\ 1-\varepsilon, T(1, 1-\varepsilon), 1, 1\ \right) \leq 0 \\ \Rightarrow \quad &\psi\left(\ 1-\varepsilon, 1-\varepsilon, 1, 1\ \right) \leq 0. \end{aligned}$$

By (ψ_2), we get $1-\varepsilon = 1$, which is a contradiction.

Hence, (x_n) is a Cauchy sequence in a complete fuzzy b-metric space. So, there exists $x \in X$ such that $\lim\limits_{n\to\infty} M(x_n, x, t) = 1$. Next, we show that $x \in \Im x \cap \Re x$, indeed, we have $x_{2n} \in \Re x_{2n-1}$ and $x_{2n+1} \in \Im x_{2n}, \quad n \in \mathbb{N}$.

Since $\lim\limits_{n\to\infty} x_{2n-1} = \lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} x_{2n+1} = x$, $\Im$ and $\Re$ are closed, then $x \in \Im x \cap \Re x$.

Unicity. Suppose that $\Im x = \{x\}$ and that there exists $y \in X$ is another common fixed point of $\Im$ and $\Re$, such that $\Re y = \{y\}$. Then by (4), we have

$$\begin{aligned} &\psi \left(\begin{array}{c} H_M\left(\Im x, \Re y, t\right), M\left(x, y, t\right), M\left(\Im x, x, t\right), \\ M\left(\Re y, y, t\right) \end{array} \right) \leq 0 \\ \Rightarrow \quad &\psi \left(\begin{array}{c} M\left(x, y, t\right), M\left(x, y, t\right), M\left(x, x, t\right), \\ M\left(y, y, t\right) \end{array} \right) \leq 0 \\ \Rightarrow \quad &\psi\left(\ M\left(x, y, t\right), M\left(x, y, t\right), 1, 1\ \right) \leq 0. \end{aligned}$$

By (ψ_2) we get $M\left(x, y, t\right) = 1$, then $x = y$.

THEOREM 2. *Let (X, M, T) be a complete fuzzy b-metric space with the constant s. Let $\Im : X \longrightarrow CL(X)$ be a closed maps and $\psi \in \Psi$ such that*

$$\psi\left(\ \delta_M\left(\Im(x), \Im(y), t\right), M\left(x, y, t\right), M\left(\Im(x), x, t\right), M\left(\Im(y), y, t\right)\ \right) \leq 0. \tag{18}$$

Then $\Im$ has a unique fixed point $x \in X$.

Proof Let $x_0 \in X$, define the sequence (x_n) of the elements from X such that: $x_{n+1} \in \Im x_n$ for every $n \in \mathbb{N}$.

According to (18), with $x = x_{n-1}$ and $y = x_n$ we have

$$\psi\left(\ \delta_M\left(\Im x_{n-1}, \Im x_n, t\right), M\left(x_{n-1}, x_n, t\right), M\left(\Im x_{n-1}, x_{n-1}, t\right), M\left(\Im x_n, x_n, t\right)\ \right) \leq 0.$$

Since $x_n \in \Im x_{n-1}$ and $x_{n+1} \in \Im x_n$ we get

$$M\left(\Im x_{n-1}, x_{n-1}, t\right) \geq M\left(x_n, x_{n-1}, t\right), \quad M\left(\Im x_n, x_n, t\right) \geq M\left(x_{n+1}, x_n, t\right)$$

and $\delta_M(\Im x_{n-1}, \Im x_n, t) \leq M(x_n, x_{n+1}, t)$.

By (ψ_0), we have

$\psi\left(\ M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t), M(x_n, x_{n-1}, t), M(x_{n+1}, x_n, t)\ \right) \leq 0.$

By (ψ_1), we have

$$M(x_{n-1}, x_n, t) < M(x_n, x_{n+1}, t), \qquad n \in \mathbb{N}^*,\ t > 0.$$

So, using the same argument as in Theorem 1, we deduce that x_n is a Cauchy sequence and converges to $x \in X$, the fixed point of $\Im$.

Unicity. Suppose that there exists $y \in X$ is another fixed point of $\Im$. Then by (18), we have

$$\psi\begin{pmatrix} \delta_M(\Im x, \Im y, t), M(x, y, t), M(\Im x, x, t), \\ M(\Im y, y, t) \end{pmatrix} \leq 0$$

by (ψ_0), we have

$$\psi\begin{pmatrix} M(x, y, t), M(x, y, t), M(x, x, t), \\ M(y, y, t) \end{pmatrix} \leq 0$$
$$\Rightarrow \quad \psi\left(\ M(x, y, t), M(x, y, t), 1, 1\ \right) \leq 0.$$

By (ψ_2) we get $M(x, y, t) = 1$, then $x = y$.

As a consequence of Theorem 2, if $\Im = f$ is single-valued mapping, then we obtain the following:

Corollary 1. *Let (X, M, T) be a complete fuzzy b-metric space with the constant s. Let $f : X \longrightarrow X$ be a continuous mapping and $\psi \in \Psi$ such that*

$$\psi\left(\ M(fx, fy, t), M(x, y, t), M(fx, x, t), M(fy, y, t)\ \right) \leq 0.$$

Then f has a unique fixed point $x \in X$.

Example 7. Let X be the subset of $\mathbb{R}^3$ defined by $X = \{A, B, C, D\}$, where $A = (1,0,0),\ B = (0,1,0),\ C = (0,0,1)$ and $D = (2,2,2)$. $T(c,d) = c.d$ for all $c, d \in [0,1]$ and (X, M, T) is a complete fuzzy $b-$metric space such that:

$$M(x, y, t) = e^{\frac{-d(x,y)}{t}}, \quad x, y \in X,\ t > 0,$$

where $d(x, y)$ denotes the Euclidean distance of $\mathbb{R}^3$.

Let $\Im : X \to X$ be given by $\Im(A) = \Im(B) = \Im(C) = A, \Im(D) = B$.
Show that for all $x, y \in X$

$$\psi\left(\; M\left(\Im x, \Im y, t\right), M\left(x, y, t\right), M\left(\Im x, x, t\right), M\left(\Im y, y, t\right) \; \right) \leq 0.$$

with ψ as in Example 4, and $\beta(t) = -ln(t)$.

Indeed: If $x, y \in \{A, B, C\}$ we have $M\left(\Im x, \Im y, t\right) = M\left(A, A, t\right) = 1$.
So $\beta(M\left(\Im x, \Im y, t\right)) = 0 \leq \lambda\beta(M\left(x, y, t\right)),\ \lambda \in (0, 1)$.
If $x \in \{A, B, C\}$ and $y = D$, we find

$$M\left(\Im x, \Im y, t\right) = e^{\frac{-\sqrt{2}}{t}} \text{ and } M\left(x, y, t\right) = e^{\frac{-3}{t}}.$$

So, $\beta(M\left(\Im x, \Im y, t\right)) \leq \lambda\beta(M\left(x, y, t\right)),\ \lambda \in (\frac{\sqrt{2}}{3}, 1)$.

Now, all the hypotheses of Corollary 1 are satisfied and thus $\Im$ has a unique fixed point, which is $x = A$.

REMARK 2. From Corollary 1 and Example 6 we obtain Theorem 3.1 (Shen et al., 2012) Let $(X, M, *)$ be an M-complete fuzzy metric space and T a self-map of X and suppose that $\varphi : [0, 1] \longrightarrow [0, 1]$ satisfies the foregoing properties $(P1)$ and $(P2)$. Furthermore, let k be a function from $(0, \infty) \longrightarrow (0, 1)$. If for any $t > 0$, T satisfies the following condition: $\varphi(M(Tx, Ty, t)) \leq k(t)\varphi(M(x, y, t))$, where $x, y \in X$ and $x \neq y$, then T has a unique fixed point.

Conclusion

In this paper, a new space called a Pompeiu-Hausdorff fuzzy b-metric space is constructed. Some examples in this space are presented. Additionally, some new fixed-point results in this space are formulated and proven, which extends the results of (Shen et al., 2012), and the existence and uniqueness of the common fixed point in such a space are demonstrated.

The approach proposed may pave the way for new developments in generalized metrical structures and fixed-point theory.

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Aplicaciones multivaluadas en espacios b-métricos difusos de Pompeiu-Hausdorff y un resultado de punto fijo común

Noreddine Makran[a], *Stojan* N. Radenović[b]

[a] Universidad Mohammed Premier, Facultad de Ciencias, Departamento de Matemáticas, Oujda, Reino de Marruecos, **autor de correspondencia**

[b] Universidad de Belgrado, Facultad de Ingeniería Mecánica, Belgrado, República de Serbia

CAMPO: matemáticas, ciencias computacionales
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: El estudio de la teoría de conjuntos difusos surgió a partir de la presencia de la incertidumbre como parte esencial de los problemas del mundo real, lo que llevó a Zadeh a abordar el problema de la indeterminación. Zadeh introdujo el concepto de lógica difusa. A diferencia de la teoría lógica tradicional, en la que un elemento pertenece o no al conjunto, en la lógica difusa la afiliación del elemento al conjunto se expresa como un número del intervalo $[0, 1]$.

Métodos: La teoría de un punto fijo en espacios métricos difusos puede verse de diferentes maneras, una de las cuales implica el uso de la lógica difusa. Los espacios métricos difusos, que son tipos específicos de espacios topológicos con características "geométricas" agradables, poseen una serie de propiedades atractivas y se utilizan comúnmente tanto en ciencias puras como aplicadas. Los espacios métricos y sus diversas generalizaciones ocurren con frecuencia en aplicaciones de ciencias de la computación. Por esta razón, en este artículo se construye un nuevo espacio llamado espacio b-métrico difuso de Pompeiu-Hausdorff.

Resultados: En este espacio, también se formulan y prueban algunos nuevos resultados de punto fijo. Además, se investiga un teorema de punto fijo común general para un par de aplicaciones multivaluadas en espacios b-métricos difusos de Pompeiu-Hausdorff. Los hallazgos obtenidos en espacios métricos difusos, como los discutidos en el artículo de Shen, Y. et al. (2012. Fixed point theorems in fuzzy metric spaces. Applied Mathematics Letters, 25, pp.138-141), se generalizan a partir de los resultados de este documento, y se producen hallazgos específicos adicionales que se respaldan con ejemplos.

Conclusión: El estudio de la semántica denotacional y sus aplicaciones en la teoría de control utilizando espacios b-métricos difusos y espacios b-métricos difusos de Pompeiu-Hausdorff será un siguiente paso importante.

Palabras claves: espacio métrico difuso, espacio b-métrico difuso, norma t, punto fijo, relación implícita.

Многозначные отображения в нечетких b-метрических пространствах типа Помпею-Хаусдорфа и результат с общей неподвижной точкой

Нуреддин Макран[а], *Стоян* Н. Раденович[б]

[а] Университет Мохаммеда Премьера, факультет естественных наук, кафедра математики, г. Уджда, Королевство Марокко, **корреспондент**

[б] Белградский университет, машиностроительный факультет, г. Белград, Республика Сербия

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ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: Изучению теории нечетких множеств способствовало наличие неопределенности как неотъемлемой части реальных проблем, что привело Заде к решению проблемы неопределенности. Именно он ввел понятие нечеткой логики. В отличие от традиционной логической теории, где элемент либо принадлежит множеству, либо нет, в нечеткой логике принадлежность элемента к множеству выражается числом из интервала $[0, 1]$.

Методы: Теорию неподвижной точки в нечетких метрических пространствах можно рассматривать по-разному.

Один из подходов предполагает использование нечеткой логики. Нечеткие метрические пространства как специфические типы топологических пространств с приятными "геометрическими" характеристиками обладают рядом привлекательных свойств и широко используются как в чистой, так и в прикладной науке. Метрические пространства и их различные обобщения часто встречаются в приложениях вычислительных наук. В связи с вышеизложенным в данной работе сформировано новое пространство, называемое нечетким b-метрическим пространством Помпею-Хаусдорфа.

Результаты: В этом пространстве также сформулированы и доказаны некоторые новые результаты о неподвижной точке. Помимо того, исследована теорема об общей неподвижной точке для пары многозначных отображений в нечетких b-метрических пространствах типа Помпею-Хаусдорфа. Результаты, полученные в нечетких метрических пространствах, как, например, те, которые обсуждались в статье Шена и др. (Shen, Y. et al. 2012. Fixed point theorems in fuzzy metric spaces. Applied Mathematics Letters, 25, pp.138-141), обобщаются результатами данного исследования, а в дополнение приводятся конкретные выводы, подкрепленные примерами.

Выводы: Следующим важным шагом станет изучение денотационной семантики и ее применения в теории управления с использованием нечетких b-метрических пространств и нечетких b-метрических пространств Помпею-Хаусдорфа.

Ключевые слова: нечеткое метрическое пространство, нечеткое b-метрическое пространство, t-норма, фиксированная точка, неявное отношение.

Вишезначна пресликавања расплинутих б-метричких простора типа Помпеју-Хауздорф и резултат заједничке непокретне тачке

Нуредин Макран[а], *Стојан* Н. Раденовић[б]

[а] Универзитет „Мухамед Премијер”,
Природно-математички факултет, Одсек за математику,
Оујда, Краљевина Мароко, **аутор за преписку**

[б] Универзитет у Београду, Машински факултет,
Београд, Република Србија

ОБЛАСТ: математика
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Сажетак:

Увод/циљ: Проучавање теорије расплинутих скупова подстакнуто је несигурношћу као суштинском цртом проблема у реалном свету, што је навело Задеа да се позабави проблемом неодређености и уведе појам расплинуте логике. За разлику од традиционалне теорије логике, у којој један елемент припада или не припада неком скупу , у расплинутој логици припадност елемента неком скупу изражава се бројем из интервала $[0, 1]$.

Методе: Теорија непокретне тачке у расплинутим метричким просторима може се посматрати на различите начине, од којих један подразумева коришћење расплинуте логике. Расплинути метрички простори, као специфични типови тополошких простора са пријатним „геометријским” карактеристикама, имају много интересантних карактеристика и обично се користе и у чистим и у примењеним наукама. Метрички простори и њихове различите генерализације често се срећу у применама рачунарских наука. Зато је у овом раду конструисан нови простор назван расплинути б-метрички простор типа Помпеју-Хауздорф.

Резултати: У овом простору формулисани су и доказани неки нови резултати непокретне тачаке. При томе је испитана општа теорема заједничке непокретне тачке за пар вишезначног пресликавања у расплинутим б-метричким просторима типа Помпеју-Хауздорф. Налази добијени у расплинутим метричким просторима, као што су они наведени у чланку Шена и сарадника (Shen, Y. et al. 2012. Fixed point theorems in fuzzy metric spaces. Applied Mathematics Letters, 25, pp.138-141), генерализовани су резултатима овог рада који презентује и додатне специфичне налазе поткрепљене примерима.

Закључак: Следећи важан корак биће проучавање денотационе семантике и њене примене у теорији управљања помоћу расплинутих б-метичких простора и расплинутих б-метричких простора типа Помпеју-Хауздорф.

Кључне речи: расплинути метрички простор, расплинути б-метрички простор, т-норма, непокретна тачка, имплицитна релација.

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