Berezin inequalities for sums of operators and classical inequalities concerning the Berezin radius

Mehmet Gürdal[a], *Vuk* N. Stojiljković[b]

[a]Suleyman Demirel University, Department of Mathematics, Isparta, Republic of Türkiye,
e-mail: gurdalmehmet@sdu.edu.tr,
ORCID iD: https://orcid.org/0000-0003-0866-1869

[b]University of Novi Sad, Faculty of Science, Novi Sad, Republic of Serbia; Mathematical Grammar School, Belgrade, Republic of Serbia,
e-mail: vuk.stojiljkovic999@gmail.com, **corresponding author**,
ORCID iD: https://orcid.org/0000-0002-4244-4342

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Abstract:

Introduction/purpose: In this article, the author's goal is to seek to obtain new inequalities of the Berezin type.

Methods: The methods used are standard for operator theory.

Results: Various inequalities of the type given by Huban et al. and Erkan have been obtained.

Conclusions: In addition to obtaining various inequalities of the form given by Huban et al. and Erkan in particular, the authors sharpened the inequalities related to the Berezin norm.

Key words: Berezin norm, Berezin number, inequalities.

Introduction

The Berezin norm and the Berezin number of an operator have been researched for their many applications in numerical analysis, quantum physics, engineering, and other domains in the literature related to operator theory. In order to explain the Berezin number and norm, the authors first go over several ideas and traits of bounded linear operators on a Hilbert space.

Let $\mathfrak{L}(\mathfrak{H})$ denote the C^*-algebra of all bounded linear operators defined on a complex Hilbert space $\mathfrak{H}$ with the an inner product $\langle .,. \rangle$ and a corresponding norm $\|.\|$. Recall that the functional Hilbert space $\mathfrak{H} = \mathfrak{H}(\Xi)$ is a Hilbert space of complex-valued functions on a (nonempty) set Ξ such that the evaluation functionals $\varphi_\varrho(f) = f(\varrho)$, $\varrho \in \Xi$, are continuous on $\mathfrak{H}$ and for every $\varrho \in \Xi$ there exists a function $f_\varrho \in \mathfrak{H}$ such that $f_\varrho(\varrho) \neq 0$ or, equivalently, there is no $\varrho_0 \in \Xi$ such that $f(\varrho_0) = 0$ for all $f \in \mathfrak{H}$. The Riesz representation theorem ensures that for each $\varrho \in \Xi$ there is a unique element $\varkappa_\varrho \in \mathfrak{H}$ such that $f(\varrho) = \langle f, \varkappa_\varrho \rangle$ for all $f \in \mathfrak{H}$. The collection $\{\varkappa_\varrho : \varrho \in \Xi\}$ is called the reproducing kernel of $\mathfrak{H}$. For $\varrho \in \Xi$, let $\widehat{\varkappa}_\varrho := \frac{\varkappa_\varrho}{||\varkappa_\varrho||}$ be the normalized reproducing kernel of $\mathfrak{H}$. The absolute value of the positive operator is denoted by $|\mathfrak{Z}| = (\mathfrak{Z}^*\mathfrak{Z})^{1/2}$.

For a bounded linear operator $\mathfrak{Z}$ on $\mathfrak{H}$, the function $\widetilde{\mathfrak{Z}}$ defined on Ξ by $\widetilde{\mathfrak{Z}}(\varrho) := \langle \mathfrak{Z}\widehat{\varkappa}_\varrho(z), \widehat{\varkappa}_\varrho(z) \rangle$ is the Berezin symbol of $\mathfrak{Z}$, which firstly have been introduced by Berezin (Berezin, 1972). In other words, the Berezin symbol $\widetilde{\mathfrak{Z}}$ is the function on Ξ defined by restriction of the quadratic form $\langle \mathfrak{Z}x_1, x_1 \rangle$ with $x_1 \in \mathfrak{H}$ to the subset of all normalized reproducing kernels of the unit sphere in $\mathfrak{H}$. It is clear from the Cauchy-Schwarz inequality that $\widetilde{\mathfrak{Z}}$ is the bounded function on Ξ whose values lie in the numerical range of the operator $\widetilde{\mathfrak{Z}}$. The Berezin set (or range) and the Berezin number (or radius) of the operator $\mathfrak{Z}$ are defined by

$$\mathrm{Ber}(\mathfrak{Z}) := \{\widetilde{\mathfrak{Z}}(\varrho) : \varrho \in \Xi\} \text{ and } \mathrm{ber}(\mathfrak{Z}) := \sup_{\varrho \in \Xi} \left| \widetilde{\mathfrak{Z}}(\varrho) \right|,$$

respectively (see (Karaev, 2006)).

A relevant and important concept is a numerical radius which is the supremum of the absolute values of all numbers in $\mathfrak{W}(\mathfrak{Z})$, that is

$$\mathfrak{w}(\mathfrak{Z}) = \sup_{x \in \mathfrak{H}, \|x\|=1} |\langle \mathfrak{Z}x, x \rangle|.$$

It is obvious that $\mathrm{ber}(\mathfrak{Z}) \leq \mathfrak{w}(\mathfrak{Z}) \leq ||\mathfrak{Z}||$ and $\mathrm{Ber}(\mathfrak{Z}) \subset \mathfrak{W}(\mathfrak{Z})$, where $\mathfrak{w}(\mathfrak{Z})$ denotes the numerical radius and $\mathfrak{W}(\mathfrak{Z})$ is the numerical range of the operator $\mathfrak{Z}$. It is well known that

$$\frac{\|\mathfrak{Z}\|}{2} \leq \mathfrak{w}(\mathfrak{Z}) \leq \|\mathfrak{Z}\|$$

and

$$\mathrm{ber}(\mathfrak{Z}) \leq \mathfrak{w}(\mathfrak{Z}) \leq \|\mathfrak{Z}\|, \tag{1}$$

for any $\mathfrak{Z} \in \mathfrak{L}(\mathfrak{H})$.

In (Huban et al, 2022a), Huban et al. substantially improved the upper bound in (1) by showing that if $\mathfrak{Z} \in \mathfrak{L}(\mathfrak{H})$, then

$$\operatorname{ber}(\mathfrak{Z}) \leq \frac{1}{2} \||\mathfrak{Z}| + |\mathfrak{Z}^*|\|_{\operatorname{ber}} \leq \frac{1}{2} \left(\|\mathfrak{Z}\|_{\operatorname{ber}} + \|\mathfrak{Z}^2\|_{\operatorname{ber}}^{\frac{1}{2}} \right). \tag{2}$$

Another improvement for inequality (1) was provided by Huban et al. (Huban et al, 2021) as

$$\operatorname{ber}^2(\mathfrak{Z}) \leq \frac{1}{2} \left\| |\mathfrak{Z}|^2 + |\mathfrak{Z}^*|^2 \right\|_{\operatorname{ber}}, \tag{3}$$

which was further improved in (Gürdal & Başaran, 2023) by Başaran and Gürdal as

$$\operatorname{ber}^2(\mathfrak{Z}) \leq \frac{1}{6} \left\| |\mathfrak{Z}|^2 + |\mathfrak{Z}^*|^2 \right\|_{\operatorname{ber}} + \frac{1}{3} \operatorname{ber}(\mathfrak{Z}) \||\mathfrak{Z}| + |\mathfrak{Z}^*|\|_{\operatorname{ber}}. \tag{4}$$

The following inequalities for $\operatorname{ber}^2(.)$ have been obtained in (Huban et al, 2021)

$$\frac{1}{4} \left\| |\mathfrak{Z}|^2 + |\mathfrak{Z}^*|^2 \right\|_{\operatorname{ber}} \leq \operatorname{ber}^2(\mathfrak{Z}) \leq \frac{1}{2} \left\| |\mathfrak{Z}|^2 + |\mathfrak{Z}^*|^2 \right\|_{\operatorname{ber}}. \tag{5}$$

Furthermore, Huban et al. (Huban et al, 2022a) established some refinements of (2) and (5), respectively, that can be presented as

$$\operatorname{ber}^j(\mathfrak{Z}) \leq \frac{1}{2} \left\| |\mathfrak{Z}|^{2j\xi} + |\mathfrak{Z}^*|^{2j(1-\xi)} \right\|_{\operatorname{ber}} \tag{6}$$

and

$$\operatorname{ber}^{2j}(\mathfrak{Z}) \leq \left\| \xi |\mathfrak{Z}|^{2j} + (1-\xi) |\mathfrak{Z}^*|^{2j} \right\|_{\operatorname{ber}}, \tag{7}$$

where $\mathfrak{Z} \in \mathfrak{L}(\mathfrak{H})$, $0 \leq \xi \leq 1$, and $j \geq 1$.

Another important fact about the Berezin number upper bounds that are of our interest are due to Huban et al. in (Huban et al, 2021): Let $\mathfrak{Z}_1, \mathfrak{Z}_2 \in \mathfrak{L}(\mathfrak{H})$ and $r \geq 1$, then

$$\operatorname{ber}^r(\mathfrak{Z}_2^* \mathfrak{Z}_1) \leq \frac{1}{2} \left\| |\mathfrak{Z}_1|^{2r} + |\mathfrak{Z}_2|^{2r} \right\|_{\operatorname{ber}}. \tag{8}$$

For an in-depth exploration of the intricacies surrounding the Berezin symbol, interested readers are strongly encouraged to refer to (Bakherad & Garayev, 2019; Başaran et al, 2022; Chalendar et al, 2012; Garayev & Alomari, 2021; Garayev et al, 2020; Güntürk & Gürdal, 2024; Garayev et al, 2021; Stojiljković & Gürdal, 2024a,b; Gürdal et al, 2023; Gürdal & Tapdigoglu, 2023; Tapdigoglu et al, 2021; Yamancıet al, 2020; Gürdal & Stojiljkovic, 2024a; Huban et al, 2022b; Gürdal & Stojiljkovic, 2024b) and the comprehensive references provided therein.

In this paper, motivated by previously reported results (Stojiljković & Dragomir, 2024), this work aims to develop new Berezin number upper bounds for reproducing kernel Hilbert space operators by introducing new improvements to the well-known Cauchy-Schwarz inequality.

Preliminaries

We require a few well-known lemmas in order to demonstrate our extended Berezin number inequalities.

According to the traditional Schwarz inequality for positive operator, for any $x_1, x_2 \in \mathfrak{H}$

$$|\langle \mathfrak{Z}x_1, x_2\rangle|^2 \leq \langle \mathfrak{Z}x_1, x_1\rangle \langle \mathfrak{Z}x_2, x_2\rangle \tag{9}$$

if $\mathfrak{Z} \in \mathfrak{L}(\mathfrak{H})$ is a positive operators. Kato's inequality, sometimes referred to as the combined Cauchy-Schwarz inequality, was initially put out by Kato (Kato, 1952) in 1952 as a companion to the Schwarz inequality (9). It says that

$$|\langle \mathfrak{Z}x_1, x_2\rangle|^2 \leq \langle |\mathfrak{Z}|^{2\gamma}x_1, x_1\rangle\langle |\mathfrak{Z}^*|^{2(1-\gamma)}x_2, x_2\rangle,\ \gamma \in [0,1] \tag{10}$$

for any operator $\mathfrak{Z} \in \mathfrak{L}(\mathfrak{H})$ and any $x_1, x_2 \in \mathfrak{H}$. In order to generalize this result, in 1994 Furuta (Furuta, 1994) obtained the following result:

$$|\langle \mathfrak{Z}|\mathfrak{Z}|^{\gamma+\eta-1}x_1, x_2\rangle|^2 \leq \langle |\mathfrak{Z}|^{2\gamma}x_1, x_1\rangle\langle |\mathfrak{Z}^*|^{2\eta}x_2, x_2\rangle, \tag{11}$$

for any $x_1, x_2 \in \mathcal{H}$ and $\gamma, \eta \in [0,1]$ with $\gamma + \eta \geq 1$.

LEMMA 1. *(McCarthy, 1967). Let $\mathfrak{Z} \in \mathfrak{L}(\mathfrak{H})$, $\mathfrak{Z} \geq 0$ and let $x \in \mathfrak{H}$ be any unit vector. Then there is*

$$\langle \mathfrak{Z}x, x\rangle^{\mathfrak{r}} \leq \langle \mathfrak{Z}^{\mathfrak{r}}x, x\rangle \text{ for } \mathfrak{r} \geq 1, \tag{12}$$

$$\langle \mathfrak{Z}^{\mathfrak{r}}x, x\rangle \leq \langle \mathfrak{Z}x, x\rangle^{\mathfrak{r}} \text{ for } 0 < \mathfrak{r} \leq 1. \tag{13}$$

The well-known Buzano's inequality is the following outcome.

LEMMA 2. *Let $x, y, e \in \mathfrak{H}$ with $\|e\| = 1$. Then there is*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}(\|x\|\,\|y\| + |\langle x, y\rangle|). \tag{14}$$

The following result is found in (Singh Aujla & Silva, 2003) and is related to non-negative convex functions.

LEMMA 3. *Let f be a non-negative convex function on $[0,+\infty)$ and $\mathfrak{Z}_1, \mathfrak{Z}_2 \in \mathfrak{L}(\mathfrak{H})$ be positive operators. Then*

$$\left\| f\left(\frac{\mathfrak{Z}_1+\mathfrak{Z}_2}{2}\right)\right\| \leq \left\| \frac{f(\mathfrak{Z}_1)+f(\mathfrak{Z}_2)}{2}\right\|. \tag{15}$$

LEMMA 4. *Let $x, y \in \mathfrak{H}$ and $\mathfrak{A} \in \mathfrak{L}(\mathfrak{H})$.*

$$|\langle \mathfrak{A}x, y\rangle|^2 \leq \langle f^2(|\mathfrak{A}|)x, x\rangle\langle g^2(|\mathfrak{A}^*|)y, y\rangle. \tag{16}$$

is the case where f and g are two nonnegative continuous functions on $[0,+\infty)$ that fulfill $f(t)g(t) = t, t \geq 0$.

With regard to the mapping φ, Stojiljković and Gürdal (Stojiljković & Gürdal, 2024c) recently acquired the following modification.

LEMMA 5. *Let $u_k, v_k, e \in \mathfrak{H}$, $l, q > 1$, $p \geq 1$, $\frac{1}{l}+\frac{1}{q} = 1$. Let $\mathfrak{J}$ be a set such that $(0,1) \subset \mathfrak{J} \subset \mathbb{R}$. Let φ be a of mapping such that $\varphi : \mathfrak{J} \to \mathbb{R}^+$, such that the following holds $\sum_i^n \varphi(\alpha_i) = 1$. Then the following inequality holds*

$$\left|\sum_{k=1}^{n} \varphi(\alpha_k)\langle u_k, e\rangle\langle e, v_k\rangle\right|^p \leq \frac{\sum_{k=1}^{n} \varphi^{l/2}(\alpha_k)|\langle u_k, e\rangle|^{pl}}{l} + \\ + \frac{\sum_{k=1}^{n} \varphi^{q/2}(\alpha_k)|\langle e, v_k\rangle|^{pq}}{q}. \tag{17}$$

Main results

We give our first result, a consequence of Lemma 5 which is instrumental in the development of the later results.

COROLLARY 1. *Let $l, q > 1$ such that $\frac{1}{l}+\frac{1}{q} = 1$ and $p \geq 1$, $\mathfrak{Z}_i \in \mathfrak{L}(\mathfrak{H})$ and $\varphi : \mathfrak{J} \to \mathbb{R}^+$ with $\sum_{i=1}^n \varphi(\alpha_i) = 1$, then the following inequality holds:*

$$\left|\sum_{k=1}^{n} \varphi(\alpha_k)\langle \mathfrak{Z}_k \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^2\right|^p \leq \frac{\sum_{k=1}^{n} \langle |\mathfrak{Z}_k|^{2p\alpha l} \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle \varphi^{l/2}(\alpha_k)}{2l} + \\ + \frac{\sum_{k=1}^{n} \langle |\mathfrak{Z}_k^*|^{2pl(1-\alpha)} \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle \varphi^{l/2}(\alpha_k)}{2l} + \frac{\sum_{k=1}^{n} \varphi^{q/2}(\alpha_k)|\langle \mathfrak{Z}_k \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^{pq}}{q}. \tag{18}$$

Proof. If we take $v_k = \mathfrak{Z}_k^* \widehat{\varkappa}_\varrho$, $u_k = \mathfrak{Z}_k \widehat{\varkappa}_\varrho$ and $e = \widehat{\varkappa}_\varrho$ in Lemma 5 above, one obtains

$$\begin{aligned}
&\left|\sum_{k=1}^{n} \varphi(\alpha_k)\langle \mathfrak{Z}_k \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^2\right|^p \\
&\le \frac{\sum_{k=1}^n \varphi^{l/2}(\alpha_k)|\langle \mathfrak{Z}_k \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^{pl}}{l} + \frac{\sum_{k=1}^n \varphi^{q/2}(\alpha_k)|\langle \mathfrak{Z}_k \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^{pq}}{q} \\
&\le \frac{\sum_{k=1}^n \langle |\mathfrak{Z}_k|^{2p\alpha l} \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle \varphi^{l/2}(\alpha_k)}{2l} + \frac{\sum_{k=1}^n \langle |\mathfrak{Z}_k^*|^{2pl(1-\alpha)} \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle \varphi^{l/2}(\alpha_k)}{2l} + \\
&+ \frac{\sum_{k=1}^n \varphi^{q/2}(\alpha_k)|\langle \mathfrak{Z}_k \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^{pq}}{q}.
\end{aligned}$$

□

REMARK 1. Taking supremum over all $\varrho \in \Xi$, one obtains

$$\begin{aligned}
&\sup_{\varrho\in\Xi}\left|\sum_{k=1}^{n} \varphi(\alpha_k)\langle \mathfrak{Z}_k \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^2\right|^p \\
&\le \frac{1}{2l}\sup_{\varrho\in\Xi}\sum_k (\varphi(\alpha_k))^{l/2}\left\langle \left(|\mathfrak{Z}_k|^{2p\alpha l} + |\mathfrak{Z}_k^*|^{2pl(1-\alpha)}\right)\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\right\rangle + \\
&+ \sup_{\varrho\in\Xi}\frac{\sum_{k=1}^n \varphi^{q/2}(\alpha_k)|\langle \mathfrak{Z}_k \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^{pq}}{q}.
\end{aligned}$$

Setting $\mathfrak{Z}_k, \varphi_k = 0$ for $k \in \{2, 3..., n\}, q = l = 2$, we obtain the following inequality which sharpens the one given by Huban et al. (Huban et al, 2022a) (7)

$$\begin{aligned}
\operatorname{ber}^{2p}(\mathfrak{Z}) &\le \frac{\left\| |\mathfrak{Z}|^{4\alpha p} + |\mathfrak{Z}^*|^{4p(1-\alpha)} \right\|_{\operatorname{ber}}}{4} + \frac{\operatorname{ber}^{2p}(\mathfrak{Z})}{2} \\
&\le \frac{1}{2}\left\| |\mathfrak{Z}|^{4\alpha p} + |\mathfrak{Z}^*|^{4p(1-\alpha)} \right\|_{\operatorname{ber}}.
\end{aligned}$$

In particular, we obtain a refinement by setting $\alpha = \frac{1}{2}$ of inequality (7) for $s = \frac{1}{2}$.

Proof. Using (8) on the third term, we obtain the desired inequality. □

THEOREM 1. *Let* $p \ge 1, l, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha \in [0, 1], p \ge \frac{4}{l}, \mathfrak{Z}_i \in \mathfrak{L}(\mathfrak{H})$ *and* H *be such that* $H(l) = f^2(|\mathfrak{Z}_i|^{\alpha pl}) + g^2(|\mathfrak{Z}_i|^{pl\alpha}) + f^2(|\mathfrak{Z}_i^*|^{pl(1-\alpha)}) + g^2(|\mathfrak{Z}_i^*|^{pl(1-\alpha)})$*,* *then the following inequality holds:*

$$\operatorname{ber}^p\left(\sum_i \varphi(\alpha_i)\mathfrak{Z}_i\right) \tag{19}$$
$$\leq \sup_{\varrho\in\Xi}\left\{\frac{\sum_i \varphi^{l/2}(\alpha_i)|\langle \mathfrak{Z}_i\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^{\frac{pl}{2}}}{l}+\frac{\sum_i \varphi^{q/2}(\alpha_i)|\langle \mathfrak{Z}_i\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^{\frac{pq}{2}}}{q}\right\}$$

$$\leq \left\|\sum_i \frac{1}{8}\left(\frac{H\left(l\right)}{l}\varphi^{l/2}\left(\alpha_i\right)+\frac{H\left(q\right)}{q}\varphi^{q/2}\left(\alpha_i\right)\right)\right\|+$$
$$\frac{1}{2}\sum_{i=1}^n\left(\varphi^{l/2}\left(\alpha_i\right)\frac{\operatorname{ber}^{\frac{pl}{4}}\left(|\mathfrak{Z}_i^*|^{2(1-\alpha)}|\mathfrak{Z}_i|^{2\alpha}\right)}{l}+\varphi^{q/2}\left(\alpha_i\right)\frac{\operatorname{ber}^{\frac{pq}{4}}\left(|\mathfrak{Z}_i^*|^{2(1-\alpha)}|\mathfrak{Z}_i|^{2\alpha}\right)}{q}\right).$$

Proof. Start from the left-hand side

$$\left|\sum_{i=1}^n \varphi(\alpha_i)\langle \mathfrak{Z}_i\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle\right|^p$$
$$\leq \sum_{i=1}^n \varphi(\alpha_i)|\langle \mathfrak{Z}_i\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^p$$
$$\leq \left(\sum_{i=1}^n \varphi^{l/2}(\alpha_i)|\langle \mathfrak{Z}_i\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^{\frac{pl}{2}}\right)^{1/l}\left(\sum_{i=1}^n \varphi^{q/2}(\alpha_i)|\langle \mathfrak{Z}_i\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^{\frac{pq}{2}}\right)^{1/q}$$
$$\leq \frac{\sum_{i=1}^n \varphi^{l/2}(\alpha_i)|\langle \mathfrak{Z}_i\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^{\frac{pl}{2}}}{l}+\frac{\sum_{i=1}^n \varphi^{q/2}(\alpha_i)|\langle \mathfrak{Z}_i\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^{\frac{pq}{2}}}{q}.$$

We focus now on the first part, second one is done analogous. First use Kato's inequality

$$|\langle \mathfrak{Z}_i\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^{\frac{pl}{2}}$$
$$\leq \langle |\mathfrak{Z}_i|^{2\alpha}\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle^{\frac{pl}{4}}\langle \widehat{\varkappa}_\varrho,|\mathfrak{Z}_i^*|^{2(1-\alpha)}\widehat{\varkappa}_\varrho\rangle^{\frac{pl}{4}}$$
$$\leq \frac{1}{2}\left(\left\||\mathfrak{Z}_i|^{2\alpha}\widehat{\varkappa}_\varrho\right\|^{\frac{pl}{4}}\left\||\mathfrak{Z}_i^*|^{2(1-\alpha)}\widehat{\varkappa}_\varrho\right\|^{\frac{pl}{4}}+|\langle |\mathfrak{Z}_i|^{2\alpha}\widehat{\varkappa}_\varrho,|\mathfrak{Z}_i^*|^{2(1-\alpha)}\widehat{\varkappa}_\varrho|^{\frac{pl}{4}}\right)$$
$$\leq \frac{1}{4}\left(\langle |\mathfrak{Z}_i|^{4\alpha}\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle^{\frac{pl}{4}}+\langle |\mathfrak{Z}_i^*|^{4(1-\alpha)}\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle^{\frac{pl}{4}}\right)+\frac{1}{2}|\langle |\mathfrak{Z}_i^*|^{2(1-\alpha)}|\mathfrak{Z}_i|^{2\alpha}\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^{\frac{pl}{4}}$$
$$\leq \frac{1}{4}\left(\langle |\mathfrak{Z}_i|^{pl\alpha}\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle+\langle |\mathfrak{Z}_i^*|^{pl(1-\alpha)}\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle\right)+\frac{1}{2}|\langle |\mathfrak{Z}_i^*|^{2(1-\alpha)}|\mathfrak{Z}_i|^{2\alpha}\widehat{\varkappa}_\varrho,\widehat{\varkappa}_\varrho\rangle|^{\frac{pl}{4}}$$

$$\leq \frac{1}{8}\left(\langle f^2(|\mathfrak{Z}_i|^{p\alpha l})\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle + \langle g^2(|\mathfrak{Z}_i|^{pl\alpha})\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle\right) +$$
$$+ \frac{1}{8}\left(\langle f^2(|\mathfrak{Z}_i^*|^{pl(1-\alpha)})\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle + \langle g^2(|\mathfrak{Z}_i^*|^{pl(1-\alpha)})\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle\right) +$$
$$+ \frac{1}{2}|\langle|\mathfrak{Z}_i^*|^{2(1-\alpha)}|\mathfrak{Z}_i|^{2\alpha}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^{\frac{pl}{4}}.$$

The other part is obtained in an analogous way, adding them and taking a supremum over $\varrho \in \Xi$ we obtain the desired inequality. □

In particular, by setting $\mathfrak{Z}_k = 0$, $f(t) = g(t) = \sqrt{t}$, $\varphi_k = 0$ for $k \in \{2, 3..., n\}, q = l = 2$, we obtain the refinement of (6), namely

$$\mathrm{ber}^p(\mathfrak{Z}) \leq \frac{1}{4}\left\||\mathfrak{Z}|^{2p\alpha} + |\mathfrak{Z}^*|^{2p(1-\alpha)}\right\|_{\mathrm{ber}} + \frac{1}{2}\mathrm{ber}^{\frac{p}{2}}(|\mathfrak{Z}^*|^{2(1-\alpha)}|\mathfrak{Z}|^{2\alpha})$$
$$\leq \left\|\frac{|\mathfrak{Z}|^{2p\alpha} + |\mathfrak{Z}^*|^{2p(1-\alpha)}}{2}\right\|_{\mathrm{ber}}.$$

In the following we give a variation of the inequality given by Erkan and Gürdal (Erkan & Gürdal, 2024, Theorem 4).

THEOREM 2. *Let* $r \geq 1, \alpha \in [0,1], \mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3, \mathfrak{Z}_4 \in \mathfrak{L}(\mathfrak{H})$ *and* $H(\mathfrak{Z}_1) = f^2(|\mathfrak{Z}_1|^{4r\alpha}) + g^2(|\mathfrak{Z}_1|^{4r\alpha}) + f^2(|\mathfrak{Z}_1|^{4r(1-\alpha)}) + g^2(|\mathfrak{Z}_1|^{4r(1-\alpha)})$ *then*

$$\mathrm{ber}^r(\mathfrak{Z}_1^*\mathfrak{Z}_2 + \mathfrak{Z}_3^*\mathfrak{Z}_4) \leq 2^{r-4}\left\|H(\mathfrak{Z}_2) + H(\mathfrak{Z}_1) + H(\mathfrak{Z}_4) + H(\mathfrak{Z}_3)\right\|_{\mathrm{ber}}. \quad (20)$$

Proof. We proceed

$$\frac{|\langle(\mathfrak{Z}_1^*\mathfrak{Z}_2 + \mathfrak{Z}_3^*\mathfrak{Z}_4)\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^r}{2^r}$$
$$\leq \left(\frac{|\langle\mathfrak{Z}_1^*\mathfrak{Z}_2\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle| + |\langle\mathfrak{Z}_3^*\mathfrak{Z}_4\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|}{2}\right)^r$$
$$\leq \frac{|\langle\mathfrak{Z}_1^*\mathfrak{Z}_2\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^r + |\langle\mathfrak{Z}_3^*\mathfrak{Z}_4\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^r}{2}$$
$$\leq \frac{1}{8}\langle|\mathfrak{Z}_2|^{4r\alpha}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle + \langle|\mathfrak{Z}_2|^{4r(1-\alpha)}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho + \langle|\mathfrak{Z}_1|^{4\alpha r}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle +$$
$$+ \langle|\mathfrak{Z}_1|^{4r(1-\alpha)}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle + \langle|\mathfrak{Z}_4|^{4r\alpha}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle +$$
$$+ \langle|\mathfrak{Z}_4|^{4r(1-\alpha)}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho + \langle|\mathfrak{Z}_3|^{4\alpha r}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle + \langle|\mathfrak{Z}_3|^{4r(1-\alpha)}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle.$$

Proceeding to use (16) and AG inequality, we obtain the desired inequality. □

COROLLARY 2. *Setting $f(t) = g(t) = \sqrt{t}$ and $\alpha = \frac{1}{2}$ we obtain a variation of the inequality given by Erkan and Gürdal (Erkan & Gürdal, 2024)*

$$\mathrm{ber}^r(\mathfrak{z}_1^*\mathfrak{z}_2 + \mathfrak{z}_3^*\mathfrak{z}_4) \leq 2^{r-2}\left\| |\mathfrak{z}_2|^{2r} + |\mathfrak{z}_3|^{2r} + |\mathfrak{z}_4|^{2r} + |\mathfrak{z}_1|^{2r} \right\|_{\mathrm{ber}}.$$

THEOREM 3. *Let $r \geq 1, \alpha \in [0,1]$, $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{L}(\mathfrak{H})$ and f, g nonnegative such that $f(t)g(t) = t$ also let*

$$H(\mathfrak{z}_1) = f^2(|\mathfrak{z}_1|^{2r\alpha}) + g^2(|\mathfrak{z}_1|^{2r\alpha}) + f^2(|\mathfrak{z}_1^*|^{2r(1-\alpha)}) + g^2(|\mathfrak{z}_1^*|^{2r(1-\alpha)})$$

then

$$\mathrm{ber}^r(\mathfrak{z}_1 + \mathfrak{z}_2) \leq 2^{r-3}\left\| H(\mathfrak{z}_1) + H(\mathfrak{z}_2) \right\|_{\mathrm{ber}}. \tag{21}$$

Proof.

$$\begin{aligned}
&\frac{|\langle(\mathfrak{z}_1+\mathfrak{z}_2)\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^r}{2^r} \\
&\leq \left(\frac{|\langle\mathfrak{z}_1\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle| + |\langle\mathfrak{z}_2\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|}{2}\right)^r \\
&\leq \frac{|\langle\mathfrak{z}_1\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^r + |\langle\mathfrak{z}_2\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^r}{2} \\
&\leq \frac{\langle|\mathfrak{z}_1|^{2\alpha}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^{\frac{r}{2}}\langle|\mathfrak{z}_1^*|^{2(1-\alpha)}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^{\frac{r}{2}}}{2} + \frac{\langle|\mathfrak{z}_2|^{2\alpha}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^{\frac{r}{2}}\langle|\mathfrak{z}_2^*|^{2(1-\alpha)}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^{\frac{r}{2}}}{2} \\
&\leq \frac{\langle|\mathfrak{z}_1|^{2r\alpha}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle + \langle|\mathfrak{z}_1^*|^{2r(1-\alpha)}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle}{4} + \\
&+ \frac{\langle|\mathfrak{z}_2|^{2r\alpha}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle + \langle|\mathfrak{z}_2^*|^{2r(1-\alpha)}\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle}{4}.
\end{aligned}$$

The proof is finished with using (16) and AG inequality. □

COROLLARY 3. *Setting $f(t) = g(t) = \sqrt{t}$ and replacing the second slot with y and proceeding with the proof for the case $\langle(\mathfrak{z}_1+\mathfrak{z}_2)\widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle$ the proof still holds and then using triangle inequality we obtain the inequality given by Huban et al. (Huban et al, 2022a, Theorem 3.7)*

$$\|\mathfrak{z}_1 + \mathfrak{z}_2\|_{\mathrm{ber}}^r \leq 2^{r-2}\left(\left\| |\mathfrak{z}_1|^{2\alpha r} + |\mathfrak{z}_2|^{2\alpha r} \right\|_{\mathrm{ber}} + \left\| |\mathfrak{z}^*{}_1|^{2r(1-\alpha)} + |\mathfrak{z}^*{}_2|^{2r(1-\alpha)} \right\|_{\mathrm{ber}}\right).$$

We now present a variation of the inequality given by Huban et al. (Huban et al, 2022a).

THEOREM 4. *Let us define* $H(\mathfrak{z}_1) = f^2(|\mathfrak{z}_1|^{2\alpha r}) + g^2(|\mathfrak{z}_1|^{2\alpha r})$, $G(\mathfrak{z}_1) = f^2(|\mathfrak{z}^*{}_1|^{2r(1-\alpha)}) + g^2(|\mathfrak{z}^*{}_1|^{2r(1-\alpha)})$ *where* $\mathfrak{z}_1 = \mathfrak{z}_2 + i\mathfrak{z}_3$ $r \geq 2, \alpha \in [0,1], \mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3 \in \mathfrak{L}(\mathfrak{H})$ *where* $\mathfrak{z}_2, \mathfrak{z}_3$ *are the Cartesian decomposition operators of* $\mathfrak{z}_1$ *where* $\mathfrak{z}_2, \mathfrak{z}_3$ *are selfadjoint, then one obtains*

$$\mathrm{ber}^r(\mathfrak{z}_1) \leq 2^{\frac{r}{2}-3} \left\| H(\mathfrak{z}_2) + G(\mathfrak{z}_2) + H(\mathfrak{z}_3) + G(\mathfrak{z}_3) \right\|_{\mathrm{ber}}.$$

Proof.

$$\begin{aligned}
&\frac{|\langle \mathfrak{z}_1 \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle|^r}{2^{\frac{r}{2}}} \\
&= \sqrt[\frac{r}{2}]{\frac{\langle \mathfrak{z}_2 \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^2 + \langle \mathfrak{z}_3 \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^2}{2}} \\
&\leq \frac{\langle \mathfrak{z}_2 \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^r + \langle \mathfrak{z}_3 \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle^r}{2} \\
&\leq \frac{\langle |\mathfrak{z}_2|^{2\alpha r} \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle + \langle |\mathfrak{z}^*{}_2|^{2r(1-\alpha)} \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle}{4} + \\
&+ \frac{\langle |\mathfrak{z}_3|^{2\alpha r} \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle + \langle |\mathfrak{z}^*{}_3|^{2r(1-\alpha)} \widehat{\varkappa}_\varrho, \widehat{\varkappa}_\varrho\rangle}{4}
\end{aligned}$$

following a similar procedure to the one given in (21), we obtain the desired result. □

THEOREM 5. *Let* $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{L}(\mathfrak{H})$, $r \geq 2$ *and* $\alpha \in [0,1]$, *then*

$$\begin{aligned}
\mathrm{ber}^r(\mathfrak{z}_1 + \mathfrak{z}_2) \leq 2^{r-3} \left\| |\mathfrak{z}_1|^{2\alpha r} + |\mathfrak{z}_1^*|^{2r(1-\alpha)} + |\mathfrak{z}_2|^{2\alpha r} + |\mathfrak{z}_2^*|^{2r(1-\alpha)} \right\|_{\mathrm{ber}} + \\
+ 2^{r-2}(\mathrm{ber}(|\mathfrak{z}^*{}_1|^{r(1-\alpha)}|\mathfrak{z}_1|^{\alpha r}) + \mathrm{ber}(|\mathfrak{z}^*{}_2|^{r(1-\alpha)}|\mathfrak{z}_2|^{\alpha r})). \quad (22)
\end{aligned}$$

Proof. Following a similar principle to the one given in (21) with an exception of using Buzano's inequality, we obtain the desired inequality; one must only realise that in order to use the Mc-Carthy inequality there must be $r \geq 2$. □

COROLLARY 4. *Previously obtained inequality (22) refines the inequality given by Huban et al. in (Huban et al, 2022a), namely Theorem 3.6 (eq. 3.6) for* $r \geq 2$*; it can be seen by using (6) on both Berezin radius terms, from which we obtain*

$$\mathrm{ber}^r(\mathfrak{z}_1 + \mathfrak{z}_2) \leq 2^{r-3} \left\| |\mathfrak{z}_1|^{2\alpha r} + |\mathfrak{z}_1^*|^{2r(1-\alpha)} + |\mathfrak{z}_2|^{2\alpha r} + |\mathfrak{z}_2^*|^{2r(1-\alpha)} \right\|_{\mathrm{ber}} +$$

$$+ 2^{r-2}(\text{ber}(|\mathfrak{z}^*{}_1|^{r(1-\alpha)}|\mathfrak{z}_1|^{\alpha r}) + \text{ber}(|\mathfrak{z}^*{}_2|^{r(1-\alpha)}|\mathfrak{z}_2|^{\alpha r}))$$
$$\leq 2^{r-3}\left\| |\mathfrak{z}_1|^{2\alpha r} + |\mathfrak{z}_1^*|^{2r(1-\alpha)} + |\mathfrak{z}_2|^{2\alpha r} + |\mathfrak{z}_2^*|^{2r(1-\alpha)} \right\|_{\text{ber}} +$$
$$+ 2^{r-3}\left(\left\| |\mathfrak{z}_1|^{2\alpha r} + |\mathfrak{z}_1^*|^{2r(1-\alpha)} \right\|_{\text{ber}} + \left\| |\mathfrak{z}_2|^{2\alpha r} + |\mathfrak{z}_2^*|^{2r(1-\alpha)} \right\|_{\text{ber}} \right)$$
$$\leq 2^{r-2}\left\| |\mathfrak{z}_1|^{2\alpha r} + |\mathfrak{z}_1^*|^{2r(1-\alpha)} \right\|_{\text{ber}} + \left\| |\mathfrak{z}_2|^{2\alpha r} + |\mathfrak{z}_2^*|^{2r(1-\alpha)} \right\|_{\text{ber}}.$$

If we were to use the triangle inequality on eq. (3.6) (Huban et al, 2022a) we would obtain the right hand side of the above chain of inequalities, which shows that our inequality is sharper than it.

Further, setting $\mathfrak{z}_1 = \mathfrak{z}_2$ we obtain (6) on the right-hand side which shows that our inequality (22) is a refinement of (6) for $r \geq 2$.

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Desigualdades de Berezin para sumas de operadores y desigualdades clásicas relativas al radio de Berezin

Mehmet Gürdal[a], *Vuk* N. Stojiljković[b], **autor de correspondencia**

[a] Universidad Suleyman Demirel, Departamento de Matemáticas, Isparta, República de Turquía

[b] Universidad de Novi Sad, Facultad de Ciencias, Novi Sad, República de Serbia; Escuela Secundaria de Matemáticas, Belgrado, República de Serbia

CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: En este artículo, el objetivo del autor es buscar obtener nuevas desigualdades del tipo Berezin.

Métodos: Los métodos utilizados son estándar para la teoría de operadores.

Resultados: Varias desigualdades del tipo dado por Huban et al. y Erkan han sido obtenidas.

Conclusión: Además de obtener en particular diversas desigualdades de la forma dada por Huban et al. y Erkan, los autores agudizaron las desigualdades relacionadas con la norma Berezin.

Palabras claves: norma de Berezin, número de Berezin, desigualdades.

Неравенства Березина для сумм операторов и классические неравенства относительно радиуса Березина

Мехмет Гурдал[а], *Вук* Н. Соилькович[б], **корреспондент**

[а] Университет имени Сулеймана Демиреля, математический факультет, г. Испарта, Турецкая Республика

[б] Нови-Садский университет, факультет науки, г. Нови-Сад, Республика Сербия; Математическая школа, г. Белград, Республика Сербия

РУБРИКА ГРНТИ: 27.25.15 Дескриптивная теория функций, 27.39.19 Линейные операторы и операторные уравнения, 27.39.21 Спектральная теория линейных операторов

ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: Целью данной статьи является поиск новых неравенств типа Березина.

Методы: В исследовании использовались стандартные методы для теории операторов.

Результаты: Были получены различные неравенства типа неравенств, приведенных Хубаном и др. и Эрканом.

Выводы: В дополнение к полученным неравенствам типа неравенств, приведенных Хубаном и др. и Эрканом, авторы уточнили неравенства, связанные с нормой Березина.

Ключевые слова: норма Березина, число Березина, неравенства.

Березинове неједнакости за збир оператора и класичне неједнакости које се односе на Березинову норму

Мехмет Гурдал[a], *Вук* Н. Стојиљковић[б], **аутор за преписку**

[a] Универзитет „Сулејман Демирел", Катедра за математику, Испарта, Република Турска

[б] Универзитет у Новом Саду, Природно-математички факултет, Катедра за математику, Нови Сад, Република Србија; Математичка гимназија, Београд, Република Србија

ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Циљ овог рада јесте да се изведу нове неједнакости Березиновог типа.

Методе: Примењене су стандардне методе за теорију оператора.

Резултати: Добијене су разне неједнакости типа које су изнели Хубан и др. и Еркан.

Закључак: Поред добијања разних неједнакости облика које су изнели Хубан и др. и Еркан, аутори су пооштрили неједнакости везане за Березинов радијус.

Кључне речи: Березинова норма, Березинов број, неједнакости.

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