Solving a damped spring-mass system via the MA-simulation function

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Abstract:

Introduction/purpose: In an interesting article, Perveen & Imdad (2019) introduced the notion of an MA-simulation function, and utilized it to prove the existence of a fixed point for a self mapping through α-admisibilty and the continuity of the self-map in a fuzzy metric space. The purpose of this paper is to establish a unique fixed point theorem for an MA-contractive mapping by relaxing the condition of continuity and α-admissibilty of the map in a fuzzy metric space. As an application of our result, we study the existence and uniqueness of the solution to the damped spring-mass system. The article includes an example which shows the validity of our results.

Methods: The fixed point method with an MA-simulation function was used.

Results: A unique fixed point for a self map in a fuzzy metric space is obtained.

Conclusions: A fixed point of the self map is obtained without the continuity and α-admissibility of the self map via the MA-simulation function. Also, the existence and uniqueness of the solution of a damped spring-mass system in the setting of a fuzzy metric space is obtained.

Key words: fuzzy metric space, M-Cauchy sequence, fixed points, MA-simulation function.

Introduction

In 1965, the fuzzy set theory was initiated by Zadeh (1965) introducing a method to deal with uncertainty and vagueness in everyday life.The concept of fuzzy sets was used by Kramosil & Michálek (1975) to introduce fuzzy metric spaces. Later on, it was modified by George & Veeraman (1994) in order to obtain a Hausdorff topology for this class of fuzzy metric spaces. Contractive mappings in fuzzy metric spaces were studied by various authors, see, e.g. Mihet (2008), Mihet (2010), Tirado (2012) and Wardowski (2013),Gregori & Miñana (2014), and used in establishing some fixed point theorems in a fuzzy metric space in the sense of George & Veeraman. Jain et al. (2009), established fixed point results in a fuzzy metric space by using the concept of a compatible map and a weakly compatible map. Khojasteh et al. (2015) introduced simulation functions defined as follows

A mapping $\zeta : [0, +\infty) \times [0, +\infty) \to R$ is said to be a simulation function if it satisfies the following:

(1) $\zeta(0, 0) = 0$;

(2) $\zeta(t, s) < s - t$, for all $t, s > 0$;

(3) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n > 0$, then $\limsup_{n\to+\infty} \zeta(t_n, s_n) < 0$.

After this, several authors utilized a simulation function by modifying it in various spaces and proved results on a fixed point. Recently, Perveen & Imdad (2019) introduced an MA-simulation function and proved some fixed point theorem for a self map through α-admissibility and continuity of the self map. In this paper, we establish a new fixed point theorem for a self map using a new contractive condition via the MA-simulation function. The paper is organized as follows

Following the preliminaries discussions, we introduce a new contractive condition using the MA-simulation function.Then we study a fuzzy contractive mapping due to Gregori & Miñana (2014), Mihet (2010), Tirado (2012) and Wardowski (2013). Subsequently, we establish the existence of a unique fixed point of a self mapping in an M-complete fuzzy metric space and provide an illustrative example to demonstrate our result. Lastly, we apply our results to demonstrate the existence and uniqueness of the solution for the damped spring-mass system in the setting of a fuzzy metric space.

Now, let us recall the definition of a fuzzy metric space and other results given in George & Veeraman (1994).

DEFINITION 1. *(Schweizer & Sklar, 1983) A mapping $*: [0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm (t-norm for short) if $*$ is continuous and satisfies the following conditions:*

(i) *$*$ is commutative and associative, i.e. $a*b = b*a$ and $a*(b*c)=(a*b)*c$, for all $a,b,c\in[0,1]$;*

(ii) *$1*a=a$, for all $a\in[0,1]$;*

(iii) *$a*c\leq b*d$, for $a\leq b, c\leq d$, for $a,b,c,d\in[0,1]$.*

The well-known examples of the t-norm are the minimum t-norm $*_m, a*_m b = min\{a,b\}$ written as $*_m$ and the product t-norm $*, a*b=ab$.

DEFINITION 2. *(George & Veeraman, 1994) A fuzzy metric space is an ordered triple $(X,M,*)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X\times X\times(0,+\infty)$ satisfying the following conditions, for all $x,y,z\in X$ and $t,s>0$;*

(GV1) $M(x,y,t)>0$;

(GV2) $M(x,y,t)=1$ *if and only if* $x=y$;

(GV3) $M(x,y,t)=M(y,x,t)$;

(GV4) $M(x,z,t+s)\geq M(x,y,t)*M(y,z,s)$;

(GV5) $M(x,y,.):(0,+\infty)\to(0,1]$ *is continuous.*

Note that in view of the condition (GV2) we have $M(x,x,t)=1$, for all $x\in X$ and $t>0$ and $M(x,y,t)<1$, for all $x\neq y$ and $t>0$.
The following notion was introduced by George & Veeraman (1994).

DEFINITION 3. *(George & Veeraman, 1994) A sequence $\{x_n\}$ in a fuzzy metric space $(X,M,*)$ is said to be M-Cauchy, or simply Cauchy, if for each $\epsilon\in(0,1)$ and each $t>0$ there exists an $n_0\in N$, such that $M(x_n,x_m,t)>1-\epsilon$, for all $n,m\geq n_0$. Equivalently, $\{x_n\}$ is Cauchy if $\lim_{m,n\to+\infty}M(x_n,x_m,t)=1$, for all $t>0$.*

LEMMA 1. *(Grabiec, 1988) Let $(X,M,*)$ be a fuzzy metric space. Then $M(x,y,.)$ is non-decreasing for all $x,y\in X$.*

Theorem 1. *(George & Veeraman, 1994) Let $(X, M, *)$ be a fuzzy metric space and τ be the topology induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in $X, \{x_n\} \to x$ if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$.*

Definition 4. *George & Veeraman (1994) $(X, M, *)$ (or simply X) is called M-complete if every M-Cauchy sequence in X is convergent.*

Lemma 2. *(Saha et al., 2016) If $*$ is a continuous t-norm, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences such that $\alpha_n \to \alpha$ and $\gamma_n \to \gamma$ as $n \to +\infty$ then*

$$\overline{\lim_{k \to +\infty}}(\alpha_k * \beta_k * \gamma_k) = \alpha * \overline{\lim_{k \to +\infty}} \beta_k * \gamma,$$

and

$$\underline{\lim}_{k \to +\infty}(\alpha_k * \beta_k * \gamma_k) = \alpha * \underline{\lim}_{k \to +\infty} \beta_k * \gamma.$$

Lemma 3. *(Saha et al., 2016) Let $\{f(k, .) : (0, +\infty) \to (0, 1]; k = 0, 1, 2, ..., \}$ be a sequence of functions such that $f(k, .)$ is continuous and monotone increasing for each $k \geq 0$. Then $\overline{\lim}_{k \to +\infty} f(k, t)$ is a left continuous function in t and $\underline{\lim}_{k \to +\infty} f(k, t)$ is a right continuous function in t.*

Definition 5. *(Perveen & Imdad, 2019) Let $(X, M, *)$ be a fuzzy metric space. A mapping $S : X \to X$ is said to be α-admissible if there exists a function $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ such that for all $t > 0$*

$$z, y \in X, \alpha(z, y, t) \geq 1 \qquad \textit{implies} \qquad \alpha(Sz, Sy, t) \geq 1.$$

MA-Simulation function

In Perveen & Imdad (2019), the authors have defined the following MA-simulation function in a fuzzy metric space:
A mapping $\xi : (0, 1] \times (0, 1] \to \Re$ is said to be an MA-simulation function if it satisfies the following conditions:

(ξ_1) $\xi(t, s) < \frac{1}{t} - \frac{1}{s}$, for all $t, s \in (0, 1)$;

(ξ_2) if $\{t_n\}, \{s_n\}$ are a sequence in $(0, 1]$ such that $\lim_{n \to +\infty} t_n = \lim_{n \to +\infty} s_n = l \in (0, 1)$, and $t_n < s_n$, for all $n \in N$, then $\limsup_{n \to +\infty} \xi(t_n, s_n) < 0$.

The set of all MA-simulation functions is denoted by Ξ_{MA}.

Definition 6. *(Perveen & Imdad, 2019) Let $(X, M, *)$ be a fuzzy metric space and $\xi \in \Xi_{MA}$. A mapping $S : X \to X$ is said to be α-admissible*

Ξ_{MA}-contraction if there exists a function $\xi \in \Xi_{MA}$ such that for all $t > 0$, it satisfies the following:

$$z, y \in X, \alpha(z, y, t) \geq 1 \quad \textit{implies} \quad \xi(M(z, y, t), M(Sz, Sy, t)) \geq 0.$$

Now we define a new contractive condition employing an MA-simulation function as follows:

DEFINITION 7. *Let $(X, M, *)$ be a fuzzy metric space and $\xi \in \Xi_{MA}$. A mapping $S : X \to X$ is said to be Ξ_{MA} contractive if for some $\xi \in \Xi_{MA}$ and for all $t > 0$, it satisfies the following condition:*

$$x, y \in X, x \neq y, \xi(M(x, y, t), M(Sx, Sy, t)) \geq 0. \tag{1}$$

Gregori & Sapena (2002) defined the fuzzy contractive mappings as follows:
Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is called a fuzzy contractive mapping if there exists $k \in (0, 1)$ such that
$\frac{1}{M(Tx,Ty,t)} - 1 \leq k \left(\frac{1}{M(x,y,t)} - 1 \right)$, for all $x, y \in X$.

REMARK 1. The fuzzy contractive mapping defined by Gregori & Sapena (2002), is Ξ_{MA}-contractive if we take $\xi \in \Xi_{MA}$ to be $\xi(t, s) = k(\frac{1}{t} - 1) - (\frac{1}{s} - 1)$, for all $s, t \in (0, 1)$, and $k \in (0, 1)$ in (1).

Tirado (2012) defined the following contraction.
Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is a Tirado contraction if there exists $k \in (0, 1)$ such that
$1 - M(Tx, Ty, t) \leq k(1 - M(x, y, t))$, for all $x, y \in X$.

REMARK 2. Every Tirado contraction is Ξ_{MA}-contractive if we take $\xi \in \Xi_{MA}$ to be $\xi(t, s) = k(1 - t) + (s - 1)$, for all $t, s \in (0, 1)$, in (1).

Mihet (2010) defined the class Ψ of mappings as follows:
Let $\psi : (0, 1] \to (0, 1]$ such that ψ is continuous, non-decreasing and $\psi(t) > t$, for all $t \in (0, 1)$.
Let $\psi \in \Psi$. A mapping $T : X \to X$ is called a fuzzy ψ-contractive mapping

if

$$M(x,y,t) > 0 \qquad \text{implies} \qquad M(Tx,Ty,t) \geq \psi(M(x,y,t)),$$

for all $x, y \in X$ and $t > 0$.

REMARK 3. Every fuzzy ψ-contractive mapping is Ξ_{MA}-contractive, if we take $\xi \in \Xi_{MA}$ to be $\xi(t,s) = s - \psi(t)$, for all $t, s \in (0,1)$.

Wardowski (2013) defined the following class *H* of mappings as follows: Let *H* be the family of the mappings $\eta : (0,1] \to [0,+\infty)$ satisfying the following conditions:

(H-1) η transforms $(0,1]$ onto $[0,+\infty)$;

(H-2) η is strictly decreasing.

A mapping $T : X \to X$ is called fuzzy H-contractive with respect to $\eta \in H$ if there exists $k \in (0,1)$ satisfying the following condition:

$$\eta(M(Tx,Ty,t)) \leq k\eta(M(x,y,t)),$$

for all $x, y \in X$ and $t > 0$.

REMARK 4. In view of the remark in Gregori & Miñana (2014), every fuzzy H-contractive mapping with respect to $\eta \in H$ is Ξ_{MA}-contractive with respect to the function $\xi \in \Xi_{MA}$, if we define $\xi(t,s) = k(\eta(t)) - \eta(s))$, for all $t, s \in (0,1)$ in equation (1).

The following result has been established in Perveen & Imdad (2019)

Theorem 3.1 (Perveen & Imdad, 2019) Let $(X, M, *)$ be a complete fuzzy metric space and $S : X \to X$ is an α-admissible Ξ_{MA}-contraction with respect to ξ. Assume that the following conditions are satisfied

$$z, y \in X, \alpha(z,y,t) \geq 1 \qquad \text{implies} \qquad \xi(M(z,y,t), M(Sz,Sy,t) \geq 0,$$

(a) there exists $z_0 \in X$ such that $\alpha(z_0, Sz_0, t) \geq 1$;

(b) S is triangular α-admissible;

(c) S is continuous
or
if $\{z_n\}$ is a sequence in X such that $\alpha(z_n, z_{n+1}, t) \geq 1$, for all $n \in N, t > 0$ and $\{z_n\} \to z$, for some $z \in X$, there exists a sub sequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\alpha(z_{n_k}, z, t) \geq 1$, for all $k \in N$ and $t > 0$.

Then, S has a fixed point in X.

Main results

Our first new result is the next one.

Theorem 1 Let S be a self map on a complete fuzzy metric space $(X, M, *)$. For some $\xi \in \Xi_{MA}$ the map S is Ξ_{MA} contractive. Then, S has a unique fixed point in X.

Proof. First we prove the uniqueness of the fixed point. Suppose u and v be two fixed points of S. Then $u = Su$ and $v = Sv$. We show that $u = v$. Suppose, on the contrary, that $u \neq v$, then $Su \neq Sv$. Now

$$\begin{aligned} 0 &\leq \xi(M(u,v,t), M(Su,Sv,t)), \text{(using equation (1))} \\ &= \xi(M(u,v,t), M(u,v,t)), \\ &< \frac{1}{M(u,v,t)} - \frac{1}{M(u,v,t)}, \text{(using } \xi_1 \text{)} \\ &= 0. \end{aligned}$$

which is not possible. So, $u = v$. Thus, if the map S has a fixed point, then it is unique.

Now we prove the existence of the fixed point of the self map S. For $x_0 \in X$ define the sequence $\{x_n\}$ by $Sx_n = x_{n+1}$, for $n = 0, 1, 2, ...$

Now we consider the different cases for the sequence $\{x_n\}$.

CASE I Suppose $x_n = x_{n+1}$, for some $n \in N$. Now $x_n = x_{n+1} = Sx_n$ and so we have $Sx_n = x_n$. Thus, x_n is a fixed point of the map S in this case. So we can assume the consecutive terms of the sequence $\{x_n\}$ are distinct.

Again, to see the existence of the fixed point in the other case, we first show that all the terms of the sequence $\{x_n\}$ are distinct.

CASE II Suppose $x_n = x_m$, for some $m > n$, and consecutive terms of the sequence $\{x_n\}$ are distinct. Then $Sx_n = Sx_m$, i.e. , $x_{n+1} = x_{m+1}$. Also

$$\begin{aligned} 0 &\leq \xi(M(x_n, x_{n+1}, t), M(Sx_n, Sx_{n+1}, t), \quad \text{(using equation (1))} \\ &= \xi(M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t)) \end{aligned}$$

$$< \quad \frac{1}{M(x_n, x_{n+1}, t)} - \frac{1}{M(x_{n+1}, x_{n+2}, t)}, \quad \text{(using } \xi_1. \text{)}$$

i. e.

$$M(x_n, x_{n+1}, t) < M(x_{n+1}, x_{n+2}, t), \quad \text{for all t} > 0. \tag{2}$$

Repeating the above procedure m-times, we get

$$M(x_n, x_{n+1}, t) < M(x_{n+1}, x_{n+2}, t) < \cdots < M(x_m, x_{m+1}, t) = M(x_n, x_{n+1}, t),$$

which yield $M(x_n, x_{n+1}, t) < M(x_n, x_{n+1}, t)$ which is not true. So, this case does not arise and we conclude that $x_n \neq x_m$ for distinct $n, m \in N$. Thus, the elements of the sequence $\{x_n\}$ are distinct.

STEP 1 In this step, we prove that $\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = 1$, for all $t > 0$. From equation (2), we have

$$M(x_n, x_{n+1}, t) < M(x_{n+1}, x_{n+2}, t), \quad \text{for all t} > 0.$$

Thus, $\{M(x_n, x_{n+1}, t)\}$, for each $t > 0$, is a strictly increasing sequence which is bounded above by 1. Let $\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = a(t)$, for $t > 0$. We claim that $a(t) = 1$.

Suppose, if possible, on the contrary, that $a(s) < 1$, for some $s > 0$.

Taking $s_n = M(x_{n+1}, x_{n+2}, s)$ and $t_n = M(x_n, x_{n+1}, s)$ then again from equation (2), $t_n < s_n$, for all n and using the condition (ξ_2) we have

$$0 \leq \limsup_{n\to+\infty} \xi(M(x_n, x_{n+1}, s), M(x_{n+1}, x_{n+2}, s)) < 0,$$

which is not possible and we arrive at a contradiction. Hence $a(s) = 1$ i. e.

$$\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = 1, \quad \text{for all t} > 0. \tag{3}$$

Now we prove that the sequence $\{x_n\}$ is M-Cauchy. Suppose, if possible, on the contrary, that it is not true: then there exist $\eta \in (0, 1), t_0 > 0$ and the sequences $\{p(n)\}, \{q(n)\}$ ($q(n)$) being the smallest integer corresponding to $p(n)$.

$$n < p(n) < q(n), M(x_{p(n)}, x_{q(n)}, t_0) \leq 1 - \eta, M(x_{p(n)}, x_{q(n)-1}, t_0) > 1 - \eta. \tag{4}$$

STEP 2 In this step, we show that $\lim_{n\to+\infty} M(x_{p(n)-1}, x_{q(n)-1}, t_0) = 1 - \eta$. Now for all $n \geq 1, 0 < \lambda < t_0/2$, we obtain,

$$\begin{aligned} 1 - \eta \quad & \geq M(x_{p(n)}, x_{q(n)}, t_0), \qquad \text{(using(4))} \\ & \geq M(x_{p(n)}, x_{p(n)-1}, \lambda) * M(x_{p(n)-1}, x_{q(n)-1}, t_0 - 2\lambda) * M(x_{q(n)-1}, x_{q(n)}, \lambda). \end{aligned}$$

Let

$$h_1(t) = \overline{\lim}_{n\to\infty} M(x_{p(n)-1}, x_{q(n)-1}, t), t > 0.$$

Taking the limit supremum on both sides of the above inequality and using the properties of M and $*$, and by Lemma 2, we obtain

$$\begin{aligned} 1-\eta &\geq 1 * \overline{\lim}_{n\to+\infty} M(x_{p(n)-1}, x_{q(n)-1}, t_0 - 2\lambda) * 1 \qquad \text{(using(3))} \\ &= h_1(t_0 - 2\lambda). \end{aligned} \tag{5}$$

Since M is bounded with the range in $(0, 1]$ continuous and non-decreasing in the third variable t, it follows from Lemma 3, that h_1 is continuous from the left. Therefore, for $\lambda \to 0$, we obtain

$$h_1(t_0) = \overline{\lim}_{n\to+\infty} M(x_{p(n)-1}, x_{q(n)-1}, t_0) \leq 1 - \eta. \tag{6}$$

Let

$$h_2(t) = \underline{\lim}_{n\to+\infty} M(x_{p(n)-1}, x_{q(n)-1}, t), t > 0.$$

Again, for all $n \geq 1, t_0 > 0$

$$\begin{aligned} M(x_{p(n)-1}, x_{q(n)-1}, \lambda + t_0) &\geq M(x_{p(n)-1}, x_{p(n)}, \lambda) * M(x_{p(n)}, x_{q(n)-1}, t_0) \\ &\geq M(x_{p(n)}, x_{p(n)-1}, \lambda) * (1-\eta) \quad \text{(using(4))} \end{aligned}$$

Taking the limit infimum as $n \to +\infty$ in above inequality, we obtain

$$\begin{aligned} h_2(\lambda + t_0) &= \underline{\lim}_{n\to+\infty} M(x_{p(n)-1}, x_{q(n)-1}, \lambda + t_0), \\ &\geq \underline{\lim}_{n\to+\infty} M(x_{p(n)}, x_{p(n)-1}, \lambda) * (1-\eta), \\ &= 1 * (1-\eta), \qquad \text{(using(3))} \\ &= (1-\eta) \end{aligned}$$

Since M is continuous and bounded with the range in $(0, 1]$ and non-decreasing in the third variable t, it follows from Lemma 3, that h_2 is continuous from the right. Therefore, for $\lambda \to 0$, we obtain

$$\underline{\lim}_{n\to+\infty} M(x_{p(n)-1}, x_{q(n)-1}, t_0) \geq 1 - \eta. \tag{7}$$

Combining inequalities (6) and (7), we get

$$\lim_{n\to+\infty} M(x_{p(n)-1}, x_{q(n)-1}, t_0) = 1 - \eta. \tag{8}$$

STEP 3 In this step, we show that $\lim_{n\to+\infty} M(x_{p(n)}, x_{q(n)}, t_0) = 1 - \eta$.
From equation (6), we have

$$\overline{\lim}_{n\to+\infty} M(x_{p(n)}, x_{q(n)}, t_0) \leq 1 - \eta. \tag{9}$$

Also, for all $n \geq 1$ and $t_0 > 0$, we have

$$M(x_{p(n)}, x_{q(n)}, t_0+2\lambda) \geq M(x_{p(n)}, x_{p(n)-1}, \lambda)*M(x_{p(n)-1}, x_{q(n)-1}, t_0)*M(x_{q(n)-1}, x_{q(n)}, \lambda)$$

Taking the limit infimum as $n \to +\infty$ in the above inequality, using(8) and the properties of M and $*$ and by Lemma 3, we obtain

$$\begin{aligned} \underline{\lim}_{n\to+\infty} M(x_{p(n)}, x_{q(n)}, t_0 + 2\lambda) &\geq 1 * \underline{\lim}_{n\to+\infty} M(x_{p(n)-1}, x_{q(n)-1}, t_0) * 1 \\ &= 1 - \eta, \end{aligned} \tag{10}$$

Since M is bounded with the range in $(0, 1]$ continuous and non-decreasing in the third variable t, it follows by Lemma 3 that $\underline{\lim}_{n\to+\infty} M(x_{p(n)}, x_{q(n)}, t_0)$ is a continuous function of t from the right. Therefore, for $\lambda \to 0$, we obtain

$$\underline{\lim}_{n\to+\infty} M(x_{p(n)}, x_{q(n)}, t_0) \geq 1 - \eta. \tag{11}$$

Combining inequalities (9) and (11), we get

$$\lim_{n\to+\infty} M(x_{p(n)}, x_{q(n)}, t_0) = 1 - \eta. \tag{12}$$

STEP 4 In this step, we show that the sequence $\{x_n\}$ is an M-Cauchy sequence.
Taking $x = x_{p(n)-1}$ and $y = x_{q(n)-1}$ in equation (1), we have

$$\begin{aligned} 0 &\leq \xi(M(x_{p(n)-1}, x_{q(n)-1}, t), M(Sx_{p(n)-1}, Sx_{q(n)-1}, t)) \\ &= \xi(M(x_{p(n)-1}, x_{q(n)-1}, t), M(x_{p(n)}, x_{q(n)}, t)) \\ &< \frac{1}{M(x_{p(n)-1}, x_{q(n)-1}, t)} - \frac{1}{M(x_{p(n)}, x_{q(n)}, t)}, \end{aligned}$$

i.e.

$$M(x_{p(n)-1}, x_{q(n)-1}, t) < M(x_{p(n)}, x_{q(n)}, t), \text{ for all t} > 0.$$

Taking $t_n = M(x_{p(n)-1}, x_{q(n)-1}, t)$ and $s_n = M(x_{p(n)}, x_{q(n)}, t)$ then $t_n < s_n,$ for all n and using equations (8) and (12), we have

$$\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = 1-\eta.$$

Using (ξ_2) we have

$$0 \leq \limsup_{n\to+\infty} \xi(M(x_{p(n)-1}, x_{q(n)-1}, t), M(x_{p(n)}, x_{q(n)}, t)) < 0.$$

which is not possible and we arrive at a contradiction. So, the sequence $\{x_n\}$ is an M-Cauchy sequence in X which is M-complete. Therefore, there exists $u \in X$ such that

$$\{x_n\} \to u. \tag{13}$$

i. e.

$$\{Sx_n\} \to u. \tag{14}$$

STEP 5 Now we show that $Su = u$. Suppose, on the contrary, that $Su \neq u$. Then there exists a positive integer n_0 such that $Su \neq x_n$, for all $n \geq n_0$ (for otherwise we get $Su = u$). Taking $x = u$ and $y = x_n$ in equation (1) we get

$$\begin{aligned} 0 &\leq \xi(M(u, x_n, t), M(Su, Sx_n, t)) \\ &< \frac{1}{M(u, x_n, t)} - \frac{1}{M(Su, Sx_n, t)} \end{aligned}$$

i. e.

$$M(u, x_n, t) < M(Su, Sx_n, t), \text{ for all t} > 0.$$

As M is continuous and for $n \to +\infty$ and using equations (13) and (14), we obtain

$$M(u, u, t) \leq M(Su, u, t), \text{ for all t} > 0.$$

So, $M(Su, u, t) \geq 1$, and we arrive at a contradiction as $Su \neq u$. Thus $Su = u$. □

We illustrate by the following example that the necessity of the α-admissibility and the continuity of the self map S is not all required for the existence of the unique fixed point of the map.

Example (of Theorem 1): Let $X = [-1, 1]$, and define a fuzzy set M on $X \times X \times (0, +\infty)$ by:

$$M(x, y, t) = \begin{cases} 1; & \text{if } x = y, \\ \min\{1 - \frac{|x|}{2}, 1 - \frac{|y|}{2}\}, & \text{if } x \neq y. \end{cases}, \text{ for all x, y} \in X,\ t \in (0, +\infty).$$

Taking $\xi(t, s) = k(\frac{1}{s} - 1) - (\frac{1}{t} - 1),$ for all t, s $\in (0, 1]$ and $a * b = \min\{a, b\}$ then $(X, M, *)$ is an M-complete fuzzy metric space.
Define the mapping $\alpha : X \times X \times (0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1; & \text{if } x, y \in [0, 1/2], \\ 0, & \text{otherwise.} \end{cases}, \text{ for all x, y } \in X,\ t \in (0, +\infty).$$

Define the self-map S on X by

$$S(x) = \begin{cases} 0; & \text{if } -1 < x < 0, \\ 1, & \text{if } 0 \leq x \leq 1. \end{cases}, \text{ for all x, y} \in X,\ t \in (0, +\infty).$$

Thus, all the conditions of primary result theorem are satisfied and $x = 1$ is the unique fixed point of the map S.
Note that the map S is neither continuous nor α-admissible as for $x = 0$ and $y = 1/2$ we have $S(0) = 0, S(1/2) = 0$ and $\alpha(x, y, t) = \alpha(0, 1/2, t) = 1$ but $\alpha(Sx, Sy, t) = \alpha(S(0), S(1/2), t) = \alpha(1, 1, t) = 0 < 1.$

Application (to the damped spring-mass system)

In this section, by applying our theorem, we prove the existence of the unique solution for the spring mass system in an automobile suspension system. Let m be the mass of the spring and F be the external force acting on it, then the critical damped motion of this system subjected to the external force F is governed by the following initial value problem:

$$m\frac{d^2v}{dt^2} + \wp\frac{dv}{dt} - mF(r, v(r)) = 0, \tag{15}$$

$v(0) = 0, v'(0) = 0,$
where m is the mass of the spring, $\wp(> 0)$ is the damping constant and F is the external force acting on it given by $F : [0, J] \times R^+ \to R$ is continuous.

In terms of in integral equation, the above problem is

$$v(r) = \int_0^J R(r,z)F(z,v(z))dz, \qquad r \in [0,J] \tag{16}$$

where the respective Green's function $R(r,z)$ with $\mu = \frac{\wp}{m}$ is given by

$$R(r,z) = \begin{cases} \frac{1-e^{\mu(r-z)}}{\mu}, & \text{if } 0 \leq z \leq r \leq J; \\ 0, & \text{if } 0 \leq r \leq z \leq J \end{cases} \tag{17}$$

Theorem 2 Consider equation (16) with $R(r,z)$ given by (17). Suppose

(2.11) $\beta \in (C[0,J],R)$ is a lower solution of problem equation (16), i. e.

$$\beta(r) \leq \int_0^J R(r,z)F(z,v(z))dz, \tag{18}$$

(2.12) $F(z,.)$ is an increasing function on $(0,1]$ for every $z \in [0,J]$ and choose μ suitably such that
$\inf_{0\leq r\leq J} R(r,z) > 0.$

$$0 \leq \sup_{0\leq r\leq J} (\delta(\mu,r))^2 < 1/2. \tag{19}$$

$$R(r,z)F(z,1) \leq J^{-1},$$

where $\delta(\mu,r) = \frac{(1+r\mu-e^{r\mu})}{\mu^2}$

(2.13) For each $r \in [0,J]$ and $u,v \in X$, we have

$$|F(r,u(r)) - F(r,v(r))| < \lambda|u(r)-v(r)|, \quad \lambda \in (0,1). \tag{20}$$

Then equation (16) has a solution in $(C[0,1],R)$ which is a unique solution for initial value problem (15).

Proof. Taking $X = (C[0,J],R)$. Then X is a complete metric space with a sup-metric

$$d(z,y) = \sup_{t\in\Omega} |z(t)-y(t)|$$

and the space $(X,M,*)$ is a complete metric space if we take $M(z,y,t) = \frac{t}{t+d(z,y)}$, for all $z,y \in X$ and $t > 0$ and taking $p*q = pq$, for all p, q $\in (0,1]$.

Define $S : X \to X$ by

$$Sv(r) = \int_0^J R(r,z)F(z,v(z))dz, \text{ for all v} \in X. \tag{21}$$

Then v is a solution of (16), if v is a fixed point of S.
For $u, v \in X$ we have

$$\begin{aligned}
|Su(r) - Sv(r)| &= |\int_0^J R(r,z)F(z,u(z))dz - \int_0^J R(r,z)F(z,v(z))dz| \\
&\leq \int_0^J R(r,z)|F(z,u(z)) - F(z,v(z))|dz \\
&\leq \int_0^J R(r,z) \sup_{0\leq r\leq J} |F(z,u(z)) - F(z,v(z))|dz \\
&= \sup_{0\leq r\leq J} |F(z,u(z)) - F(z,v(z))| \int_0^J R(r,z)dz \\
&\leq \lambda \sup_{0\leq r\leq J} |u(z) - v(z)| \int_0^J R(r,z)dz \\
&\leq \lambda \sup_{0\leq r\leq J} |u(z) - v(z)| \frac{1 + r\mu - e^{ru}}{\mu^2}
\end{aligned}$$

i. e.

$$|Su(r) - Sv(r)| \leq \lambda \sup_{0\leq r\leq J} |u(z) - v(z)| \frac{1 + r\mu - e^{ru}}{\mu^2}.$$

Taking the supremum, we get

$$\sup_{0\leq r\leq J} |Su(r) - Sv(r)| \leq \lambda \sup_{0\leq r\leq J} |u(z) - v(z)| \sup_{0\leq r\leq J} (\delta(\mu,r))^2.$$

$$\begin{aligned}
d(Su, Sv) &\leq \lambda \sup_{0\leq r\leq J} (\delta(\mu,r))^2 d(u,v) \\
&\leq \frac{\lambda}{2} d(u,v).
\end{aligned}$$

Also

$$\begin{aligned}
\frac{1}{M(Su,Sv,t)} - 1 &= \frac{d(Su,Sv)}{t} \\
&\leq \frac{\lambda d(u,v)}{2t}
\end{aligned}$$

$$= \frac{\lambda}{2}(\frac{1}{M(u,v,t)} - 1), \text{ for all u, } v \in X, t > 0.$$

Taking $\xi(t,s) = \frac{\lambda}{2}(\frac{1}{s} - 1) - (\frac{1}{t} - 1)$, for all t, s $\in (0,1]$. All the conditions of the primary result theorem are met. As a result S has a unique fixed point in X.

□

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Resolución de un sistema resorte-masa amortiguado mediante la función de simulación MA

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CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: En un interesante artículo, Perveen & Imdad (2019) introdujeron la noción de una función de simulación MA y la utilizaron para demostrar la existencia de un punto fijo para una auto papeo mediante α-admisibilidad y la continuidad del auto mapeo en un espacio métrico difuso. El propósito de este trabajo es establecer un teorema de punto fijo único para una aplicación MA-contractiva mediante la relajación de la condición de continuidad y α-admisibilidad del mapa en un espacio métrico difuso. Como aplicación de nuestro resultado, estudiamos la existencia y unicidad de la solución para el sistema resorte-masa amortiguado. El artículo incluye un ejemplo que muestra la validez de nuestros resultados.

Métodos: Se utilizó el método de punto fijo con una función de simulación MA

Resultados: Se obtuvo un único punto fijo para una auto mapa en un espacio métrico difuso.

Conclusión: Se obtiene un punto fijo de los auto mapas sin la continuidad ni la α-admisibilidad del auto mapa mediante la función de simulación MA. Además, se obtiene la existencia y unicidad de la solución de un sistema resorte-masa amortiguado en el contexto de un espacio métrico difuso.

Palabras claves: espacio métrico difuso, secuencia M-Cauchy, puntos fijos, función de simulación MA.

Решение проблемы демпфирования системы масса-пружина с помощью тематического моделирования

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РУБРИКА ГРНТИ: 27.25.17 Метрическая теория функций, 27.39.15 Линейные пространства, снабженные топологией, порядком и другими структурами

ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: В своей интереснейшей статье Perveen & Imdad (2019) ввели понятие функции математического моделирования, используя его для доказательства существования неподвижной точки при самоотображении через α-допустимость и непрерывность отображения в нечетком метрическом пространстве. Цель данной статьи − представить уникальную теорему о неподвижной точке для математического отображения, ослабив условие непрерывности и α-допустимости самоотображения в нечетком метрическом пространстве. В целях подтверждения результатов исследования были изучены существование и уникальность решения проблемы демпфи-

рования системы масса-пружина. В статье приведен пример, который доказывает достоверность результатов исследования.

Методы: В статье использован метод нподвижной точки с функцией МА-моделирования.

Результаты: Получена уникальная неподвижная точка для самоотображения в нечетком метрическом пространстве.

Выводы: Неподвижная точка самоотображения получена без учета непрерывности и α-допустимости самоотображения с помощью функции МА-моделирования. Помимо того, подвержено существование и уникальность решения проблемы демпфирования системы масса-пружина в условиях нечеткого метрического пространства.

Ключевые слова: нечеткое метрическое пространство, последовательность Коши, неподвижные точки, МА-имитационное моделирование.

Решавање пригушеног система опруга-маса помоћу МА симулационе функције

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Сажетак:

Увод/циљ: У свом занимљивом раду Perveen & Imdad (2019) увели су појам МА симулационе функције користећи га за доказивање постојања фиксне тачке за самопресликавање кроз α-прихватљивост и континуитет самопресликавања у расплинутим метричким просторима. Циљ овог рада јесте да установи теорему јединствене фиксне тачке за МА контрактивно пресликавање релаксацијом услова континуитета и α-прихватљивости мапе у расплинутом метричком простору. Ради примене добијених резултата, испитано је постојање и јединственост решења пригуше-

ног система опруга-маса. Наводи се и пример који показује валидност резултата.

Методе: Коришћена је метода фиксне тачке са MA симулационом функцијом.

Резултати: Добијена је јединствена фиксна тачка за самопресликавање у расплинутом метричком простору.

Закључак: Фиксна тачка самопресликавања добијена је без континуитета и α-прихватљивости самопресликавања путем MA симулационе функције. Утврђено је, такође, постојање и јединственост решења пригушеног система опруга-маса при постављању расплинутог метричког система.

Кључне речи: расплинути метрички простор, M-Кошијев низ, фиксне тачке, MA симулациона функција.

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