Paired-Chatterjea type contractions: Novel fixed point results and continuity properties in metric spaces

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Abstract:

Introduction/purpose: The paper deals with Paired-Chatterjea type contraction mappings as an extension of traditional Chatterjea type contractions that operates on three points rather than two, in the framework of standard metric spaces.

Methods: The concept of Chatterjea type contraction mappings is employed in a metric space on three points rather than two using the idea of paired contraction mappings.

Results: A series of corresponding properties has been discussed. Furthermore, it is established that Paired-Chatterjea type mappings form

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a distinct class from traditional Chatterjea type mappings and obtain at least one fixed point in the absence periodic points of prime period 2 within complete metric spaces. It is also demonstrated that how additional criteria to these mappings, such as continuity and asymptotic regularity, broaden the scope of fixed point results. Extending beyond Chatterjea's foundational contributions, two additional fixed point results applicable to Paired-Chatterjea type mappings in metric spaces are established, even in scenarios where completeness is not required.

Conclusions: Paired-Chatterjea type mappings are generally discontinuous; they exhibit continuity at fixed points similar to Kannan and Chatterjea type mappings. In the absence of a periodic point of prime period 2, these mappings have a fixed point within the complete metric space.

Keywords: metric space, fixed point, Chatterjea type mappings, Paired contraction, Paired-Chatterjea type mapping.

Introduction

The study of fixed point theory (*FPT*), a captivating area in mathematics, delves into the existence and properties of fixed points (*FPs*). In mathematics, a *FP* of a function is an element that stays the same when the function is executed on it. Formally, if $\Im$ is a function, then ζ is a *FP* if $\Im(\zeta) = \zeta$.

Numerous findings regarding *FPs* and their diverse applications in various mathematical and scientific domains have been documented. To yield fresh and intriguing results, two principal approaches can be pursued. Firstly, by modifying the characteristics of the operators involved, achieved through the imposition or relaxation of specific constraints (Kannan, 1968; Chatterjea, 1972; Chand & Rohen, 2024; Petrov, 2023; Bisht & Petrov, 2024). Secondly, by altering the framework, or the abstract space structure itself, which includes variations like metric spaces, symmetric or non-symmetric spaces, b-metric spaces, S-metric spaces, and G-metric spaces (Chand & Rohen, 2023; Chand et al., 2024; Bimol et al., 2024).

The Kannan's *FP* theorem is a notable result in *FPT*, specifically within metric spaces. It ensures the existence of a *FP* for mappings, even when these mappings are discontinuous. It was introduced by R. Kannan in 1968 (Kannan, 1968) and provides conditions under which a mapping possesses a unique *FP*.

THEOREM 1. *Let* $(\mho, \partial)$ *be a complete metric space, and suppose that* $\Im : \mho \to \mho$ *is a self-map for which we obtain a constant* $0 \leq \lambda < \frac{1}{2}$ *satisfying:*

$$\partial(\Im\zeta, \Im\varepsilon) \leq \lambda(\partial(\zeta, \Im\zeta) + \partial(\varepsilon, \Im\varepsilon)) \text{ for all } \zeta, \varepsilon \in \mho. \quad (1)$$

Then $\Im$ *has a unique FP.*

Another noteworthy outcome in *FPT* is the Chatterjea *FP* theorem; it also provides a *FP* for mappings that are discontinuous too. It was introduced by S.K. Chatterjea in 1972 (Chatterjea, 1972) and provides conditions under which a mapping has a unique *FP*.

THEOREM 2. *Let* $(\mho, \partial)$ *be a complete metric space, and suppose that* $\Im : \mho \to \mho$ *is a self-map for which we obtain a constant* $0 \leq \eta < \frac{1}{2}$

$$\partial(\Im\zeta, \Im\varepsilon) \leq \eta(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta)) \text{ for all } \zeta, \varepsilon \in \mho. \quad (2)$$

Then there is a unique FP for $\Im$.

DEFINITION 1. *A self-map* $\Im : \mho \to \mho$ *is termed as a Chatterjea type map if a constant can be found* $0 \leq \eta < \frac{1}{2}$ *such that*

$$\partial(\Im\zeta, \Im\varepsilon) \leq \eta(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta)), \quad (3)$$

satisfied for all $\zeta, \varepsilon \in \mho$.

Like Kannan's fixed-point theorem (Theorem 1), the mapping $\Im$ is guaranteed to be continuous at the *FP* (Rhoades, 1988) by condition (3). Moreover, the uniqueness of the *FP* is the only thing that unites Chatterjea mappings, Kannan mappings, and the Banach Contraction Principle; otherwise, they are independent of one another.

Within the field of *FPT*, there are different types of generalizations of the Chatterjea theorem that can be distinguished from one another. The contractive property of the mapping is loosened in the first instance as illustrated, for example, in references (Chandok & Postolache, 2013; Debnath et al., 2021; Kadelburg & Radenovic, 2016; Subrahmanyam, 2018; Bisht & Petrov, 2024). The second instance involves relaxing the topology, as examined, for instance, in reference (Agarwal et al., 2018). And the third instance concerns theorems developed for multivalued mappings of the Chatterjea type, which are covered in works such as (Choudhury et al., 2019; Tassaddiq et al., 2022). Lastly, the fourth instance provides a thorough investigation of several generalization paths in this field by examining notable extensions in a more flexible or comprehensive framework of the metric space (Harjani et al., 2011; Karahan & Isik, 2019; Kohsaka & Suzuki, 2017; Malčeski et al., 2016; Berinde & Păcurar, 2021).

In 2024, D. Chand and Y. Rohen (Chand & Rohen, 2024) presented a novel class of mappings, referred to as a three-point version of the Banach contraction, and termed them Paired-Contraction: the mapping is known as Paired-contraction mapping, defined as follows.

DEFINITION 2. *If* $(\mho, \partial)$ *is a metric space with* $|\mho| \geq 3$*. The self-mapping* $\Im : \mho \rightarrow \mho$ *is referred to as a Paired contraction mapping if there is a constant* $\lambda \in [0,1)$ *so that the inequality*

$$\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) \leq \lambda(\partial(\zeta, \varepsilon) + \partial(\varepsilon, \xi)), \tag{4}$$

holds for all pairwise distinct $\zeta, \varepsilon, \xi \in \mho$*.*

Geometric refinement in metric *FPT* is obtained through the use of different combinations of unique distances, which are important. For 6 points $\zeta, \varepsilon, \xi, \Im\zeta, \Im\varepsilon, \Im\xi$ specified in Definition 2, there are ${}^{6}C_{2}$ pairs that, when taken two at a time, yield 15 unique distances. One of these combinations, specifically $\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi)$, appears in the left part of (4). However, the expression in the right part combines two distances, that is, $\partial(\zeta, \varepsilon) + \partial(\varepsilon, \xi)$ for the pairwise distinct points ζ, ε and ξ. This motivates the development of various classes of three-point version mappings, similar to the well-established two-point counterparts.

Inspired by the insights from (Chand & Rohen, 2024), we introduce a Paired-Chatterjea type mapping on three point analogue by using four distances among the points $\zeta, \varepsilon, \xi, \Im\zeta, \Im\varepsilon, \Im\xi$, represented as $\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\varepsilon, \Im\xi) + \partial(\xi, \Im\zeta)$ for the pairwise distinct points ζ, ε, ξ respectively.

These mappings led to the establishment of a fixed-point theorem. While the proof draws inspiration from Banach's classical theorem, the key distinction lies in these mappings - instead of using two points to define it, three points are used. Moreover, we need an assumption for these mappings in order to prevent periodic points with prime period 2. Notably, normal contraction mappings are a significant subset of these mappings.

In the second Section, Paired-Chatterjea Type Mappings, we study the relationship between Paired-Chatterjea type mappings, Chatterjea type mappings, Kannan type mappings and Paired-Contraction mappings. In addition, we introduce Example 1, a Paired-Chatterjea type mapping that deviates from a Chatterjea type mapping.

In the third Section, Fixed Point Results of the Paired-Chatterjea Type Mappings, the primary outcome of this article, Theorem 3, which gives a fixed-point theorem for Paired-Chatterjea type mappings, is proved. Notably, this result states that for Paired-Chatterjea type mappings there exists at least one *FP* and there exist at most two *FPs*. In addition, we also establish that Paired-Chatterjea type mappings exhibit continuity at their *FPs*. We also included two examples (2) and (3) to support our findings.

In the fourth Section, Results on Paired Chatterjea-Style Mappings within Incomplete Metric Spaces, based on Kannan's work (Kannan, 1969), we offer two new *FP* results for Paired-Chatterjea type mappings. In the first theorem (Theorem 4), we eliminate the necessity of the metric space $\mho$ being complete. In the second theorem (Theorem 5), we need the mapping $\Im$ to be continuous in space, and condition (5) to only hold on a dense subset M of the space $\mho$. The concept of continuity and discontinuity at the fixed point is further elaborated in the recent papers (Savaliya et al., 2024; Jachymski, 1994; Pant et al., 2021).

Paired-Chatterjea type mappings

In this section, we will introduce the Paired-Chatterjea type contraction and discuss Paired-Chatterjea type mappings and also establish the connections between Paired-Chatterjea type mappings, Paired-Contractions mappings, Kannan-type mappings and Chatterjea-type mappings.

DEFINITION 3. *If* $(\mho, \partial)$ *is a metric space with* $|\mho| \geq 3$*. A mapping* $\Im : \mho \to \mho$ *is known as a Paired-Chatterjea type mapping on* $\mho$ *if the following inequality:*

$$\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) \leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\xi, \Im\varepsilon) + \partial(\varepsilon, \Im\xi)), \quad (5)$$

holds for the constant $\lambda \in [0, \frac{1}{2})$ *and pairwise distinct points* $\zeta, \varepsilon, \xi \in \mho$.

REMARK 1. It is essential that $\zeta, \varepsilon, \xi \in \mho$ be pairwise distinct. Without this condition, the definition would reduce to that of a standard definition of Chatterjea-type mapping of a two point version.

PROPOSITION 1. *Chatterjea type mappings are Paired-Chatterjea type mappings.*

Proof. In a metric space $(\mho, \partial)$ with $|\mho| \geq 3$, let $\Im : \mho \to \mho$ represent a Chatterjea type mapping. Take three $\zeta, \varepsilon, \xi \in \mho$ pairwise distinct points. Examine inequality (2) for the pairs (ζ, ε), (ε, ξ), we get inequalities, we obtain:

$$\partial(\Im\zeta, \Im\varepsilon) \leq \eta(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta)), \tag{6}$$

and

$$\partial(\Im\varepsilon, \Im\xi) \leq \eta(\partial(\varepsilon, \Im\xi) + \partial(\xi, \Im\varepsilon)), \tag{7}$$

Adding inequalities (6) and (7), we have

$$\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) \leq \eta(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\varepsilon, \Im\xi) + \partial(\xi, \Im\varepsilon)). \tag{8}$$

This supports the desired claim. □

PROPOSITION 2. *Kannan type mappings are Paired-Chatterjea type mappings for the contracting coefficient $\delta \in [0, \frac{1}{4})$.*

Proof. Let $(\mho, \partial)$ be a metric space with $|\mho| \geq 3$. Consider three pairwise distinct points $\zeta, \varepsilon, \xi \in \mho$, examine inequality (3) for the pairs (ζ, ε), (ε, ξ),, we get

$$\partial(\Im\zeta, \Im\varepsilon) \leq \delta(\partial(\zeta, \Im\zeta) + \partial(\varepsilon, \Im\varepsilon)), \tag{9}$$

and

$$\partial(\Im\varepsilon, \Im\xi) \leq \delta(\partial(\varepsilon, \Im\varepsilon) + \partial(\xi, \Im\xi)). \tag{10}$$

Adding these inequalities, we have

$$\begin{gathered}
\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) \leq \delta(\partial(\zeta, \Im\zeta) + \partial(\varepsilon, \Im\varepsilon) + \partial(\varepsilon, \Im\varepsilon) + \partial(\xi, \Im\xi)), \\
(\text{Using triangle inequality}) \\
\leq \delta \left\{ \begin{array}{c} \partial(\zeta, \Im\varepsilon) + \partial(\Im\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\Im\zeta, \Im\varepsilon) \\ +\partial(\varepsilon, \Im\xi) + \partial(\Im\varepsilon, \Im\xi) + \partial(\xi, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) \end{array} \right\}, \\
\implies (1-2\delta)(\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi)) \leq \delta(\partial(\zeta, \Im\varepsilon) + \\
\partial(\varepsilon, \Im\zeta) + \partial(\varepsilon, \Im\xi) + \partial(\xi, \Im\varepsilon)) \\
\implies \partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) \leq \\
\frac{\delta}{1-2\delta}(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\varepsilon, \Im\xi) + \partial(\xi, \Im\varepsilon)).
\end{gathered}$$

Since $0 \leq \delta < \frac{1}{4}$, we get $0 \leq \lambda = \frac{\delta}{1-2\delta} < \frac{1}{2}$. Thus, $\Im$ is a Paired-Chatterjea type mapping. □

PROPOSITION 3. *In a metric space* $(\mho, \partial)$*, where* $|\mho| \geq 3$*, a Paired contraction(PC) mapping* $\Im : \mho \to \mho$ *with contracting coefficient* $\alpha \in [0, \frac{1}{3})$ *represents a Paired-Chatterjea Type mapping.*

Proof. Consider three pairwise distinct points $\zeta, \varepsilon, \xi \in \mho$. Since $\Im$ is a Paired contraction mapping, then we have

$$\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) \leq \alpha(\partial(\zeta, \varepsilon) + \partial(\varepsilon, \xi)).$$

Utilizing triangle inequality many times to the above inequality, we get

$$\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) \leq \alpha \begin{Bmatrix} \partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\Im\zeta, \Im\varepsilon) \\ +\partial(\varepsilon, \Im\xi) + \partial(\xi, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) \end{Bmatrix},$$

$$(1-\alpha)(\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi)) \leq$$
$$\alpha(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\varepsilon, \Im\xi) + \partial(\xi, \Im\varepsilon)),$$
$$(\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi)) \leq$$
$$\frac{\alpha}{1-\alpha}(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\varepsilon, \Im\xi) + \partial(\xi, \Im\varepsilon)).$$

As $0 \leq \alpha < \frac{1}{3}$, we have $0 \leq \lambda = \frac{\alpha}{1-\alpha} < \frac{1}{2}$. Consequently, $\Im$ is a mapping of the Paired-Chatterjea type. □

PROPOSITION 4. *In a metric space* $(\mho, \partial)$*, where* $|\mho| \geq 3$*, and let* $\Im : \mho \to \mho$ *be a Paired-Chatterjea type mapping. We obtain the following inequality for any* $\varepsilon \in \mho$ *if* ζ *is an accumulation point of* $\mho$ *and* $\Im$ *is a continuous mapping at* ζ*:*

$$\partial(\Im\zeta, \Im\varepsilon) \leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta)). \quad (11)$$

Proof. Suppose ζ is an accumulation point in $\mho$, and let $\varepsilon \in \mho$ be another element. If $\varepsilon = \zeta$, then there is nothing to prove. Let $\varepsilon \neq \zeta$. Given that ζ is an accumulation point, we can assert the existence of a sequence (ξ_n) where $\xi_n \neq \zeta$, $\xi_n \neq \varepsilon$, and all ξ_n are distinct for every n. Hence, by (5) we conclude that

$$\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi_n) \leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\xi_n, \Im\varepsilon) + \partial(\varepsilon, \Im\xi_n)),$$

satisfied for each $n \in N$. Given that at ζ, $\Im$ is continuous and $\xi_n \to \zeta$ as $n \to +\infty$, therefore, $\Im\xi_n \to \Im\zeta$. We get

$$\begin{aligned}\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\zeta) &\leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta)),\\ \implies 2\partial(\Im\zeta, \Im\varepsilon) &\leq \lambda(2\partial(\zeta, \Im\varepsilon) + 2\partial(\varepsilon, \Im\zeta)),\\ \implies \partial(\Im\zeta, \Im\varepsilon) &\leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta)),\end{aligned}$$

which is the desired inequality. □

COROLLARY 1. *In a metric space $(\mho, \partial)$ where, $|\mho| \geq 3$, and let $\Im : \mho \to \mho$ be a continuous Paired-Chatterjea mapping, then $\Im$ is a Chatterjea type mapping if all points in $\mho$ are accumulation points.*

Proof. As per proposition (2), the following inequalities are established:

$$\partial(\Im\zeta, \Im\varepsilon) \leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta)), \text{ for all } \varepsilon \in \mho, \tag{12}$$

$$\partial(\Im\varepsilon, \Im\zeta) \leq \lambda(\partial(\varepsilon, \Im\zeta) + \partial(\zeta, \Im\varepsilon)), \text{ for all } \zeta \in \mho. \tag{13}$$

Adding equations (12) and (13), we get

$$\partial(\Im\zeta, \Im\varepsilon) \leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta)) \text{ for all } \zeta, \varepsilon \in \mho. \tag{14}$$

Since $\lambda \in [0, \frac{1}{2})$, thereby concluding the proof. □

EXAMPLE 1. Let $\mho = \{\zeta, \varepsilon, \xi\}$ and the metric ∂ defined on this set by $\partial(\zeta, \zeta) = \partial(\varepsilon, \varepsilon) = \partial(\xi, \xi) = 0$, $\partial(\zeta, \xi) = \partial(\varepsilon, \xi) = 2$, $\partial(\zeta, \varepsilon) = 1$. Let a mapping $\Im : \mho \to \mho$ be such that $\Im\zeta = \zeta$, $\Im\varepsilon = \varepsilon$ and $\Im\xi = \varepsilon$.

Now, we have

$$\partial(\Im\zeta, \Im\varepsilon) = \partial(\zeta, \varepsilon) = 1, \text{ and } \partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) = \partial(\zeta, \varepsilon) + \partial(\varepsilon, \zeta) = 2.$$

It is evident that for every $\eta \in [0, 1)$, inequality (2) does not hold. Therefore, $\Im$ is not a Chatterjea type mapping.

But, we get that inequality (5) holds for $\lambda = \frac{2}{5}$ and for all $\zeta, \varepsilon, \xi \in \mho$, implies that $\Im$ is a Paired-Chatterjea type mapping.

Fixed point results of Paired-Chatterjea type mappings

A point $\zeta \in \mho$ is termed a periodic point of period n if applying the mapping $\Im$ repeatedly n times returns ζ to its original position, i.e., $\Im^n(\zeta) = \zeta$, where $\mho$ is a metric space and ζ is a self-map on it. The smallest positive integer n that satisfies this condition is known as the prime period of ζ. If $\Im(\Im(\zeta)) = \zeta$ and $\Im\zeta \neq \zeta$, then ζ has prime period 2.

We will now demonstrate the main result of this paper, which is stated and proven as follows:

THEOREM 3. *Let $(\mho, \partial)$ be a complete metric space, where $|\mho| \geq 3$, and consider the mapping $\Im : \mho \to \mho$ that is a Paired-Chatterjea contraction mapping and has no periodic elements of prime period 2. Under these conditions, there exists at least one FP of $\Im$, and the number of FPs can be at most two.*

Proof. Consider an arbitrary point $\zeta \in \mho$, and set the sequence $\{\zeta_n\}$ as $\zeta_0 = \zeta$, $\zeta_1 = \Im\zeta_0$ and $\zeta_n = \Im\zeta_{n-1} = \Im^n\zeta_0$. If for some n, ζ_n is a *FP* then there is nothing to prove. Assuming that ζ_n is not a *FP* for any $n = 0, 1, 2, ...$, we can deduce that $\zeta_{n-1} \neq \zeta_n \neq \zeta_{n+1}$. Since, $\Im$ had no periodic point with prime period 2 implies $\zeta_{n+1} \neq \zeta_{n-1}$. Therefore, ζ_{n-1}, ζ_n and ζ_{n+1} are pairwise distinct. Placing $\zeta = \zeta_{n-1}$, $\varepsilon = \zeta_n$ and $\xi = \zeta_{n+1}$ in (5), then we get

$$\begin{gathered}\partial(\Im\zeta_{n-1}, \Im\zeta_n) + \partial(\Im\zeta_n, \Im\zeta_{n+1}) \leq \\ \lambda\left\{\partial(\zeta_{n-1}, \Im\zeta_n) + \partial(\zeta_n, \Im\zeta_{n-1}) + \partial(\zeta_n, \Im\zeta_{n+1}) + \partial(\zeta_{n+1}, \Im\zeta_n)\right\}, \\ \partial(\zeta_n, \zeta_{n+1}) + \partial(\zeta_{n+1}, \zeta_{n+2}) \leq \\ \lambda\left\{\partial(\zeta_{n-1}, \zeta_{n+1}) + \partial(\zeta_n, \zeta_n) + \partial(\zeta_n, \zeta_{n+2}) + \partial(\zeta_{n+1}, \zeta_{n+1})\right\}. \end{gathered} \tag{15}$$

Utilizing the triangle inequality in (15), we have

$$\begin{gathered}\partial(\zeta_n, \zeta_{n+1}) + \partial(\zeta_{n+1}, \zeta_{n+2}) \leq \\ \lambda\left\{\partial(\zeta_{n-1}, \zeta_n) + \partial(\zeta_n, \zeta_{n+1}) + \partial(\zeta_n, \zeta_{n+1}) + \partial(\zeta_{n+1}, \zeta_{n+2})\right\}, \\ (1-\lambda)(\partial(\zeta_n, \zeta_{n+2}) + \partial(\zeta_{n+1}, \zeta_{n+2})) \leq \lambda\left\{\partial(\zeta_{n-1}, \zeta_n) + \partial(\zeta_n, \zeta_{n+1})\right\}, \\ \partial(\zeta_n, \zeta_{n+2}) + \partial(\zeta_{n+1}, \zeta_{n+2}) \leq \frac{\lambda}{1-\lambda}\left\{\partial(\zeta_{n-1}, \zeta_n) + \partial(\zeta_n, \zeta_{n+1})\right\}, \\ \partial(\zeta_n, \zeta_{n+2}) + \partial(\zeta_{n+1}, \zeta_{n+2}) \leq \alpha\left\{\partial(\zeta_{n-1}, \zeta_n) + \partial(\zeta_n, \zeta_{n+1})\right\}. \end{gathered} \tag{16}$$

Where $\alpha = \frac{\lambda}{1-\lambda}$. Since $0 \leq \lambda < \frac{1}{2}$, therefore $0 \leq \alpha = \frac{\lambda}{1-\lambda} < 1$. From inequality (16), we get

$$\partial(\zeta_n, \zeta_{n+2}) + \partial(\zeta_{n+1}, \zeta_{n+2}) \leq \alpha\left\{\partial(\zeta_{n-1}, \zeta_n) + \partial(\zeta_n, \zeta_{n+1})\right\}$$

$$
\begin{aligned}
&\leq \alpha^2 \{\partial(\zeta_{n-2}, \zeta_{n-1}) + \partial(\zeta_{n-1}, \zeta_n)\} \\
&. \\
&. \\
&. \\
&\leq \alpha^n \{\partial(\zeta_0, \zeta_1) + \partial(\zeta_1, \zeta_2)\} .
\end{aligned}
$$

If we consider $K_0 = \partial(\zeta_0, \zeta_1) + \partial(\zeta_1, \zeta_2$, $K_1 = \partial(\zeta_1, \zeta_2) + \partial(\zeta_2, \zeta_3)$,...$K_n = \partial(\zeta_n, \zeta_{n+1}) + \partial(\zeta_{n+1}, \zeta_{n+2}$, then it follows that

$$
K_n \leq \alpha K_{n-1} \leq \alpha^2 K_{n-2} \leq \ldots \leq \alpha^n K_n. \tag{17}
$$

Suppose there is the smallest number $j \geq 3$ such that $\zeta_j = \zeta_i$ for some i where $0 \leq i \leq j-2$. In the present situation, it appears that $\zeta_{j+1} = \zeta_{i+1}$ and $\zeta_{j+2} = \zeta_{i+2}$. Therefore

$$
\begin{aligned}
K_i &= \partial(\zeta_i, \zeta_{i+1}) + \partial(\zeta_{i+1}, \zeta_{i+2}) \\
&= \partial(\zeta_j, \zeta_{j+1}) + \partial(\zeta_{j+1}, \zeta_{j+2}) \\
&= K_i.
\end{aligned}
$$

This results in a conflict with equation (17). Therefore, such values of i and j cannot exist.

Now, let us show that $\{\zeta_n\}$ is a Cauchy sequence. Based on the previous arguments, it is evident that:

$$
\partial(\zeta_n, \zeta_{n+1}) \leq K_n \leq \alpha^n K_0. \tag{18}
$$

Taking into consideration equation (18), for $(m-n) \in \mathbb{N}$, assuming $m > n$ for the sake of simplicity and utilizing repeatedly triangle inequality, we find that

$$
\begin{aligned}
\partial(\zeta_n, \zeta_m) &\leq \partial(\zeta_n, \zeta_{n+1}) + \partial(\zeta_{n+1}, \zeta_{n+2}) + \ldots + \partial(\zeta_{m-1}, \zeta_m) \\
&\leq \alpha^n K_0 + \alpha^{n-1} K_0 + \ldots + \alpha^{m-1} K_0 \\
&= \alpha^n (1 + \alpha + \alpha^2 + \ldots + \alpha^{m-1}) K_0 \\
&= \alpha^n \left(\frac{1-\alpha^m}{1-\alpha} \right) K_0.
\end{aligned}
$$

Under the assumption that $\alpha \in [0,1)$, we can observe that $\partial(\zeta_n, \zeta_m) < \alpha^n \frac{1}{1-\alpha} K_0$. Consequently, as we let $m, n \to \infty$, we find that $\partial(\zeta_n, \zeta_m) \to 0$.

This proves that sequence $\{\zeta_n\}$ is a Cauchy. This sequence has a limit $\zeta^* \in \mho$ according to the completeness of $(\mho, \partial)$.

Note that any consecutive three elements of $\{\zeta_n\}$ are distinct from each other. If $\zeta^* \neq \zeta_k$ for any $k \in \{1, 2, 3, \ldots\}$, then inequality (5) is satisfied for any three elements ζ^*, ζ_{n-1} and ζ_n. Assume that $\zeta_k = \zeta^*$ for the smallest possible $k \in \{1, 2, 3, \ldots\}$. If $m > k$ exists such that $\zeta_m = \zeta^*$, then the sequence, beginning from k, is cyclic and cannot be a Cauchy. Therefore, for $n-1 > k$, the points ζ^*, ζ_{n-1}, and ζ_n are distinct from each other.

Now, we demonstrate that the limit point ζ^* is a *FP* of $\Im$ that is $\Im\zeta^* = \zeta^*$. If there exists a $k \in \{1, 2, 3, \ldots\}$ such that $\zeta_k = \zeta^*$, then assuming $n-1 > k$, and applying the triangle inequality with inequality (5) follows that:

$$\begin{aligned}
\partial(\zeta^*, \Im\zeta^*) &\leq \partial(\zeta^*, \zeta_n) + \partial(\zeta_n, \Im\zeta^*) \\
&\leq \partial(\zeta^*, \zeta_n) + \partial(\Im\zeta_{n-1}, \Im\zeta^*) \\
&\leq \partial(\zeta^*, \zeta_n) + \partial(\Im\zeta_{n-1}, \Im\zeta^*) + \partial(\Im\zeta^*, \Im\zeta_n) \\
&\leq \partial(\zeta^*, \zeta_n) + \lambda(\partial(\zeta_{n-1}, \Im\zeta^*) + \partial(\zeta^*, \Im\zeta_{n-1}) + \partial(\zeta_n, \Im\zeta^*) + \partial(\zeta^*, \Im\zeta_n)) \\
&\leq \partial(\zeta^*, \zeta_n) + \lambda(\partial(\zeta_{n-1}, \Im\zeta^*) + \partial(\zeta^*, \zeta_n) + \partial(\zeta_n, \Im\zeta^*) + \partial(\zeta^*, \zeta_{n+1})).
\end{aligned}$$

Considering the continuity of the metric ∂ and letting $n \to \infty$, we obtain

$$\partial(\zeta^*, \Im\zeta^*) \leq 2\lambda\partial(\zeta^*, \Im\zeta^*). \tag{19}$$

Since $0 \leq \lambda < \frac{1}{2}$, we have $0 \leq \lambda < 1$ which implies from inequality (19) that $\partial(\zeta^*, \Im\zeta^*) = 0$ that is $\Im\zeta^* = \zeta^*$.

Assume there are three distinct *FPs* ζ, ε and ξ implies that $\Im\zeta = \zeta$, $\Im\varepsilon = \varepsilon$ and $\Im\xi = \xi$. Consequently, it follows from equation (5) that:

$$\partial(\zeta, \varepsilon) + \partial(\varepsilon, \xi) \leq 2\lambda(\partial(\zeta, \varepsilon) + \partial(\varepsilon, \xi)),$$

which is a contradiction for any $\lambda \in [0, \frac{1}{2})$. It concludes the theorem's proof.

□

REMARK 2. If under the supposition of Theorem 3, ζ^* is a *FP* of the mapping $\Im$ which is the limit of a Picard sequence defined as $\zeta_n = \Im\zeta_{n-1}$ for all $n \in \{1, 2, 3, \ldots\}$, where $\zeta_0 \in \mho$. Then ζ^* is a unique *FP*.

On the contrary, suppose that $\Im$ has another *FP* ζ^{**} such that $\zeta^{**} \neq \zeta^*$. Then, we have $\zeta_n \neq \zeta^{**}$ for all $n \in \{1, 2, 3, \ldots\}$. Hence, ζ^*, ζ^{**} and ζ_n are distinct from each other for all $n = 1, 2, 3, \ldots$. Think about the ratio:

$$\Gamma_n = \frac{\partial(\Im\zeta^*, \Im\zeta^{**}) + \partial(\Im\zeta^{**}, \Im\zeta_n)}{\partial(\zeta^*, \Im\zeta^{**}) + \partial(\zeta^{**}, \Im\zeta^*) + \partial(\zeta^{**}, \Im\zeta_n) + \partial(\zeta_n, \Im\zeta^{**})}$$
$$= \frac{\partial(\zeta^*, \zeta^{**}) + \partial(\zeta^{**}, \zeta_{n+1)}}{\partial(\zeta^*, \zeta^{**}) + \partial(\zeta^{**}, \zeta^*) + \partial(\zeta^{**}, \zeta_{n+1}) + \partial(\zeta_n, \zeta^{**})}.$$

Taking into consideration that $\partial(\zeta^*, \zeta_{n+1}) \to 0$, $\partial(\zeta^{**}, \zeta_{n+1}) \to \partial(\zeta^{**}, \zeta^*)$ and $\partial(\zeta_n, \zeta^*) \to 0$, we get $\Gamma_n \to \frac{1}{2}$ as $n \to \infty$, this is against condition (5).

In general, Chatterjea-type mappings are continuous at their *FPs* under reasonable conditions. This follows from the contraction property and the fact that for a sequence converging to the *FP*, the images of the sequence under the mapping also converge to the *FP*. The continuity at the *FP* guarantees that the *FP* is stable under small perturbations of initial conditions, which is a crucial property in iterative schemes used in practical applications.

In the next proposition we will prove that Paired-Chatterjea type mappings are also continuous at their *FPs*.

PROPOSITION 5. *Paired-Chatterjea type mappings are continuous at their FPs.*

Proof. Consider a metric space $(\mho, \partial)$, where $|\mho| \geq 3$. Let $\Im : \mho \to \mho$ be a Paired-Chatterjea type mapping and ζ^* be a *FP* of $\Im$. If $\{\zeta_n\}$ is a sequence such that $\zeta_n \to \zeta^*$. To show continuity at ζ^* of $\Im$, we need to show that $\Im\zeta_n \to \Im\zeta^*$. We consider the following cases to complete the proof.

Case 1: $\zeta_n \neq \zeta_{n+1}$ and $\zeta_n \neq \zeta^*$ for all n;
Utilizing inequality (5), we get

$$\partial(\Im\zeta_n, \Im\zeta^*) + \partial(\Im\zeta^*, \Im\zeta_{n+1}) \leq$$
$$\lambda(\partial(\zeta^*, \Im\zeta_n) + \partial(\zeta_n, \Im\zeta^*) + \partial(\zeta^*, \Im\zeta_{n+1}) + \partial(\zeta_{n+1}, \Im\zeta^*)).$$

By utilizing triangle inequality and *FP* property, we get

$$\partial(\Im\zeta_n, \Im\zeta^*) + \partial(\Im\zeta^*, \Im\zeta_{n+1}) \leq$$
$$\lambda(\partial(\Im\zeta^*, \Im\zeta_n) + \partial(\zeta_n, \zeta^*) + \partial(\Im\zeta^*, \Im\zeta_{n+1}) + \partial(\zeta_{n+1}, \zeta^*)).$$

Therefore, we have

$$(1-\lambda)(\partial(\Im\zeta_n, \Im\zeta^*) + \partial(\Im\zeta^*, \Im\zeta_{n+1})) \le \lambda(\partial(\zeta_n, \zeta^*) + \partial(\zeta_{n+1}, \zeta^*)).$$

Which implies that

$$\partial(\Im\zeta_n, \Im\zeta^*) + \partial(\Im\zeta^*, \Im\zeta_{n+1}) \le \frac{\lambda}{1-\lambda}(\partial(\zeta_n, \zeta^*) + \partial(\zeta_{n+1}, \zeta^*)).$$

Since $\partial(\zeta_n, \zeta^*) \to 0$ and $\partial(\zeta_{n+1}, \zeta^*) \to 0$, we have

$$\partial(\Im\zeta_n, \Im\zeta^*) + \partial(\Im\zeta^*, \Im\zeta_{n+1}) \to 0,$$

and, hence, $\partial(\Im\zeta_n, \Im\zeta^*) \to 0$ implies that $\Im\zeta_n \to \Im\zeta^*$.

Case 2: $\zeta_n \neq \zeta^*$ for all n, but $\zeta_n = \zeta_{n+1}$ is possible.

Consider the subsequence $\{\zeta_{n_k}\}$ obtained by removing corresponding repeating terms of $\{\zeta_n\}$ so that $\zeta_{n_k} \neq \zeta_{n_{k+1}}$ for all k. Clearly $\zeta_{n_k} \to \zeta^*$, as it was just proved that $\Im\zeta_{n_k} \to \Im\zeta^* = \zeta^*$. The difference between $\Im\zeta_{n_k}$ and $\Im\zeta_n$ is that $\Im\zeta_n$ can be achieved by adding the corresponding repeating terms to $\Im\zeta_{n_k}$. It implies that $\Im\zeta_n \to \Im\zeta^*$.

Case 3: If $\zeta_n = \zeta^*$ for all $n > N$, where N is a fixed natural number, then it is evident that $\Im\zeta_n \to \Im\zeta^*$.

Case 4: Let $\{\zeta_n\}$ be an arbitrary sequence such that $\zeta_n \to \zeta^*$, but in a manner different from the previously discussed cases:

Let us consider (ζ_{n_k}) achieved by removing ζ^*(if exist) from the sequence (ζ_n). Hence, $\zeta_{n_k} \to \zeta^*$. It was just demonstrated that $\Im\zeta_{n_k} \to \Im\zeta^*$ for such a sequence. Again, we observe that $\Im\zeta_n$ can be derived from $\Im\zeta_{n_k}$ by adding the term ζ^* at certain positions. Hence, it is clear that $\Im\zeta_n = \Im\zeta^*$. □

EXAMPLE 2. Let us consider $\mho = \{a, b, c, d, e, f\}$ as shown in Figure 1. And let the metric d on $\mho$ be defined by

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$$\mathsf{d}(a,b)=\mathsf{d}(a,c)=\mathsf{d}(b,c)=\mathsf{d}(b,d)=\mathsf{d}(c,e)=\mathsf{d}(d,e)=\mathsf{d}(d,f)=\mathsf{d}(e,f)=1,$$

and

$$\mathsf{d}(a,d)=\mathsf{d}(a,e)=\mathsf{d}(b,e)=\mathsf{d}(c,d)=\mathsf{d}(b,f)=\mathsf{d}(c,f)=2 \text{ and } \mathsf{d}(a,f)=3.$$

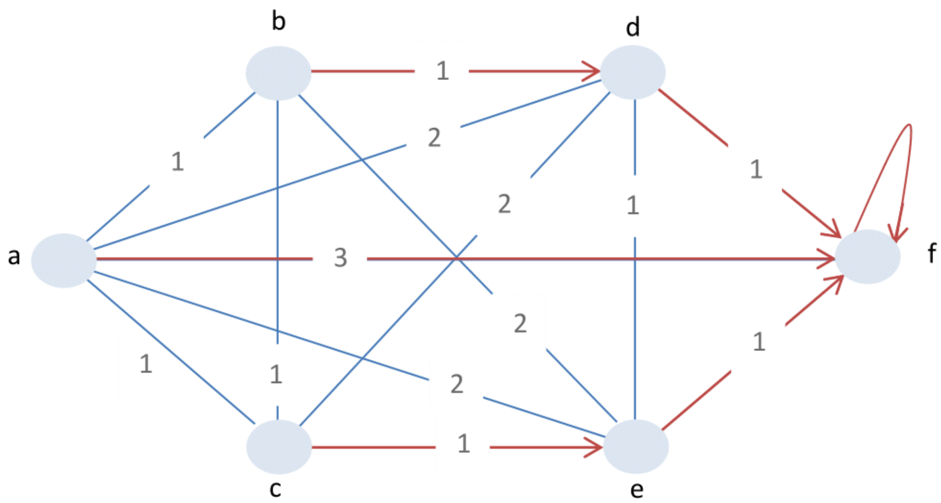

Figure 1 – A Paired-Chatterjea type mapping

Let $\Im:\mho\to\mho$ be such that $\Im a=\Im d=\Im e=\Im f=f$, $\Im b=d$ and $\Im c=e$.(Note: We denote left-side by $L(\zeta,\varepsilon,\xi)$ and right-side by $R(\zeta,\varepsilon,\xi)$ of inequality (5)) We have

$$\begin{aligned}&L(a,d,e)=L(a,e,d)=L(d,a,e)=L(a,d,f)=L(a,f,d)=L(d,a,f)\\&=L(a,e,f)=L(a,f,e)=L(e,a,f)=L(d,e,f)=L(d,f,e)=L(e,d,f)=0,\end{aligned}$$

$$\begin{aligned}&L(a,d,b)=L(b,a,d)=L(a,e,b)=L(b,a,e)=L(a,f,b)=L(b,a,f)\\&=L(a,d,c)=L(c,a,d)=L(a,e,c)=L(c,a,e)=L(a,f,c)=L(c,a,f)\\&=L(d,b,e)=L(b,e,d)=L(b,d,f)=L(b,f,d)=L(b,e,f)=L(b,f,e)\\&=L(c,d,e)=L(c,e,d)=L(c,d,f)=L(c,f,d)=L(c,e,f)=L(c,f,e)=1,\end{aligned}$$

$$L(a,b,c)=L(a,c,b)=L(b,a,c)=L(a,b,d)=L(a,b,e)=L(a,b,f)$$

$$= L(a,c,d) = L(a,c,e) = L(a,c,f) = L(b,c,d) = L(b,d,c) = L(c,b,d)$$
$$= L(b,c,e) = L(b,e,c) = L(c,b,e) = L(b,c,f) = L(b,f,c) = L(c,b,f)$$
$$= L(d,b,e) = L(d,b,f) = L(e,b,f) = L(d,c,e) = L(d,c,f) = L(e,c,f) = 2.$$

And

$$R(a,b,c) = R(a,c,b) = R(b,a,c) = R(b,a,d) = R(b,a,e)$$
$$= R(c,a,d) = R(c,a,e) = R(d,a,e) = 8,$$

$$R(a,b,e) = R(a,e,b) = R(a,b,f) = R(f,a,b) = R(a,c,d)$$
$$= R(a,d,c) = R(a,c,f) = R(c,a,f) = R(d,a,f) = R(b,c,d) = R(b,c,e)$$
$$= R(b,c,f) = R(c,b,f) = R(e,a,f) = 7,$$

$$R(a,b,d) = R(a,d,b) = R(a,f,b) = R(a,e,c) = R(a,f,c) = R(a,d,e)$$
$$= R(a,e,d) = R(c,b,d) = R(b,c,e) = R(b,f,c) = R(e,b,f) = R(d,c,f) = 6,$$

$$R(b,d,c) = R(b,e,c) = R(b,e,d) = R(d,b,e) = R(d,b,f)$$
$$= R(d,c,e) = R(e,c,f) = 5,$$

$$R(a,f,d) = R(a,e,f) = R(a,f,e) = R(b,d,e) = R(b,f,d) = R(b,e,f)$$
$$= R(b,f,e) = R(c,d,e) = R(d,e,c) = R(c,d,f) = R(c,f,d) = R(c,f,e) = 4,$$

$$R(b,d,f) = R(c,e,f) = R(d,e,f) = R(e,d,f) = 3, \text{and} d(d,f,e) = 2.$$

We note that

$$L(\zeta,\varepsilon,\xi) \leq \frac{2}{5} R(\zeta,\varepsilon,\xi),$$

for all pairwise distinct points $\zeta,\varepsilon,\xi \in \mho$. Therefore, $\Im$ is a Paired-Chatterjea type mapping.

Since

$$\mathsf{d}(\Im b, \Im d) = \mathsf{d}(b,d) = 1,$$

and

$$\mathsf{d}(b, \Im d) + \mathsf{d}(d, \Im b) = \mathsf{d}(b,f) + \mathsf{d}(d,d) = 2,$$

it does not satisfy (3) for any $0 \leq \eta < \frac{1}{2}$, so $\Im$ is not a Chatterjea type mapping.

Also

$$\mathsf{d}(\Im a, \Im b) + \mathsf{d}(\Im b, \Im c) = \mathsf{d}(f, d) + \mathsf{d}(d, e) = 2,$$

and

$$\mathsf{d}(a, b) + \mathsf{d}(b, c) = 2,$$

it does not satisfy (4) for any $0 \le \lambda < 1$, so $\Im$ is not a Paired contraction mapping.

We observe that $\Im$ possesses a unique *FP*.

EXAMPLE 3. Let $\mho = R$, $\partial(\zeta, \varepsilon) = |\zeta - \varepsilon|$ and $\Im : \mho \to \mho$ defined as

$$\Im\zeta = \begin{cases} 0, & if\ \zeta < \frac{5}{2}, \\ 1, & if\ \zeta \ge \frac{5}{2}. \end{cases}$$

To prove a Paired-Chatterjea type mapping, we will discuss all possible cases. When ζ, ε, ξ are pairwise distinct and $\zeta < \frac{5}{2}, \varepsilon < \frac{5}{2}, \xi < \frac{5}{2}$ or $\zeta \ge \frac{5}{2}, \varepsilon \ge \frac{5}{2}, \xi \ge \frac{5}{2}$, then inequality (5) satisfy trivially as $L(\zeta, \varepsilon, \xi) = 0$. The other possible cases are as follows:

Case 1: $\zeta, \varepsilon \ge \frac{5}{2}, \xi < \frac{5}{2}$,

$$L(\zeta, \varepsilon, \xi) = \partial(1, 1) + \partial(1, 0) = 1,$$

$$R(\zeta, \varepsilon, \xi) = \partial(\zeta, 1) + \partial(\varepsilon, 1) + \partial(\varepsilon, 0) + \partial(\xi, 1) = |\zeta - 1| + \\ |\varepsilon - 1| + |\varepsilon| + |\xi - 1| = \zeta + 2\varepsilon - 2 + |\xi - 1| \ge \frac{11}{2}.$$

Case 2: $\zeta, \xi \ge \frac{5}{2}, \varepsilon < \frac{5}{2}$,

$$L(\zeta, \varepsilon, \xi) = \partial(1, 0) + \partial(0, 1) = 2,$$

$$R(\zeta, \varepsilon, \xi) = |\zeta| + |\varepsilon - 1| + |\varepsilon - 1| + |\xi| = \zeta + \xi + 2|\varepsilon - 1| \ge 5.$$

Case 3: $\varepsilon, \xi \ge \frac{5}{2}, \zeta < \frac{5}{2}$,

$$L(\zeta, \varepsilon, \xi) = \partial(0, 1) + \partial(1, 1) = 1,$$

$$R(\zeta, \varepsilon, \xi) = |\zeta - 1| + |\varepsilon| + |\varepsilon - 1| + |\xi - 1| = |\zeta - 1| + 2\varepsilon + \xi - 2 \ge \frac{11}{2}.$$

Case 4: $\zeta \geq \frac{5}{2}, \varepsilon, \xi < \frac{5}{2}$,

$$L(\zeta,\varepsilon,\xi) = \partial(1,0) + \partial(0,0) = 1,$$

$$R(\zeta,\varepsilon,\xi) = |\zeta| + |\varepsilon - 1| + |\varepsilon| + |\xi| = \zeta + |\varepsilon| + |\varepsilon - 1| + |\xi| \geq \frac{7}{2}.$$

Case 5: $\varepsilon \geq \frac{5}{2}, \zeta, \xi < \frac{5}{2}$,

$$L(\zeta,\varepsilon,\xi) = \partial(0,1) + \partial(1,0) = 2,$$

$$R(\zeta,\varepsilon,\xi) = |\zeta - 1| + |\varepsilon| + |\varepsilon| + |\xi - 1| = |\zeta - 1| + 2\varepsilon| + |\xi - 1| \geq 5.$$

Case 6: $\xi \geq \frac{5}{2}, \zeta, \varepsilon < \frac{5}{2}$,

$$L(\zeta,\varepsilon,\xi) = \partial(0,0) + \partial(0,1) = 1,$$

$$R(\zeta,\varepsilon,\xi) = |\zeta| + |\varepsilon| + |\varepsilon - 1| + |\xi| = |\zeta| + |\varepsilon| + |\varepsilon - 1| + \xi \geq \frac{7}{2}.$$

It can be seen, in all cases, that

$$L(\zeta,\varepsilon,\xi) \leq \frac{2}{5} R(\zeta,\varepsilon,\xi).$$

Hence, $\Im$ is a Paired-Chatterjea type mapping. We observe that $\Im$ possesses a unique *FP*.

Now, for $\zeta = 1.9, \varepsilon = 2, \xi = 2.1$, we have

$$\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) = \partial(0,1) + \partial(1,1) = 1,$$

and

$$\partial(\zeta,\varepsilon) + \partial(\varepsilon,\xi) = \partial(1.9,2) + \partial(2,2.1) = 0.1 + 0.1 = 0.2,$$

so, we get that $\Im$ is not a Paired contraction mapping.

Also, for $\zeta = 2, \varepsilon = 3$,

$$\partial(\Im\zeta, \Im\varepsilon) = \partial(0, \frac{5}{2}) = \frac{5}{2},$$

and

$$\partial(\zeta, \Im\zeta) + \partial(\varepsilon, \Im\varepsilon) = \partial(2,0) + \partial(3, \frac{5}{2}) = \frac{5}{2},$$

hence, $\Im$ is not Kannan type mapping.

Moreover,

$$\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) = \partial(2, \frac{5}{2}) + \partial(3,0) = \frac{7}{2},$$

also, we get that $\Im$ is not a Chatterjea type mapping.

We observe that $\Im$ is a discontinuous mapping.

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Results on Paired Chatterjea-style mappings within incomplete metric spaces

In the theorem corresponding to Theorem 1 in (Kannan, 1969) we presented here, exclude the completeness requirement for the metric space and includ two assumptions, denoted as (PC_1) and (PC_2):

THEOREM 4. *Let $(\mho, \partial)$ be a complete metric space, where $|\mho| \geq 3$, and consider the mapping $\Im : \mho \to \mho$ that is a Paired-Chatterjea contraction mapping and has no periodic elements of prime period 2. If $\Im$ holds following two assumptions:*

(PC_1) $\Im$ is continuous at $\zeta^* \in \mho$.

(PC_2) There exists $\zeta_0 \in \mho$ such that the sequence $\{\zeta_n\}$, where $\zeta_n = \Im\zeta_{n-1}$ for $n = 1, 2, ...$, has a convergent subsequence $\{\zeta_{n_k}\}$ to ζ^*.

Then ζ^* is a FP of $\Im$. Moreover, $\Im$ exhibit no more than two FPs.

Proof. Given that $\Im$ is continuous at ζ^* and $\zeta_{n_k} \to \zeta^*$, it follows that $\Im\zeta_{n_k} = \zeta_{n_k+1} \to \Im\zeta^*$. Note that while ζ_{n_k+1} is a subsequence of ζ_n, it is not necessarily a subsequence of ζ_{n_k}. Let us consider, for the sake of contradiction, that $\zeta^* \neq \Im\zeta^*$. Let us examine two balls:

$$B_1 = B_1(\zeta^*, r) \text{ and } B_2 = B_2(\Im\zeta^*, r),$$

where $r < \frac{1}{3}\partial(\zeta^*, \Im\zeta^*)$. Therefore, a positive integer N exists such that, for each $i > N$, we have

$$\zeta_{n_i} \in B_1 \text{ and } \zeta_{n_i+1} \in B_2,$$

which implies that,

$$\partial(\zeta_{n_i}, \zeta_{n_i+1}) > r \text{ for } i > N. \tag{20}$$

If the mapping $\Im$ has no fixed points in the sequence $\{\zeta_n\}$, then we can apply the arguments presented in Theorem 3. For $n = 3, 4, ...$, by (17), we get

$$\partial(\zeta_n, \zeta_{n+1}) \leq \alpha^n K_0,$$

where $K_0 = \partial(\zeta_0, \zeta_1) + \partial(\zeta_1, \zeta_2)$ and $\alpha = \frac{\lambda}{1-\lambda} \in [0, 1)$. Therefore,

$$\partial(\zeta_{n_i}, \zeta_{n_i+1}) \leq \alpha^{n_i} K_0.$$

Which tends to 0 as $i \to \infty$, which leads to a contradiction to (20). Therefore, it follows that $\Im\zeta^* = \zeta^*$.

From the final paragraph of Theorem 3, it can be proven there exist at most two fixed points. □

In the next theorem corresponding to Theorem 2 in (Kannan, 1969), we consider $\Im$ as a Paired-Chatterjea type mapping defined on an everywhere dense subset of $\mho$. The mapping $\Im$ is continuous on $\mho$, although this continuity is not necessarily restricted to the point ζ^*.

THEOREM 5. *Let $(\mho, \partial)$ be a metric space with $|\mho| \geq 3$, and consider a mapping $\Im : \mho \to \mho$ that has no periodic point with prime period 2 and is a Paired-Chatterjea type mapping on $(\mathfrak{M}, \partial)$, where $\mathfrak{M}$ is an everywhere dense subset of $\mho$. If $\Im$ holds following two assumptions:*

(PC_1) $\Im$ is continuous at $\zeta^* \in \mho$.

(PC_2) There exists $\zeta_0 \in \mho$ such that the sequence $\{\zeta_n\}$, where $\zeta_n = \Im\zeta_{n-1}$ for $n = 1, 2, ...,$ has a convergent subsequence $\{\zeta_{n_k}\}$ to ζ^*.

Then ζ^* is a FP of $\Im$. Moreover, $\Im$ exhibits at most two FPs.

Proof. The proof will be based on Theorem 4, provided that $\Im$ is demonstrated to be a Paired-Chatterjea type mapping on $\mho$. Consider three pairwise distinct points ζ, ε, ξ from $\mho$. To demonstrate that $\Im$ is a Paired-Chatterjea type mapping, we will examine three distinct cases:

Case 1. $\zeta, \varepsilon \in \mathfrak{M}$ and $\xi \in \mho/\mathfrak{M}$;

Since $\mathfrak{M}$ is an everywhere dense subset of $\mho$, we can find a sequence $\{\xi_n\} \subset \mathfrak{M}$ such that $\xi_n \to \xi$, $\xi_n \neq \zeta$, $\xi_n \neq \varepsilon$ for every n, and $\xi_i \neq \xi_j$ for $i \neq j$. Consequently,

$$\begin{aligned}\partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi) &\leq \partial(\Im\zeta, \Im\varepsilon) + \partial(\Im\varepsilon, \Im\xi_n) + \partial(\Im\xi_n, \Im\xi)\\ &\text{(Using triangle inequality)}\\ &\leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\xi_n, \Im\varepsilon) + \partial(\varepsilon, \Im\xi_n)) + \partial(\Im\xi_n, \Im\xi)\\ &\leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\xi, \Im\varepsilon) + \partial(\xi, \xi_n) + \partial(\varepsilon, \Im\xi) + \partial(\Im\xi, \Im\xi_n))\\ &\quad + \partial(\Im\xi_n, \Im\xi)\\ &\leq \lambda(\partial(\zeta, \Im\varepsilon) + \partial(\varepsilon, \Im\zeta) + \partial(\xi_n, \Im\varepsilon) + \partial(\varepsilon, \Im\xi))\\ &\quad + \lambda\partial(\xi_n, \xi) + (1 + \lambda)\partial(\Im\xi_n, \Im\xi).\end{aligned}$$

As $n \to \infty$, we have $\partial(\xi_n, \xi) \to 0$ and $\partial(\Im\xi_n, \Im\xi) \to 0$. Therefore, inequality (5) is established.

Case 2. $\zeta \in \mathfrak{M}$ and $\varepsilon, \xi \in \mho/\mathfrak{M}$;

Let $\{\varepsilon_n\}$, $\{\xi_n\}$ be sequences in $\mathfrak{M}$ such that $\varepsilon_n \to \varepsilon$ and $\xi_n \to \xi$. (Here and in the next case, we will consider the points ζ, ε, ξ and all points of the sequences including the limit points of the sequences are pairwise distinct.) Then, by applying the triangle inequality and (5) on $\mathfrak{M}$, we obtain

$$\begin{gathered}\partial(\Im\zeta,\Im\varepsilon)+\partial(\Im\varepsilon,\Im\xi)\leq\partial(\Im\zeta,\Im\varepsilon_n)+\partial(\Im\varepsilon_n,\Im\varepsilon)+\partial(\Im\varepsilon,\Im\varepsilon_n)+\\ \partial(\Im\varepsilon_n,\Im\xi_n)+\partial(\Im\xi_n,\Im\xi)=(\partial(\Im\zeta,\Im\varepsilon_n)+\partial(\Im\varepsilon_n,\Im\xi_n))+\\ 2\partial(\Im\varepsilon_n,\Im\varepsilon)+\partial(\Im\xi_n,\Im\xi)\\ \leq\lambda(\partial(\zeta,\Im\varepsilon_n)+\partial(\varepsilon_n,\Im\zeta)+\partial(\varepsilon_n,\Im\xi_n)+\partial(\xi_n,\Im\varepsilon_n))\\ +2\partial(\Im\varepsilon_n,\Im\varepsilon)+\partial(\Im\xi_n,\Im\xi)\\ \leq\lambda(\partial(\zeta,\Im\varepsilon)+\partial(\varepsilon,\Im\zeta)+\partial(\varepsilon,\Im\xi)+\partial(\xi,\Im\zeta))+2\lambda\partial(\varepsilon_n,\varepsilon)+\lambda\partial(\xi_n,\xi)\\ +2(1+\lambda)\partial(\Im\varepsilon_n,\Im\varepsilon)+(1+\lambda)\partial(\Im\xi_n,\Im\xi).\end{gathered}$$

Once again, inequality (5) is obtained by letting $n\to\infty$ as $\partial(\varepsilon_n,\varepsilon)\to 0$, $\partial(\xi_n,\xi)\to 0$, $\partial(\Im\varepsilon_n,\Im\varepsilon)\to 0$ and $\partial(\Im\xi_n,\Im\xi)\to 0$.

Case 3. $\zeta,\varepsilon,\xi\in\mho/\mathfrak{M}$, and let $\{\zeta_n\}$, $\{\varepsilon_n\}$ and $\{\xi_n\}$ are sequences in $\mathfrak{M}$ such that $\zeta_n\to\zeta$, $\varepsilon_n\to\varepsilon$ and $\xi_n\to\xi$. Consequently,

$$\begin{gathered}\partial(\Im\zeta,\Im\varepsilon)+\partial(\Im\varepsilon,\Im\xi)\\ \leq\partial(\Im\zeta,\Im\zeta_n)+\partial(\Im\zeta_n,\Im\varepsilon_n)+\partial(\Im\varepsilon_n,\Im\varepsilon)+\partial(\Im\varepsilon,\Im\varepsilon_n)\\ +\partial(\Im\varepsilon_n,\Im\xi_n)+\partial(\Im\xi_n,\Im\xi)=(\partial(\Im\zeta_n,\Im\varepsilon_n)+\partial(\Im\varepsilon_n,\Im\xi_n))\\ +\partial(\Im\zeta,\Im\zeta_n)+2\partial(\Im\varepsilon,\Im\varepsilon_n)+\partial(\Im\xi,\Im\xi_n)\\ \leq\lambda(\partial(\zeta_n,\Im\varepsilon_n)+\partial(\varepsilon_n,\Im\zeta_n)+\partial(\varepsilon_n,\Im\xi_n)+\partial(\xi_n,\Im\varepsilon_n))\\ +\partial(\Im\zeta,\Im\zeta_n)+2\partial(\Im\varepsilon,\Im\varepsilon_n)+\partial(\Im\xi,\Im\xi_n)\\ \leq\lambda(\partial(\zeta,\Im\varepsilon)+\partial(\varepsilon,\Im\zeta)+\partial(\varepsilon,\Im\xi)+\partial(\xi,\Im\varepsilon))\\ +\lambda(\partial(\zeta_n,\zeta)+2\partial(\varepsilon_n,\varepsilon)+\partial(\xi_n,\xi))\\ +(1+\lambda)(\partial(\Im\zeta,\Im\zeta_n)+2\partial(\Im\varepsilon,\Im\varepsilon_n)+\partial(\Im\xi,\Im\xi_n)).\end{gathered}$$

Again, allowing $n\to\infty$, inequality (5) is obtained as $\partial(\zeta_n,\zeta)\to 0$, $\partial(\varepsilon_n,\varepsilon)\to 0$, $\partial(\xi_n,\xi)\to 0$, $\partial(\Im\zeta_n,\Im\zeta)\to 0$, $\partial(\Im\varepsilon_n,\Im\varepsilon)\to 0$ and $\partial(\Im\xi_n,\Im\xi)\to 0$.

Hence, $\Im$ is a Paired-Chatterjea type mapping on $\mho$. Therefore, mapping $\Im$ satisfies all assumptions of Theorem 4, which completes the proof. □

EXAMPLE 4. Let $\mho = [0,1)$ equipped with standard metric $\partial(\zeta,\varepsilon) = |\zeta - \varepsilon|$. It is an incomplete metric space. Let a self map $\Im$ be defined on $\mho$ as:

$$\Im\zeta = \begin{cases} \frac{\zeta}{4}, & if\ \zeta \in \mathbb{Q} \cap [0,1); \\ \\ \frac{\zeta}{2}, & if\ \zeta \in \mathbb{Q}^c \cap [0,1). \end{cases}$$

Here, it is clear that $\Im$ is a discontinuous map over $\mho = [0,1)$ except at $\zeta = 0$. Consequently, the Banach contraction principle is not applicable. Let us consider a nonempty subset $\mathfrak{M} = \mathbb{Q} \cap [0,1)$ of $\mho$ which is everywhere dense in $\mho$. For any three points $\zeta, \varepsilon, \xi \in \mathfrak{M}$, we have

$$\partial(\zeta,\Im\varepsilon) + \partial(\varepsilon,\Im\zeta) + \partial(\varepsilon,\Im\xi) + \partial(\xi,\Im\varepsilon) = \partial(\zeta,\frac{\varepsilon}{4}) + \partial(\varepsilon,\frac{\zeta}{4}) + \partial(\varepsilon,\frac{\xi}{4}) + \partial(\xi,\frac{\varepsilon}{4}) = |\zeta - \frac{\varepsilon}{4}| + |\varepsilon - \frac{\zeta}{4}| + |\varepsilon - \frac{\xi}{4}| + |\xi - \frac{\varepsilon}{4}| \tag{21}$$

and

$$\partial(\Im\zeta,\Im\varepsilon) + \partial(\Im\varepsilon,\Im\xi) = \partial(\frac{\zeta}{4},\frac{\varepsilon}{4}) + \partial(\frac{\varepsilon}{4},\frac{\xi}{4}) = |\frac{\zeta}{4} - \frac{\varepsilon}{4}| + |\frac{\varepsilon}{4} - \frac{\xi}{4}|. \tag{22}$$

Now,

$$|\frac{\zeta}{4} - \frac{\varepsilon}{4}| + |\frac{\varepsilon}{4} - \frac{\xi}{4}| = \frac{1}{4}|\zeta - \frac{\zeta}{4} + \frac{\zeta}{4} - \frac{\varepsilon}{4} + \frac{\varepsilon}{4} - \varepsilon| + \frac{1}{4}|\varepsilon - \frac{\varepsilon}{4} + \frac{\varepsilon}{4} - \frac{\xi}{4} + \frac{\xi}{4} - \xi|$$
$$\leq \frac{1}{4}\left\{|\zeta - \frac{\varepsilon}{4}| + |\frac{\zeta}{4} - \frac{\varepsilon}{4}| + |\varepsilon - \frac{\zeta}{4}|\right\} + \frac{1}{4}\left\{|\varepsilon - \frac{\xi}{4}| + |\frac{\varepsilon}{4} - \frac{\xi}{4}| + |\xi - \frac{\varepsilon}{4}|\right\}$$

$$\implies \frac{3}{4}\left\{|\frac{\zeta}{4} - \frac{\varepsilon}{4}| + |\frac{\varepsilon}{4} - \frac{\xi}{4}|\right\} \leq \frac{1}{4}\left\{|\zeta - \frac{\varepsilon}{4}| + |\varepsilon - \frac{\zeta}{4}| + |\varepsilon - \frac{\xi}{4}| + |\xi - \frac{\varepsilon}{4}|\right\}$$

$$\implies |\frac{\zeta}{4} - \frac{\varepsilon}{4}| + |\frac{\varepsilon}{4} - \frac{\xi}{4}| \leq \frac{1}{3}\left\{|\zeta - \frac{\varepsilon}{4}| + |\varepsilon - \frac{\zeta}{4}| + |\varepsilon - \frac{\xi}{4}| + |\xi - \frac{\varepsilon}{4}|\right\}. \tag{23}$$

Therefore, from (21), (22) and (23), we can conclude that $\Im$ is a Paired-Chatterjea type mapping with the parameter $\lambda = \frac{1}{3} \in [0, \frac{1}{2})$, on $(\mathfrak{M}, \partial)$.

Hence, it can be easily verified that the mapping $\Im$ satisfies all the assumptions of Theorem 5 with $\zeta^* = 0$, which implies that there exists a fixed point of the mapping $\Im$ on $\mho$. In this example, "0" is the only fixed point of the mapping $\Im$.

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Contracciones de tipo Paired-Chatterjea sincronizada: nuevos resultados de punto fijo y propiedades de continuidad en espacios métricos

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CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: El artículo trata sobre las aplicaciones de contracciones de tipo Paired-Chatterjea como una extensión de las contracciones de tipo Chatterjea tradicionales que operan en tres puntos en lugar de dos, en el marco de espacios métricos estándar.

Métodos: Se emplea el concepto de Mapeo de tipos de Chatterjea sincronizado en un espacio métrico en tres puntos en lugar de dos utilizando la idea de Mapeo de tipos de Chatterjea sincronizados.

Resultados: Se ha discutido una serie de propiedades correspondientes. Además, se establece que los mapeos de tipo de Chatterjea sincronizada constituyen una clase distinta al de mapeo de tipos de Chatterjea tradicionales y obtienen al menos un punto fijo en ausencia de puntos periódicos de periodo primo 2 dentro de espacios métricos completos. También se demuestra cómo criterios adicionales a estas aplicaciones, como la continuidad y la regularidad asintótica, amplían el alcance de los resultados de punto fijo. Más allá de las contribuciones fundamentales de Chatterjea, se establecen dos resultados adicionales de punto fijo aplicables a las aplicaciones de mapeo de tipos

de Chatterjea sincronizada en espacios métricos, incluso en escenarios donde no se requiere la completitud.

Conclusión: Los mapeos de tipos de Chatterjea sincronizada son generalmente discontinuas; presentan continuidad en puntos fijos, similar a las aplicaciones de tipo Kannan y Chatterjea. En ausencia de un punto periódico de periodo primo 2, estas aplicaciones tienen un punto fijo dentro del espacio métrico completo.

Palabras claves: espacio métrico, punto fijo, aplicaciones de tipo Chatterjea, contracción sincronizada, Mapeo de tipos de Chatterjea sincronizada.

Парное сжатие типа Чаттерджи: новые результаты с неподвижной точкой и свойства непрерывности в метрических пространствах

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ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: В статье рассматриваются парные отображения Чаттерджи, которые являются расширением традиционных сжимающих отображений типа Чаттерджи и оперирующие тремя точками вместо двух в рамках стандартных метрических пространств.

Методы: Сжимающие отображения типа Чаттерджи используются в метрическом пространстве в трех точках вместо двух, основываясь на идее о парных сжимающих отображениях.

Результаты: В статье также представлен ряд свойств соответствующих отображений. Установлено, что парные отображения Чаттерджи образуют особый класс по сравнению с традиционными отображениями типа Чаттерджи, которые обладают хотя бы одной неподвижной точкой при отсутствии периодических точек простого периода 2 в рамках полных метрических пространств. В ходе исследования было выявлено, что дополнительные критерии для этих отображений, такие как непрерывность и асимптотическая регулярность, расширяют диапазон результатов с фиксированной точкой. Расширяя вклад Чаттерджи, были получены два дополнительных результата с неподвижной точкой, применимые к парным отображениям Чаттерджи даже в неполных метрических пространствах.

Выводы: Парные отображения типа Чаттерджи, как правило, прерывисты. Они демонстрируют непрерывность в неподвижных точках, аналогичных отображениям типа Каннан и Чаттерджи. При отсутствии периодической точки с простым периодом 2, неподвижная точка этих отображений располагается внутри полного метрического пространства.

Ключевые слова: метрическое пространство, неподвижная точка, отображения типа Чаттерджи, парное сжатие, парные отображения Чаттерджи.

Врста упарених пресликавања типа Chatterjee: нови резултати фиксне тачке и својства континуитета у метричким просторима

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Сажетак:

Увод/циљ: У раду је разматрано увођење упарених пресликавања типа Chatterjee која представљају проширење традиционалних контрактивних пресликавања овог типа, а која делују на три тачке уместо на две, у оквиру стандардних метричких простора.

Методе: Контрактивна пресликавања типа Chatterjee користе се у метричком простору на три тачке, уместо на две применом идеје упарених контрактивних пресликавања.

Резултати: Размотрен је низ својстава предметних пресликавања. Установљено је да упарена пресликавања типа Chatterjee чине посебну класу у односу на традиционална пресликавања овог типа која поседују најмање једну фиксну тачку у одсуству периодичних тачака простог периода 2 унутар комплетних метричких простора. Такође, показано је да додатни критеријуми за ова пресликавања, као што су непрекидност и асимптотска регуларност, проширују опсег резултата фиксне тачке. Проширујући доприносе Chatterjeea, успостављена су два додатна резултата фиксне тачке примењива на упарена пресликавања типа Chatterjee у метричким просторима, чак и у ситуацијама где није потребна комплетност.

Закључак: Упарена пресликавања типа Chatterjee која су најчешће дисконтинуална, показују непрекидност у фиксним тачкама слично пресликавањима типа Kannan и Chatterjee. У одсуству периодичне тачке простог периода 2, ова пресликавања имају фиксну тачку унутар комплетног метричког простора.

Кључне речи: метрички простор, фиксна тачка, пресликавања типа Chatterjee, упарена контракција, упарена пресликавања типа Chatterjee.

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