Fixed-circle and fixed-disc problems in metric spaces

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Abstract:

Introduction/purpose: The purpose of this paper is to establish existence theorems for fixed circles and fixed discs in metric spaces using different types of contractive conditions. By considering self-mappings on metric spaces, classical fixed point results are extended to these geometric fixed structures. Several examples are provided to illustrate and validate the theoretical results.
Methods: Self-mappings defined on metric spaces are considered and various types of contractive conditions introduced. Analytical techniques from fixed point theory are used to derive sufficient conditions for the existence of ϕ-fixed circles and ϕ-fixed discs. The theoretical results are supported by carefully constructed examples that satisfy the proposed contractions and demonstrate the applicability of the obtained theorems.
Results: The study successfully establishes ϕ-fixed circles and ϕ-fixed disc results for Caristi-type contractions and another class of contractions within the framework of metric spaces. Additionally, supportive examples are provided.
Conclusion: This paper establishes new existence theorems for ϕ-fixed circles and ϕ-fixed discs in metric spaces using Caristi-type and related contractive conditions. These results extend classical fixed point theory beyond single fixed points to broader geometric fixed structures, thereby enriching the theory of metric fixed points. The provided examples demonstrate the applicability and effectiveness of the proposed results and indicate their potential for further generalizations.

Key words: Fixed circle, fixed disc, metric space, Caristi contraction

Introduction

The study of fixed point theory has been a fundamental area in mathematical analysis, with applications spanning across various branches of mathematics. In metric spaces, the investigation of fixed points has led to numerous significant results, beginning with Banach's contraction principle in 1922. "While traditional fixed point theory focuses on single points, recent developments have expanded to include the study of fixed sets, particularly fixed circles and fixed discs. A fixed circle of a self-mapping T on a metric space X is a circle C such that $T(C) = C$. Similarly, a fixed disc is a disc D where $T(D) = D$." Several researchers have contributed to this area. Özgür, & Taş (2019) established the first results on fixed circles using Caristi-type contractions in metric spaces. Later, Özgür, & Taş (2021) extended these results to more general spaces. Recent work by [Mlaiki et al. (2023), Taş (2018)] has introduced new techniques for studying fixed disc properties. However, the existence of fixed circles and fixed discs under certain types of contractive conditions remains unexplored. Moreover, the relationship between different contractive conditions and their impact on the existence of fixed sets needs further investigation. The fixed-circle problem was also studied in the setting of S-metric spaces in [Özgür et al. (2017), Özgür et al. (2018)]. In recent years, the fixed-disc problem have been studied with this perspective on metric and some generalized metric spaces (see [Özgür (2019), Taş et al. (2021)] for more details). This paper presents new existence theorems for fixed circles and fixed discs using various contractive conditions. The obtained results extend previous work by considering [Özgür, & Taş (2021), Taş (2018)]. Examples are also provided demonstrating that the conditions given here are optimal. This paper is organized as follows: Section 1 presents preliminary and definitions. Section 2 establishes the main results on fixed circles, which contain theorems related to fixed discs, and provides illustrative examples. The definition of the fixed circle has been generalized by replacing the radius r of the circle with the function $\phi(r)$ where $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. The results have been also generalized by using a Caristi-type contraction in metric spaces, and Section 3 concludes the paper with some remarks and future directions.

Definition 1. Özgür, & Taş (2019) *Let (X,d) be a metric space and $C_{x_0,r} = \{x \in X : d(x_0,x) = r\}$ be a circle, for a self-mapping $T : X \to X$, if $Tx = x$ for every $x \in C_{x_0,r}$ then the circle is a fixed circle of T.*

Theorem 1. Özgür, & Taş (2019) *Let (X,d) be a metric space and $C_{x_0,r}$ be any circle on X. Let us define the mapping $\psi : X \to [0,\infty)$*

$$\psi(x) = d(x,x_0) \tag{1}$$

for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying

1. $d(x,Tx) \le \psi(x) - \psi(Tx)$
2. $d(Tx,x_0) \ge r$

for each $x \in C_{x_0,r}$ then the circle $C_{x_0,r}$ is a fixed circle of T.

Theorem 2. Özgür, & Taş (2019) *Let (X,d) be a metric space and $C_{x_0,r}$ be any circle on X. Let the mapping ψ be defined as equation (1) for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying*

(1*) $d(x,Tx) \le \psi(x) + \psi(Tx) - 2r$

(2*) $d(Tx,x_0) \le r$

for each $x \in C_{x_0,r}$ then the circle $C_{x_0,r}$ is a fixed circle of T.

Theorem 3. Özgür, & Taş (2019) *Let (X,d) be a metric space and $C_{x_0,r}$ be any circle on X. Let the mapping ψ be defined as equation (1) for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying:*

(1**) $d(x,Tx) \le \psi(x) - \psi(Tx)$

(2**) $hd(x,Tx) + d(Tx,x_0) \le r$

for each $x \in C_{x_0,r}$ then the circle $C_{x_0,r}$ is a fixed circle of T.

In 2019, Özgür (2019) defined a new contractive type mapping on metric spaces, and that includes fixed disc results via a simulation function on metric spaces.

Definition 2. Taş et al. (2021) *Let (X,d) be a metric space, $D_{x_0,r} = \{x \in X : d(x_0,x) \le r\}$ $(r \in \mathbb{R}^+ \cup \{0\})$ a disc and a self-mapping $T : X \to X$, if $Tx = x$ for every $x \in D_{x_0,r}$ then the disc is called a fixed disc of T.*

Definition 3. Özgür (2019) *Let $\zeta \in \mathcal{Z}$ be any simulation function. T is said to be a $\mathcal{Z}_c$-contraction with respect to ζ if there exists an $x_0 \in X$ such that*

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the following condition holds for all $x \in X$:

$$d(Tx, x) > 0 \Rightarrow \zeta(d(Tx, x), d(Tx, x_0)) \geq 0.$$

If T is a $\mathcal{Z}_c$-contraction with respect to ζ, then there exists

$$d(Tx, x) < d(Tx, x_0), \tag{2}$$

for all $x \in X$ with $Tx = x$. Indeed, if $Tx = x$ then inequality (2) is satisfied . If $Tx \neq x$ then $d(Tx, x) > 0$. By the definition of a $\mathcal{Z}_c$-contraction and the condition of ζ, we get

$$0 \geq \zeta(d(Tx, x), d(Tx, x_0)) < d(Tx, x_0) - d(Tx, x)$$

and so equation (2) is satisfied. In all fixed disc results, they use the number $r \in \mathbb{R}^+ \cup \{0\}$ defined by

$$r = \inf_{x \in X} \{d(x, Tx) | Tx \neq x\}.$$

THEOREM 4. Özgür (2019) *If T is a $\mathcal{Z}_c$-contraction with respect to ζ with $x_0 \in X$ and the condition $0 < d(Tx, x_0) \leq r$ holds for all $x \in D_{x_0,r} - \{x_0\}$ then the $D_{x_0,r}$ is a fixed disc of T.*

DEFINITION 4. Özgür (2019) *Let (X, d) be a metric space and $T : X \to X$ be a self-mapping and $\zeta \in \mathcal{Z}$. T is said to be a $\acute{C}iri\acute{c}$-type $\mathcal{Z}_c$-contraction with respect to ζ if there exists an $x_0 \in X$ such that the following condition holds for all $x \in X$;*

$$d(x, Tx) > 0 \Rightarrow \zeta(d(Tx, x), m^*(x, x_0)) \geq 0$$

where $m^*(x, x_0) = max\{d(x, x_0), d(x, Tx), d(x_0, Tx_0), \frac{d(x,Tx_0)+d(x_0,Tx)}{2}\}$.

THEOREM 5. Özgür (2019) *Let (X, d) be a metric space and $T : X \to X$ a $\acute{C}iri\acute{c}$-type $\mathcal{Z}_c$-contraction with respect to ζ with $x_0 \in X$, If the condition $0 < d(Tx, x_0) \leq r$ holds for all $x \in D_{x_0,r} - \{x_0\}$ then $D_{x_0,r}$ is a fixed disc of T.*

Main results

In this section, we have generalized the definition of the fixed circle by replacing the radius r of the circle with the function $\phi(r)$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$. We have also generalized theorem (1) by using the Caristi-type contraction in metric spaces.

DEFINITION 5. *Let* (X,d) *be a metric space and* $C_{x_0,\phi(r)} = \{x \in X : d(x_0,x) = \phi(r)\}$ *where* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a circle, for a self-mapping* $T : X \to X$*, if* $Tx = x$ *for every* $x \in C_{x_0,\phi(r)}$ *then the circle is a* ϕ*-fixed circle of* T*.*

THEOREM 6. *Let* (X,d) *be a metric space and* $C_{x_0,\phi(r)}$ *be any circle on* X*. Let us define the mapping* $\psi : X \to [0,+\infty)$

$$\psi(x) = d(x,x_0) \tag{3}$$

and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *for all* $x \in X$*. If there exists a self-mapping* $T : X \to X$ *satisfying*

1. $d(x,Tx) \leq \psi(x) - \psi(Tx)$
2. $d(Tx,x_0) \geq \phi(r)$

for each $x \in C_{x_0,\phi(r)}$ *then the circle* $C_{x_0,\phi(r)}$ *is a* ϕ*-fixed circle of* T*.*

Proof. Let us assume that the mapping ψ is defined by $\psi : X \to [0,+\infty)$ and $\psi(x) = d(x,x_0)$. Let $x \in C_{x_0,\phi(r)}$ be any arbitrary point. We show that $Tx = x$, whenever $x \in C_{x_0,\phi(r)}$ using condition (1)

$$\begin{aligned} d(x,Tx) \leq & \ \psi(x) - \psi(Tx) \\ = & \ d(x,x_0) - d(Tx,x_0) \\ = & \phi(r) - d(Tx,x_0) \end{aligned} \tag{4}$$

because of condition(2), the point Tx should be lying on the exterior of the circle $C_{x_0,\phi(r)}$. Then there are two cases. If $d(Tx,x_0) > \phi(r)$ then using (4) is a contradiction. Now therefore it should be $d(Tx,x_0) = \phi(r)$. In this case, by using (4) we get

$$\begin{aligned} d(x,Tx) &\leq \phi(r) - \phi(r) \\ &= 0. \end{aligned}$$

Hence, we obtain $Tx = x$ for all $x \in C_{x_0,\phi(r)}$. Consequently, $C_{x_0,\phi(r)}$ is a ϕ-fixed circle of T. □

EXAMPLE 1. Let $X = [\frac{1}{2},\infty]$, (X,d) be a metric space and let us consider a circle $C_{x_0,\phi(r)}$ and define the mapping $T : X \to X$

$$T(x) = \begin{cases} x; & x \in C_{x_0,\phi(r)} \\ 2x; & otherwise \end{cases}$$

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and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by

$$\phi(r) = \frac{1}{r}.$$

Solution: If $x \in C_{x_0,\phi(r)}$ then $Tx = x$.

1. $d(x, Tx) \leq \psi(x) - \psi(Tx)$,

$$\begin{aligned} d(x,Tx) &\leq \psi(x) - \psi(Tx) \\ d(x,Tx) &= d(x,x) = 0 \\ \psi(x) - \psi(Tx) &= d(x,x_0) - d(Tx,x_0) \\ &= d(x,x_0) - d(x,x_0) \\ &= 0. \end{aligned}$$

2. $d(Tx, x_0) = d(x, x_0) = \phi(r)$.

holds for each $x \in X$ and for all $r \geq 1$.
Then it can be easily seen that conditions (1) and (2) are satisfied. Clearly $C_{x_0,\phi(r)}$ is a ϕ-fixed circle of T.

EXAMPLE 2. Let $X = \mathbb{R}^+$ and (X, d) be a metric space and let us consider a circle $C_{x_0,\phi(r)}$ and let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ define as

$$\phi(r) = \begin{cases} \frac{1}{r}; & \text{if } r \geq 2 \\ \frac{1}{5}; & \text{otherwise} \end{cases}$$

and define the mapping $T : X \rightarrow X$ as

$$Tx = \begin{cases} \frac{1}{2}; & \text{if } x \in C_{x_0,\phi(r)} \\ 5; & otherwise \end{cases}$$

if $x \in C_{x_0,\phi(r)}$ then $Tx = \frac{1}{2}$ and if $r \geq 2$ and $x_0 = 0$ then

1. $d(x, Tx) \leq \psi(x) - \psi(Tx)$

$$\begin{aligned} d(x,Tx) &\leq \psi(x) - \psi(Tx) \\ d(x,\frac{1}{2}) &\leq d(x,0) - d(Tx,0) \\ &= x - \frac{1}{2} \end{aligned}$$

2. $d(Tx, x_0) = d(\frac{1}{2}, 0) = \frac{1}{2} \geq \frac{1}{r}$.

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for $x_0 = 0$ and for each $x \in C_{x_0,\frac{1}{r}}$, hence T satisfies conditions (1) and (2). Clearly $C_{x_0,\frac{1}{r}}$ is a ϕ-fixed circle of T. But it does not satisfy condition (2) in theorem (1).

EXAMPLE 3. Let $X = [0,2]$ and (X,d) be a metric space and let us consider a circle $C_{x_0,\phi(r)}$ and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$\phi(r) = \begin{cases} \frac{1}{2r-1}; & \text{if } r \geq 1 \\ \frac{1}{3}; & \text{otherwise} \end{cases}$$

and define the mapping $T : X \to X$ as

$$Tx = \begin{cases} \frac{x}{10}; & \text{if } x \in C_{x_0,\phi(r)} \\ \frac{1}{2}; & otherwise. \end{cases}$$

If $x \in C_{1,\frac{1}{2r-1}}$ then $Tx = \frac{x}{10}$,

1. $d(x,Tx) \leq \psi(x) - \psi(Tx)$

$$d(x,\frac{x}{10}) \leq d(x,1) - d(\frac{x}{10},1)$$
$$\frac{9x}{10} = \frac{9x}{10}.$$

2. $d(Tx,x_0) = d(\frac{x}{10},1) = \frac{x}{10} - 1 \ngeq \frac{1}{2r-1}$.

Hence, T satisfies condition (1) but does not satisfy condition (2). Clearly, T does not have a ϕ-fixed circle.

EXAMPLE 4. Let $X = \mathbb{R}^+$ and (X,d) be a metric space and let us consider a circle $C_{x_0,\phi(r)}$ and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$\phi(r) = \begin{cases} \frac{1}{r}; & \text{if } r \geq 1 \\ \frac{2}{3}; & \text{otherwise} \end{cases}$$

and define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} 2; & \text{if } x \in C_{x_0,\phi(r)} \\ \frac{12}{5}; & otherwise. \end{cases}$$

If $x \in C_{x_0,\phi(r)}$ then $Tx = 2$, $r \geq 1$.

1. $d(x, Tx) \leq \psi(x) - \psi(Tx)$

$$1. \quad d(x, Tx) \leq \psi(x) - \psi(Tx)$$
$$d(x, 2) \leq \frac{1}{r} - d(Tx, x_0)$$
$$x \nleq \frac{1}{r}, \quad \forall \;\; r > 1$$

2. $d(Tx, x_0) = d(2, 0) = 2 \geq \frac{1}{r} = \phi(r)$
   $d(Tx, x_0) \geq \phi(r), \quad \forall \quad r \geq \quad 1.$

Hence, the self-mapping T satisfies condition (2) but does not satisfy condition (1). Then, clearly T does not have a ϕ-fixed circle.

THEOREM 7. *Let (X, d) be a metric space and $C_{x_0,\phi(r)}$ be any circle on X. Let the mapping ψ be defined as equation (3) and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying*

(1)* $d(x, Tx) \leq \psi(x) + \psi(Tx) - 2\phi(r)$

(2)* $d(Tx, x_0) \leq \phi(r)$

for each $x \in C_{x_0,\phi(r)}$ then the circle $C_{x_0,\phi(r)}$ is a ϕ-fixed circle of T.

Proof. Let us assume that the mapping ψ is defined as $\psi : X \to [0, \infty)$ and let $x \in C_{x_0,\phi(r)}$ be any arbitrary point.
Now, using condition (1^*), we get

$$\begin{aligned} d(x, Tx) &\leq \psi(x) + \psi(Tx) - 2\phi(r) \\ &= d(x, x_0) + d(Tx, x_0) - 2\phi(r) \\ &= \phi(r) + d(Tx, x_0) - 2\phi(r) \\ &= d(Tx, x_0) - \phi(r) \end{aligned} \tag{5}$$

Because of condition (2*), the point Tx should be lying on or be interior of the circle $C_{x_0,\phi(r)}$. Then there are two cases. If $d(Tx, x_0) < \phi(r)$ then by using (5) we get a contradiction. It should be $d(Tx, x_0) = \phi(r)$. If $d(Tx, x_0) = \phi(r)$ then by using (5), we get

$$\begin{aligned} d(x, Tx) &\leq d(Tx, x_0) - \phi(r) \\ &\leq \phi(r) - \phi(r) \\ &= 0 \end{aligned} \tag{6}$$

Hence, $Tx = x$. Consequently, $C_{x_0,\phi(r)}$ is a ϕ-fixed circle of T. □

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Example 5. Let (X, d) be a metric space. Let us consider a circle $C_{x_0,\phi(r)}$ where $\phi(r)$ is defined as

$$\phi(r) = \begin{cases} \frac{1}{2r-1}; & r \in \{1, 2, 3\} \\ \frac{1}{3}; & \text{otherwise} \end{cases}$$

and define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} x; & \text{if } x \in C_{x_0,\phi(r)} \\ \frac{1}{5}; & \text{otherwise} \end{cases}$$

for all $x \in X$.

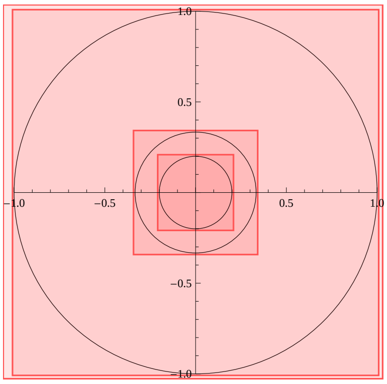

Figure 1

Figure 1 represents the circle $C_{0,\phi(r)}$ where $r \in \{1, 2, 3\}$. If $x \in C_{x_0,\phi(r)}$ then $Tx = x$. Then it is easily seen that conditions (1*) and (2*) are satisfied. Hence, clearly $C_{x_0,\phi(r)}$ is a ϕ-fixed circle of T.

Example 6. Let$X = \mathbb{R}^+$ and (X, d) be a metric space. Let us consider a circle $C_{x_0,\phi(r)}$ where $\phi(r)$ is defined as

$$\phi(r) = \begin{cases} \frac{1}{2r-1}; & r \in \{1, 2, 3\} \\ \frac{1}{3}; & \text{otherwise} \end{cases}$$

and define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} \frac{1}{6}; & \text{if } x \in C_{x_0,\phi(r)} \\ 5; & \text{otherwise} \end{cases}$$

for all $x \in X$. If $x \in C_{x_0,\phi(r)}$ then $Tx = \frac{1}{6}$ and $x_0 = 0$.

1. $d(x, Tx) \leq \psi(x) + \psi(Tx) - 2\phi(r)$

$$\begin{aligned} d(x, Tx) &\leq \psi(x) + \psi(Tx) - 2\phi(r) \\ \psi(x) + \psi(Tx) - 2\phi(r) &= d(x, x_0) + d(Tx, x_0) - 2\phi(r) \\ &= \phi(r) + d(\frac{1}{6}, 0) - 2\phi(r) \\ &= \frac{1}{6} - \frac{1}{2r-1} \\ &\ngeq d(x, \frac{1}{6}). \end{aligned}$$

2. $d(Tx, x_0) = d(\frac{1}{6}, 0) = \frac{1}{6} \leq \frac{1}{2r-1} = \phi(r)$.

Then the self-mapping T satisfies condition (2*) but does not satisfy condition (1*). Clearly, T does not ϕ-fix the circle $C_{0,\phi(r)}$.

Example 7. Let $X = [0, \frac{3}{2}]$ and (X, d) be the usual metric space with $d(x, y) = |x - y|$ and define the self-mapping $T : X \to X$ as

$$T(x) = \begin{cases} 2; & \text{if } x \in C_{0,\phi(r)} \\ \frac{5}{2}; & \text{otherwise} \end{cases}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$\phi(r) = \begin{cases} \frac{1}{2r-1}; & \text{if } r \geq 1 \\ \frac{1}{3}; & \text{otherwise} \end{cases}.$$

If $x \in C_{0,\phi(r)}$ then $T(x) = 2$ for $r \geq 1$

1. $d(x, Tx) \leq \psi(x) + \psi(Tx) - 2\phi(r)$

$$\begin{aligned} d(x, Tx) &\leq \psi(x) + \psi(Tx) - 2\phi(r) \\ d(x, 2) &\leq d(x, x_0) + d(Tx, x_0) - 2\phi(r) \\ |x - 2| &\leq \phi(r) + d(2, 0) - 2\phi(r) \\ &\leq 2 - \frac{1}{2r-1} \\ &\leq \frac{4r-3}{2r-1} \quad for \quad r \geq 1. \end{aligned}$$

2. $d(Tx, x_0) = d(2, 0) = 2 \nleq \frac{1}{2r-1} = \phi(r), \quad for \quad r \geq 1.$

Hence, the self-mapping T satisfies condition (1*) but does not satisfy condition (2*). Then, clearly T does not ϕ-fix the circle $C_{0,\phi(r)}$.

THEOREM 8. *Let (X, d) be a metric space and $C_{x_0,\phi(r)}$ be any circle on X. Let the mapping ψ be defined as equation (3) and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying*

*(1**)* $d(x, Tx) \le \psi(x) - \psi(Tx)$

*(2**)* $hd(x, Tx) + d(Tx, x_0) \ge \phi(r)$

for each $x \in C_{x_0,\phi(r)}$ and some $h \in [0, 1)$, then $C_{x_0,\phi(r)}$ is a ϕ-fixed circle of T.

Proof. We consider the mapping $\psi : X \to [0, +\infty)$ and $\psi(x) = d(x, x_0)$. Assume that $x \in C_{x_0,\phi(r)}$ and $Tx = x$ then using conditions (1**) and (2**). Now we obtain

$$\begin{aligned} d(x, Tx) &\le \psi(x) - \psi(Tx) \\ &= d(x, x_0) - d(Tx, x_0) \\ &= \phi(r) - d(Tx, x_0) \\ &\le hd(x, Tx) + d(Tx, x_0) - d(Tx, x_0) \\ &= hd(x, Tx), \end{aligned}$$

which is a contradiction with our assumption since $h \in [0, 1)$. Therefore we get $Tx = x$ and $C_{x_0,\phi(r)}$ is a ϕ-fixed circle of T. □

EXAMPLE 8. Let $X = [0, 1]$ and (X, d) be the usual metric space. Let us consider the circle $C_{\frac{1}{2},\frac{1}{2}} = \{0, 1\}$ where $\phi(r) = \frac{1}{2}$, $x_0 = \frac{1}{2}$ and define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} \frac{1}{2}; & \text{if } x \in C_{\frac{1}{2},\frac{1}{2}} \\ 1; & \text{otherwise} \end{cases}$$

for all $x \in X$. If $x \in C_{\frac{1}{2},\frac{1}{2}}$ then $Tx = \frac{1}{2}$.

1. $d(x, Tx) \le \psi(x) - \psi(Tx)$

$$\begin{aligned} d(x, Tx) &\le \psi(x) - \psi(Tx) \\ \psi(x) - \psi(Tx) &= d(x, x_0) - d(x, Tx_0) \\ &= d(x, \frac{1}{2}) - d(\frac{1}{2}, \frac{1}{2}) \end{aligned}$$

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$$\Rightarrow d(x, \frac{1}{2}) \geq d(x, \frac{1}{2}).$$

2. $hd(x, Tx) + d(Tx, x_0) \geq \phi(r)$

$$h \cdot d(x, Tx) + d(Tx, x_0) = h \cdot d(x, \frac{1}{2}) + d(\frac{1}{2}, \frac{1}{2}) \ngeq \frac{1}{2}.$$

Hence, the self-mapping T satisfies condition (1**) where $h \in [0, 1)$, but does not satisfy condition (2**). Clearly T does not ϕ-fix the circle $C_{\frac{1}{2}, \frac{1}{2}}$.

Now, we generalize the definition of fixed discs and the results of fixed disc.

DEFINITION 6. *Let (X, d) be a metric space, $D_{x_0, \phi(r)} = \{x \in X : d(x_0, x) \leq \phi(r)\}$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a disc and T be a self-mapping on X, if $Tx = x$ for every $x \in D_{x_0, \phi(r)}$ then the disc is called a ϕ-fixed disc of T.*

THEOREM 9. *If T is a $\mathcal{Z}_c$-contraction with respect to ζ with $x_0 \in X$ and the condition $0 < d(Tx, x_0) \leq \phi(r)$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ holds for all $x \in D_{x_0, \phi(r)} - \{x_0\}$ then the disc $D_{x_0, \phi(r)}$ is a ϕ-fixed disc of T.*

Proof. Let $\phi(r) = 0$. In this case $x \in D_{x_0, \phi(r)} = \{x_0\}$. I f $Tx_0 \neq x_0$ then $d(Tx_0, x_0) > 0$ and using the definition of $\mathcal{Z}_c$ -contraction we get

$$\zeta(d(Tx_0, x_0), d(Tx_0, x_0)) \geq 0.$$

This is a contradiction by the condition of ζ, $(\zeta(t, s) < s - t)$ for all $s, t > 0$. Hence, $Tx_0 = x_0$.
Now, assume $\phi(r) \neq 0$. Let $x \in D_{x_0, \phi(r)}$ be such that $Tx \neq x$. By the definition of $\phi(r)$ we have $0 < \phi(r) \leq d(x, Tx)$ and using the condition of ζ, we obtain

$$\begin{aligned}\zeta(d(Tx, x), d(Tx, x_0)) &< d(Tx, x_0) - d(Tx, x)\\ &\leq \phi(r) - d(Tx, x)\\ &\leq \phi(r) - \phi(r)\\ &= 0\end{aligned}$$

which is a contradiction with the property of T. It should be $Tx = x$, so T ϕ-fixes the disc $D_{x_0, \phi(r)}$. □

EXAMPLE 9. Let $X = \mathbb{R}$ and (X,d) be the usual metric space with $d(x,y) = |x-y|$. Let us define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} x; & x \in [-1,1] \\ x^2+1 & otherwise \end{cases}$$

for all $x \in \mathbb{R}$ and define the mapping $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\phi(r) = \frac{1}{\sqrt{r}}$ where $\phi(r)$ is a disc radius. The function $\zeta : [0,\infty)^2 \to \mathbb{R}$ defined as $\zeta(t,s) = \frac{2}{3}s - t$. Indeed, it is clear that

$$0 < d(Tx, x_0) = d(x,0) = |x| \leq \frac{1}{\sqrt{r}} = \phi(r)$$

hence $0 < d(Tx,x_0) \leq \phi(r)$, holds for all $x \in D_{x_0,[-1,1]} - \{0\}$ and $r \in [\frac{1}{9}, 1]$, we have

$$\begin{aligned} \zeta(d(Tx,x), d(Tx,x_0)) &= \zeta(d(x^2+1,x), d(x^2+1,0)) \\ &= \zeta(|x^2+1-x|, |x^2+1-0|) \\ &= \frac{2}{3}|x^2+1| - |x^2+1-x| \\ &\leq \frac{1}{3}|3x - x^2 - 1|. \end{aligned}$$

for all $x \in \mathbb{R}$ such that $d(Tx,x) > 0$. Hence T is a $\mathcal{Z}_c$-contraction with the radius $\phi(r) = \frac{1}{\sqrt{r}}$ and the center is 0. Consequently, T is a ϕ-fixed disc $D_{x_0,\phi(r)} = [-1,1]$, but does not hold theorem (4) in the condition $(0 < d(Tx,x_0) \leq r)$.

THEOREM 10. *Let (X,d) be a metric space and $T : X \to X$ a Ćirić-type $\mathcal{Z}_c$-contraction with respect to ζ with $x_0 \in X$, If the condition $0 < d(Tx,x_0) \leq \phi(r)$ holds for all $x \in D_{x_0,\phi(r)} - \{x_0\}$ then $D_{x_0,\phi(r)}$ is a ϕ-fixed disc of T.*

Proof. Let $\phi(r) = 0$. In this case, $x \in D_{x_0,\phi(r)} = \{x_0\}$ and the Ćirić-type $\mathcal{Z}_c$-contraction theory produces $Tx_0 = x$. Indeed, If $Tx_0 \neq x$ then we have $d(x_0,Tx_0) > 0$. By the definition of the Ćirić-type $\mathcal{Z}_c$-contraction, we get

$$\zeta(d(x_0,Tx_0), m^*(x_0,x_0) \geq 0 \tag{7}$$

Since

$$m^*(x_0,x_0) = max\{d(x_0,x_0), d(x_0,Tx_0), d(x_0,Tx_0), \frac{d(x_0,Tx_0)+d(x_0,Tx_0)}{2}\}$$

$$= max\{d(x,x_0), d(x_0,Tx_0), d(x_0,Tx_0), d(x_0,Tx_0)\}$$
$$= d(x_0,Tx_0).$$

Now, we find

$$\zeta(d(x_0,Tx_0), m^*(x_0,x_0)) = \zeta(d(x_0,Tx_0), d(x_0,Tx_0)) < 0$$

by the condition of ζ such that $\zeta(t,s) < s-t$. This is a contradiction to equation (7). Hence, it should be $Tx_0 = x_0$. Assume that $\phi(r) \neq 0$. Let $x \in D_{x_0,\phi(r)}$ be such that $Tx \neq x$. Then we have

$$m^*(x,x_0) = max\{d(x,x_0), d(x,Tx), d(x_0,Tx_0), \frac{d(x,Tx_0)+d(x_0,Tx)}{2}\}$$
$$= max\{d(x,x_0), d(x,Tx), \frac{d(x,Tx_0)+d(x_0,Tx)}{2}\}$$

By the assumption, we have

$$\zeta(d(Tx,x), m^*(x,x_0)) \geq 0$$

and

$$\zeta(d(Tx,x), max\{d(x,x_0), d(x,Tx), \frac{d(x,x_0)+d(x_0,Tx)}{2}\} \geq 0 \tag{8}$$

Now, we have the following three cases:
Case 1. Let $max\{d(x,x_0), d(x,Tx), \frac{d(x,x_0)+d(x_0,Tx)}{2}\} = d(x,x_0)$ from equation (8) we get

$$\zeta(d(Tx,x), d(x,x_0)) \geq 0.$$

Using the condition ζ s.t. $(\zeta(t,s) < s-t)$ and consider the definition of $\phi(r)$, we find

$$\zeta(d(Tx,x), d(x,x_0)) < d(x,x_0) - d(Tx,x) < \phi(r) - \phi(r) = 0$$

which is a contradiction.
Case 2. Let $max\{d(x,x_0), d(x,Tx), \frac{d(x,x_0)+d(x_0,Tx)}{2}\} = d(x,Tx)$ from equation (8) we get

$$\zeta(d(Tx,x), d(x,Tx)) \geq 0.$$

Using the condition ζ s.t. $(\zeta(t,s) < s - t)$, again we obtain a contradiction. Case 3. Let $max\{d(x,x_0), d(x,Tx), \frac{d(x,x_0)+d(x_0,Tx)}{2}\} = \frac{d(x,x_0)+d(x_0,Tx)}{2}$ from equation (8) we get

$$\zeta(d(Tx,x), \frac{d(x,x_0) + d(x_0,Tx)}{2}) \geq 0.$$

Using the condition ζ s.t. $(\zeta(t,s) < s - t)$, we obtain

$$\zeta(d(Tx,x), \frac{d(x,x_0) + d(x_0,Tx)}{2}) < \frac{d(x,x_0) + d(x_0,Tx)}{2} - d(Tx,x)$$
$$\leq \phi(r) - d(Tx,x) \leq \phi(r) - \phi(r) = 0.$$

Again this is a contradiction with the $\acute{C}iri\acute{c}$-type $\mathcal{Z}_c$-contractive property of T. In all the above cases, we have a contradiction. Hence, it should be $Tx = x$ and consequently, T ϕ-fixes the disc $D_{x_0,\phi(r)}$. □

Conclusion and future work

This study contributes to the field by broadening the scope of fixed point theory to include fixed circles and discs, incorporating a more flexible definition via the function $\phi(r)$. The use of Caristi-type contractions further enhances the applicability of the results within metric spaces. These advancements provide a robust framework for exploring geometric properties in fixed point theory. Future work may focus on extending these ideas to fuzzy metric spaces and probabilistic metric spaces, as well as investigating applications in dynamic systems and optimization problems.

References

Özgür, N. Y. & Taş, N. 2019. Some fixed-circle theorems on metric spaces, *Bulletin of the Malaysian Mathematical Sciences Society*, 42(4), pp. 1433-1449. Available at: https://doi.org/10.1007/s40840-017-0555-z.

Özgür, N. & Taş, N. 2021. On The Geometry of ϕ-Fixed Points. *Conference Proceedings Of Science And Technology*, 4(2), pp. 226-231.

Özgür, N. & Taş, N. 2021. Geometric properties of fixed points and simulation functions, *arXiv preprint arXiv:2102.05417*, Feb 10. Available at: https://doi.org/10.48550/arXiv.2102.05417.

Karapınar E. 2016. Fixed points results via simulation functions. *Filomat*, 30(8), pp. 2343-2350. Available at: https://www.jstor.org/stable/24899250.

Özgür, N. 2019. Fixed-disc results via simulation functions. *Turkish J. Math.*, 43(6), pp. 2794-2805. Available at: https://www.jstor.org/stable/24899250.

Taş, N. Mlaiki N., Aydi H.,Özgür N. 2021. Fixed-disc results on metric spaces, *Filomat*, 35(2), pp. 447-457. Available at: https://doi.org/10.2298/FIL2102447T.

Taş, N., Özgür, N. Y., & Mlaiki, N. 2018. New types of Fc-contractions and the fixed-circle problem, *Mathematics*, 10(6), pp. 188. Available at: https://doi.org/10.3390/math6100188.

Mlaiki, N., Özgür, N., Taş, N., & Santina, D. 2023. On the Fixed Circle Problem on Metric Spaces and Related Results. *Axioms*, 12(4), pp. 401. Available at: https://doi.org/10.3390/axioms12040401.

Taş N. 2018. Various types of fixed-point theorems on S-metric spaces. *BAUN Fen. Bil. Enst. Dergisi.*, December 20(2), pp. 211-223. Available at: doi:10.25092/baunfbed.426665.

Özgür, N.Y. Taş, N., & Çelik, U. 2017. New fixed-circle results on S-metric spaces. *Bull. Math. Anal. Appl.*, 9, pp. 10-23. Available at: https://hdl.handle.net/20.500.12462/6665.

Özgür, N.Y., Taş, N. 2018. Fixed-circle problem on S-metric spaces with a geometric viewpoint. *Facta Univ. (NIS) Ser. Math. Inform.*, 34, pp. 459-472. Available at: https://doi.org/10.48550/arXiv.1704.08838.

Проблеми фиксне тачке и фиксног диска у метричким просторима

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Сажетак:

Увод/циљ: Циљ овог рада јесте да се установе теореме постојања за непокретне кругове и непокретне дискове у метричким просторима коришћењем различитих врста услова за контракцију. Применом пресликавања у самог себе на метричким просторима, резултати класичне непокретне тачке проширују се на ове фиксне геометријске структуре. Наведено је неколико примера за илустрацију и потврду теоријских резултата.

Методе: Разматрају се пресликавања у самог себе на ме-

тричким просторима и уводе различите врсте услова за контракцију. Аналитичке технике из теорије непокретне тачке користе се за извођење довољних услова за постојање ϕ-непокретних кругова и ϕ-непокретних дискова. Теоријски резултати су поткрепљени пажљиво конструисаним примерима који задовољавају предложене контракције и потврђују примењивост добијених теорема.
Резултати: Успешно су уведени резултаи ϕ-непокретних кругова и ϕ-непокретних дискова за Каристијеве контракције, као и друга класуа контракција у оквиру метричких простора. Наводе се и одговарајући примери.
Закључак: Овај рад уводи нове теореме постојања за ϕ-непокретне кругове и ϕ-непокретне дискове у метричким просторима помоћу Каристијевих и одговарајућих услова за контракцију. Његови резултати проширују класичну теорију непокретне тачке са појединачних непокретних тачака на шире геометријске непокретне структуре и на тај начин обогаћују теорију метричких непокретних тачака. Наведени примери указују на применљивост и ефикасност предложених резултата, као и на њихов потенцијал за будућа уопштавања.
Кључне речи: непокретни круг, непокретни диск, метрички простор, Каристијева контракција

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