ORIGINAL SCIENTIFIC PAPERS

Sombor index of thorny graphs

Ivan Gutman

University of Kragujevac, Faculty of Science,
Kragujevac, Republic of Serbia,
e-mail: gutman@kg.ac.rs,
ORCID iD: https://orcid.org/0000-0001-9681-1550

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Abstract:

Introduction/purpose: The thorny graph of a graph G is obtained by attaching pendent vertices to the vertices of G. A mathematical study of the Sombor index of thorny graphs is undertaken.

Methods: Combinatorial graph theory is applied.

Results: A general expression for the Sombor index of thorny graphs is obtained, as well as lower and upper bounds. Several special cases of this general expression are pointed out.

Conclusion: The paper contributes to the theory of the Sombor index.

Keywords: degree (of vertex), Sombor index, thorny graph.

Introduction

In this paper, G will denote a simple graph with $n > 1$ vertices and m edges. Its vertices are labeled by $v_1, v_2, \ldots, v_n$. The degree (= number of first neighbors) of a vertex v_i will be denoted by $d(v_i)$. A vertex of degree one is said to be pendent. An edge whose one end vertex is of degree one is also said to be pendent.

For additional details of graph theory, see (Harary, 1969; Bondy & Murty, 1976).

In the last few years, a vertex-degree-based graph invariant, named the Sombor index, has attracted much attention in mathematics, see e.g. (Li et al., 2024; Rather et al., 2024; Wang et al., 2024) and found numerous applications, both in chemistry, see e.g. (Hayat et al., 2024; Rauf & Ahmad, 2024), network theory, see e.g. (Hamid et al., 2022; Imran et al., 2024), and in other areas, see e.g., (Alqahtani et al., 2024; Anwar et al., 2024; Jamil et al., 2025).

The Sombor index of the graph G is defined as (Gutman, 2021a)

$$SO = SO(G) = \sum_{e_{ij}} \sqrt{d(v_i)^2 + d(v_j)^2}$$

where e_{ij} denotes the edge connecting the vertices v_i and v_j, and the summation goes over all edges of G. The basic mathematical properties of the Sombor index can be found in the review (Liu et al., 2022).

Let $p_1, p_2, \dots, p_n$ be non-negative integers. Then the *thorny graph* of the graph G, denoted by G^*, is obtained by attaching p_i pendent vertices to the vertex v_i, for all $i = 1, 2, \dots, n$. Thus, the number of vertices and edges of G^* is

$$n^* = n + \sum_{i=1}^{n} p_i \qquad \text{and} \qquad m^* = m + \sum_{i=1}^{n} p_i$$

respectively.

If $p_1 = p_2 = \cdots = p_n$, then G^* is said to be a thorn-regular graph.

Thorny graphs have been extensively studied, see for example (Bonchev & Klein, 2002; De, 2012; Lakshmi & Parvathi, 2023; Marinescu-Ghemeci, 2010; Walikar et al., 2006). The Sombor index of thorny graphs was considered only in (Krishnan & Narayan, 2023) and (Lakshmi & Parvathi, 2023), but only for special cases when G is a complete graph, star, wheel, path, and similar, and only when G^* is thorn-regular. In the present paper, we examine the general case, namely when G is an arbitrary (simple, not necessarily connected) graph, and $p_1, p_2, \dots, p_n$ are arbitrary non-negative integers.

Main results

In this section we state some general properties of thorny graphs. In order to avoid trivialities, it is assumed that such graphs possess at least one edge, i.e., more than one vertex.

THEOREM 1. *Let G be a simple graph on $n > 1$ vertices, and $p_1, p_2, \dots, p_n$ be non-negative integers. Let G^* be the thorny graph of G. Then the Sombor* *index of G^* satisfies the relation*

$$SO(G^*) = \sum_{e_{ij}} \sqrt{[d(v_i)+p_i]^2 + [d(v_j)+p_j]^2} + \sum_{i=1}^{n} p_i \sqrt{[d(v_i)+p_i]^2 + 1}\,. \tag{1}$$

Proof. For $i = 1, 2, \ldots, n$, the degree of the vertex v_i of the thorny graph G^* is equal to $d(v_i)+p_i$. This implies the first term on the right-hand side of (1), in which the summation goes over all edges of the graph G. In addition to these edges, in G^* there are p_i pendent edges attached to the vertex v_i, for each $i = 1, 2, \ldots, n$. Their contributions to the Sombor index of G^* are collected in the second term on the right-hand side of (1). □

The first Zagreb index

$$M_1 = M_1(G) = \sum_{e_{ij}} \left[d(v_i) + d(v_j)\right] = \sum_{i=1}^{n} d(v_i)^2$$

is one of the oldest and best studied vertex-degree-based topological indices (Gutman & Trinajstić, 1972; Nikolić et al., 2003; Gutman & Das, 2004).

Using the inequalities

$$\frac{1}{\sqrt{2}}(a+b) \leq \sqrt{a^2+b^2} < a+b$$

the following well-known estimates for the Sombor index are straightforwardly obtained (Milovanović et al., 2021; Gutman, 2021b, 2024):

$$\frac{1}{\sqrt{2}}\, M_1(G) \leq SO(G) < M_1(G)\,. \tag{2}$$

The equality on the left-hand side holds if and only if G is a regular graph.

By the same argument, we now have

$$\begin{aligned}\sum_{e_{ij}} \sqrt{[d(v_i)+p_i]^2 + [d(v_j)+p_j]^2} \;\; &< \;\; \sum_{e_{ij}} \left[d(v_i)+p_i+d(v_j)+p_j\right] \\ &= \;\; \sum_{e_{ij}} \left[d(v_i)+d(v_j)\right] + \sum_{e_{ij}} \left[p_i+p_j\right]\end{aligned}$$

$$= \quad M_1(G) + \sum_{i=1}^{n} p_i \, d(v_i) \tag{3}$$

where we applied the identity (Došlić et al., 2011)

$$\sum_{e_{ij}} [f(v_i) + f(v_j)] = \sum_{i=1}^{n} d(v_i) \, f(v_i)$$

valid for any vertex-dependent function f.

The lower bound, analogous to (3) is

$$\sum_{e_{ij}} \sqrt{[d(v_i)+p_i]^2 + [d(v_j)+p_j]^2} > \frac{1}{\sqrt{2}} \left[M_1(G) + \sum_{i=1}^{n} p_i \, d(v_i) \right] . \tag{4}$$

The equality in (4) cannot occur since no graph G^* is regular.

For the other term in (1) we have

$$\begin{aligned} \sum_{i=1}^{n} p_i \sqrt{[d(v_i)+p_i]^2+1} &< \sum_{i=1}^{n} p_i \, [d(v_i)+p_i+1] \\ &= \sum_{i=1}^{n} p_i \, d(v_i) + \sum_{i=1}^{n} p_i(p_i+1) \, . \end{aligned} \tag{5}$$

and

$$\sum_{i=1}^{n} p_i \sqrt{[d(v_i)+p_i]^2+1} \geq \frac{1}{\sqrt{2}} \left[\sum_{i=1}^{n} p_i \, d(v_i) + \sum_{i=1}^{n} p_i(p_i+1) \right] . \tag{6}$$

The equality in (6) would occur for thorn-regular graphs of a regular graph.

Combining the estimates (3)–(6) we arrive at:

THEOREM 2. *Let G be a simple graph on $n > 1$ vertices, and $p_1, p_2, \ldots, p_n$ be non-negative integers. Let G^* be the thorny graph of G. Then the Sombor index of G^* is bounded as*

$$\begin{aligned} &\frac{1}{\sqrt{2}} \left[M_1(G) + 2\sum_{i=1}^{n} p_i \, d(v_i) + \sum_{i=1}^{n} p_i(p_i+1) \right] < SO(G^*) < \\ &M_1(G) + 2\sum_{i=1}^{n} p_i \, d(v_i) + \sum_{i=1}^{n} p_i(p_i+1) \, . \end{aligned} \tag{7}$$

The bounds (7) should be compared with inequalities (2). In fact, (2) is the special case of (7) when $p_1 = p_2 = \cdots = p_n = 0$.

Corollaries

In this section, we list a few interesting special cases of Theorem 1.

COROLLARY 1. *Let G be a regular graph on n vertices, of the degree r. Then the Sombor index of a thorn-regular graph of G is*

$$SO(G^*) = \frac{\sqrt{2}\,nr}{2}(r+p) + np\,\sqrt{(r+p)^2+1}$$

Proof. Setting $d(v_i) = r$ and $p_i = p$ for all $i = 1, 2, \ldots, n$, the left-hand term in (1) becomes $\sqrt{2}m(r+p)$. For regular graphs, $2m = nr$. □

COROLLARY 2. *If $p_i = \lambda\, d(v_i)$ holds for all $i = 1, 2, \ldots, n$, then*

$$SO(G^*) = (\lambda+1)\,SO(G) + \lambda(\lambda+1)\sum_{i=1}^{n} d(v_i)\,\sqrt{d(v_i)^2 + \frac{1}{(\lambda+1)^2}}\,.$$

If λ is sufficiently large, then

$$SO(G^*) \approx (\lambda+1)\,SO(G) + \lambda(\lambda+1)\,M_1(G)\,.$$

COROLLARY 3. *If $p_i + d(v_i) = D$ holds for all $i = 1, 2, \ldots, n$, then*

$$SO(G^*) = \sqrt{2}\,Dm + \sqrt{D^2+1}\,(n^* - n)$$

where n and m are the numbers of the vertices and edges of G, whereas n^* is the number of the vertices of G^*.

It is worth noting that the thorny graphs considered in Corollary 3 are of chemical interest. Namely, for $D = 3$, if G is the molecular graph of an unsaturated conjugated molecule, then G^* is its plerogram (= hydrogen filled molecular graph) (Gutman & Vidovic, 1998; Gutman et al., 1998). If $D = 4$, then G^* is the plerogram of the molecular graph of a saturated hydrocarbon.

References

Alqahtani, M., Kaviyarasu, M. & Rajeshwari, M. 2024. Site Selection for Thermal Power Plant Based on Sombor Index in Neutrosophic Graph. *European Journal of Pure and Applied Mathematics*, 17(4), pp. 2586–2620. Available at: https://doi.org/10.29020/nybg.ejpam.v17i4.5461.

Anwar, S., Azeem, M., Jamil, M.K., Almohsen, B. & Shang, Y. 2024. Single-valued neutrosophic fuzzy Sombor numbers and their applications in trade flows between different countries via sea route. *The Journal of Supercomputing*, 80(14), pp. 19976–20019. Available at: https://doi.org/10.1007/s11227-024-06169-8.

Bonchev, D. & Klein, D.J. 2002. On the Wiener Number of Thorn Trees, Stars, Rings, and Rods. *Croatica Chemica Acta*, 75(2), pp. 613–620 [online]. Available at: https://hrcak.srce.hr/127540 [Accessed: 25 January 2025].

Bondy, J.A. & Murty, U.S.R. 1976. *Graph theory with applications*. The Macmillan Press Ltd. ISBN: 0-444-19451-70.

De, N. 2012. On Eccentric Connectivity Index and Polynomial of Thorn Graph. *Applied Mathematics*, 3(8), art.ID:21477. Available at: https://doi.org/10.4236/am.2012.38139.

Došlić, T., Réti, T. & Vukičević, D. 2011. On the vertex degree indices of connected graphs. *Chemical Physics Letters*, 512(4-6), pp. 283–286. Available at: https://doi.org/10.1016/j.cplett.2011.07.040.

Gutman, I. 2021a. Geometric Approach to Degree–Based Topological Indices: Sombor Indices. *MATCH Communications in Mathematical and in Computer Chemistry*, 86(1), pp. 11–16 [online]. Available at: https://match.pmf.kg.ac.rs/electronic_versions/Match86/n1/match86n1_11-16.pdf [Accessed: 25 January 2025].

Gutman, I. 2021b. Some basic properties of Sombor indices. *Open Journal of Discrete Applied Mathematics (ODAM)*, 4(1), pp. 1–3. Available at: https://www.doi.org/10.30538/psrp-odam2021.0047.

Gutman, I. 2024. Improved Estimates of Sombor Index. *Iranian Journal of Mathematical Chemistry*, 15(1), pp. 1–5. Available at: https://www.doi.org/10.22052/ijmc.2023.253825.1782.

Gutman, I. & Das, K.C. 2004. The first Zagreb index 30 years after. *MATCH Communications in Mathematical and in Computer Chemistry*, 50(1), pp. 83–92 [online]. Available at: https://match.pmf.kg.ac.rs/electronic_versions/Match50/match50_83-92.pdf [Accessed: 25 January 2025].

Gutman, I., Popović, L. et al. 1998. Graph representation of organic molecules Cayley's plerograms vs. his kenograms. *Journal of the Chemical Society, Faraday Transactions*, 94(7), pp. 857–860. Available at: https://www.doi.org/10.1039/A708076J.

Gutman, I. & Trinajstić, N. 1972. Graph theory and molecular orbitals. Total φ-electron energy of alternant hydrocarbons. *Chemical Physics Letters*, 17(4), pp. 535–538. Available at: https://doi.org/10.1016/0009-2614(72)85099-1.

Gutman, I. & Vidovic, D. 1998. Relations between Wiener-type topological indices of plerograms and kenograms. *Journal of the Serbian Chemical Society*, 63(9), pp. 695–702 [online]. Available at: https://www.shd.org.rs/JSCS/Start.html [Accessed: 25 January 2025].

Hamid, K., Waseem Iqbal, M., Abbas, Q., Arif, M., Brezulianu, A. & Geman, O. 2022. Discovering irregularities from computer networks by topological mapping. *Applied Sciences*, 12(8), art.number:12051. Available at: https://doi.org/10.3390/app122312051.

Harary, F. 1969. *Graph Theory (on Demand Printing Of 02787), 1st Edition*. Boca Raton: CRC Press. Available at: https://doi.org/10.1201/9780429493768.

Hayat, S., Arshad, M. & Khan, A. 2024. Graphs with given connectivity and their minimum Sombor index having applications to QSPR studies of monocarboxylic acids. *Heliyon*, 10(1), e23392. Available at: https://doi.org/10.1016/j.heliyon.2023.e23392.

Imran, M., Azeem, M., Jamil, M.K. & Deveci, M. 2024. Some operations on intuitionistic fuzzy graphs via novel versions of the sombor index for internet routing. *Granular Computing*, 9(2), art.number:53. Available at: https://doi.org/10.1007/s41066-024-00467-5.

Jamil, M.K., Anwer, S., Azeem, M. & Gutman, I. 2025. Intuitionistic fuzzy Sombor indices: A novel approach for improving the performance of vaccination centers. *Communications in Combinatorics and Optimization*, 10(3), pp. 563–593. Available at: https://doi.org/10.22049/cco.2023.28767.1709.

Krishnan, V.L. & Narayan, P. 2023. Analysis of Sombor and Harmonic Indices of Thorn Cog-Graphs. *Mathematical Modelling of Engineering Problems*, 10(5), pp. 1777–1784. Available at: https://doi.org/10.18280/mmep.100529.

Lakshmi, K.V. & Parvathi, N. 2023. An Analysis of Thorn Graph on Topological Indices. *IAENG International Journal of Applied Mathematics*, 53(3), pp. 313–322 [online]. Available at: https://www.iaeng.org/IJAM/issues_v53/issue_3/IJAM_53_3_38.pdf [Accessed: 25 January 2025].

Li, Y., Deng, H. & Tang, Z. 2024. Sombor index of maximal outerplanar graphs. *Discrete Applied Mathematics*, 356, pp. 96–103. Available at: https://doi.org/10.1016/j.dam.2024.05.019.

Liu, H., Gutman, I., You, L. & Huang, Y. 2022. Sombor index: review of extremal results and bounds. *Journal of Mathematical Chemistry*, 60(5), pp. 771–798. Available at: https://doi.org/10.1007/s10910-022-01333-y.

Marinescu-Ghemeci, R. 2010. Radio number for some thorn graphs. *Discussiones Mathematicae Graph Theory*, 30(2), pp. 201–222. Available at: https://doi.org/10.7151/dmgt.1487.

Milovanović, I., Milovanović, E. & Matejić, M. 2021. On some mathematical properties of Sombor indices. *Bulletin of the International Mathematical Virtual Institute*, 11(2), pp. 341–353 [online]. Available at: http://www.imvibl.org/buletin/bulletin_imvi_11_2_2021/bulletin_imvi_10_2_2020_341_353.pdf [Accessed: 25 January 2025].

Nikolić, S., Kovačević, G., Miličević, A. & Trinajstić, N. 2003. The Zagreb indices 30 years after. *Croatica Chemica Acta*, 76(2), pp. 113–124 [online]. Available at: https://hrcak.srce.hr/103086 [Accessed: 25 January 2025].

Rather, B.A., Imran, M. & Pirzada, S. 2024. Sombor index and eigenvalues of comaximal graphs of commutative rings. *Journal of Algebra and its Applications*, 23(06), art.number:2450115. Available at: https://doi.org/10.1142/S0219498824501159.

Rauf, A. & Ahmad, S. 2024. On sombor indices of tetraphenylethylene, terpyridine rosettes and QSPR analysis on fluorescence properties of several aromatic hetero-cyclic species. *International Journal of Quantum Chemistry*, 124(1), e27261. Available at: https://doi.org/10.1002/qua.27261.

Walikar, H.B., Ramane, H.S., Sindagi, L., Shirakol, S.S. & Gutman, I. 2006. Hosoya polynomial of thorn trees, rods, rings, and stars. *Kragujevac Journal of Science*, 28, pp. 47–56 [online]. Available at: https://www.pmf.kg.ac.rs/KJS/en/volumes/kjs28/kjs28walikarramane47.pdf [Accessed: 25 January 2025].

Wang, Z., Gao, F., Zhao, D. & Liu, H. 2024. Sharp upper bound on the Sombor index of bipartite graphs with a given diameter. *Journal of Applied Mathematics and Computing*, 70(1), pp. 27–46. Available at: https://doi.org/10.1007/s12190-023-01955-8.

Índice de Sombor de gráficos espinosos

Ivan Gutman

Universidad de Kragujevac, Facultad de Ciencias,
Kragujevac, República de Serbia

CAMPO: matemáticas (clasificación de materias de matemáticas: primaria 05c07, secundaria 05c09)
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: El gráfico espinoso de un gráfico G se obtiene uniendo vértices colgantes a los vértices de G. Se realiza un estudio matemático del índice de Sombor de gráficos espinosos.

Métodos: Se aplica la teoría de gráficos combinatorios.

Resultados: Se obtiene una expresión general para el índice de Sombor de gráficos espinosos, así como sus límites inferior y superior. Se señalan varios casos especiales de esta expresión general.

Conclusión: El artículo contribuye a la teoría del índice de Sombor.

Palabras claves: grado (de vértice), índice de Sombor, gráfico espinoso.

Сомборский индекс тернистого графа

Иван Гутман

Крагуевацкий университет, естественно-математический факультет, г. Крагуевац, Республика Сербия

РУБРИКА ГРНТИ: 27.29.19 Краевые задачи и задачи на собственные значения для обыкновенных дифференциальных уравнений и систем уравнений
ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: Тернистый граф графа G получается путем присоединения висячей вершины к вершинам G. В данной статье представлено математическое исследование Сомборского индекса тернистых графов.

Методы: В исследовании применены комбинаторика и теория графов.

Результаты: В результате исследования получена новая общая формула для Сомборского индекса тернистых графов, а также нижняя и верхняя грани. Указано на несколько частных случаев относительно результатов.

Выводы: Данная статья вносит вклад в теорию Сомборского индекса.

Ключевые слова: степень (вершины), Сомборский индекс, тернистый граф.

Сомборски индекс трновитог графа

Иван Гутман

Универзитет у Крагујевцу, Природно-математички факултет, Крагујевац, Република Србија

ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Трновити граф графа G добија се додавањем висећих чворова на чворове графа G. Проучаване су математичке особине Сомборског индекса трновитих графова.

Методе: Примењивани су поступци комбинаторне теорије графова.

Резултати: Нађена је нова општа формула за Сомборски индекс трновитих графова, као и доње и горње границе. Истакнуто је неколико специјалних случајева ових резултата.

Закључак: Рад доприноси теорији Сомборског индекса.

Кључне речи: степен чвора, Сомборски индекс, трновити граф.

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