REVIEW PAPERS

b-metric like spaces: a survey of concepts and applications

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Abstract:

Introduction/Purpose: This paper presents a comprehensive survey of fixed-point results in metric-like spaces, with a particular emphasis on b-metric-like spaces. The purpose of the study was to explore the generalization of several important concepts in fixed point theory, including partial metric spaces, metric-like spaces, and b-metric spaces. This research aimed to contribute to the advancement of fixed-point theory in these generalized settings.

Methods: The study involved reviewing foundational and recent research on fixed-point results in b-metric-like spaces. It focused on summarizing key contributions from the early stages of the field to the present, providing an extensive compilation of results. Additionally, the paper addressed open problems arising from the best proximity results in orthogonal 0-complete b-metric-like spaces.

Results: The results highlighted significant contributions to the theory of fixed points in metric-like spaces, particularly in the context of b-metric-like spaces. The study offered affirmative answers to some open problems concerning the best proximity results in orthogonal 0-complete b-metric-like spaces, further enriching the theoretical framework.

Conclusions: This paper is concluded by advancing the understanding of fixed point theory in generalized metric-like spaces. It provides re-

searchers with valuable insights and references, which could facilitate further exploration and development in this field.

Key words: fixed point theory, b-metric-like spaces, metric-like spaces, partial metric spaces, metric spaces, best proximity results, orthogonal 0-complete spaces, generalizations.

Introduction and preliminaries

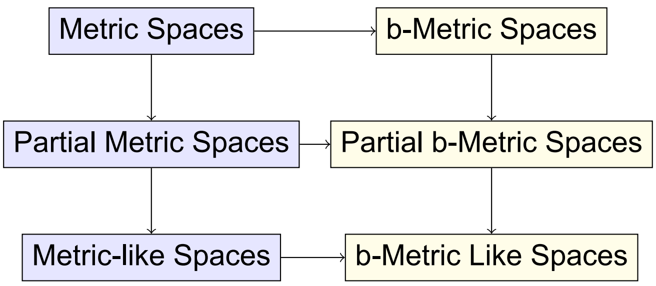
Let (X, d) be a metric space, and let $\mathcal{G} : X \to X$ be a mapping. The function $\mathcal{G}$ is defined as a contraction on X if there exists a constant k such that $0 \leq k < 1$ and for all $x, y \in X$, the following inequality holds:

$$d(\mathcal{G}(x), \mathcal{G}(y)) \leq k\, d(x, y).$$

This definition essentially means that $\mathcal{G}$ brings points closer together, with the constant k providing a bound on how much closer. The celebrated Banach Fixed-Point Theorem, established by Stefan Banach in 1922, asserts that any contraction mapping on a complete metric space has a unique fixed point. This powerful result has become a cornerstone of modern analysis and has inspired numerous extensions and generalizations, see (Abkar & Gabeleh, 2013; Chen et al., 2015; Ćirić, 1974; Czerwik, 1993; Gregori & Sapena, 2002; Karapinar, 2014; Kirk, 2003; Mattews, 1994; Mihet, 2008; Mirkov et al., 1994; Moussaoui et al., 2022, 2024; Păcurar & Rus, 2010; Petrusel, 2005; Radenović, 2016a; Suzuki, 2008, 2009; Yang et al., 2020).

Researchers have sought to broaden Banach's theorem by considering more general classes of spaces and mappings. The generalizations primarily follow two avenues: either modifying the underlying metric space or replacing the traditional contraction condition with a more flexible form. Specifically, instead of using a constant k, the contraction condition can be expressed in terms of a function φ that maps $(0, +\infty)$ to itself, i.e., by replacing $kd(x, y)$ with $\varphi(d(x, y))$. These extensions have led to the introduction of new types of generalized metric spaces, such as partial metric spaces, metric-like spaces, b-metric spaces, partial b-metric spaces, and b-metric-like spaces. Each of these spaces provides a different framework for studying contraction mappings, which may hold under conditions that relax or modify traditional distance properties. Berinde and Păcurar Berinde & Păcurar (2022) provided a concise overview of the foundational progress in fixed point theory on b-metric spaces, compiled a collection of key references, and analyzed recent advancements in the field. In 2012, Amini-Harandi Amini-Harandi (2012) developed the concept of metric-like spaces as a broadening of partial metric spaces and established foundational fixed point results that unify and extend previous work. Aleksić et al. Aleksić et al. (2020) simplified proofs for key results in b-metric spaces, showing that contractive conditions ensure b-Cauchyness of Picard sequences, improving the existing results. Monika in Monika (2019) compiles a comprehensive overview of fixed point results in metric-like spaces. The study includes significant findings from their inception to recent advancements. In 2013, Alghamdi et al. Alghamdi et al. (2013) defined b-metric-like spaces, proved fixed point theorems, and applied their results to integral equations. Chunfang Chen et al. Chen et al. (2015) explore fixed point theorems in b-metric-like spaces, generalizing the existing results and applying them to an integral equation. Kastriot Zoto et al. Zoto et al. (2018) introduced (s, p, α)-quasi-contractions and (s, p)-weak contractions, establishing fixed point results in b-metric-like spaces.

These generalized structures have proven useful in various areas of mathematics, particularly in fixed-point theory, where they provide more flexible settings for studying existence and uniqueness theorems. They also find applications in the areas such as computer science, optimization, and dynamical systems. The interplay between these important and diverse generalizations is presented as follows:



Let us now review the definitions of the six distinct classes of spaces, which encompass the classic metric space and five extended classes of generalized metric spaces.

The concept of a **metric space** was introduced by **Maurice Fréchet** Fréchet (1906) in 1906 during his doctoral work, titled *Sur quelques points du calcul fonctionnel*. This idea marked the beginning of the formal study of abstract spaces equipped with a distance function, forming a foundational aspect of modern analysis and topology.

DEFINITION 1. *Fréchet (1906) Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^+$ is said to be a metric on X if the following conditions holds for all $x, y, z \in X$:*

($\mathcal{M}$1) $d(x, y) = 0$ *if and only if* $x = y$,

($\mathcal{M}$2) $d(x, y) = d(y, x)$, *and*

($\mathcal{M}$3) $d(x, z) \leq d(y, x) + d(y, z)$

A pair (X, d) satisfying the above assumptions is a metric space.

The concept of a partial metric space was first introduced by Matthews Mattews (1994), as defined below.

DEFINITION 2. *A mapping $p : X \times X \to \mathbb{R}^+$, where X is a nonempty set, is called a partial metric on X if, for any $x, y, z \in X$, the following conditions hold:*

$(\mathcal{P}1)$ $x = y$ *if and only if* $p(x, x) = p(y, y) = p(x, y)$,

$(\mathcal{P}2)$ $p(x, x) \leq p(x, y)$,

$(\mathcal{P}3)$ $p(x, y) = p(y, x)$, *and*

$(\mathcal{P}4)$ $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is then referred to as a partial metric space.

Recently, Amini-Harandi (Amini-Harandi, 2012) see also (Amini, 2012) proposed the concept of a metric-like space, which serves as a notable generalization of both partial metric spaces and dislocated metric spaces Aage & Salunke (2008); Sarma & Kumari (2012); Zoto & Hoxha (2012). This new framework extends the traditional notions of distance between elements in a space by relaxing some of the standard conditions of metric spaces. In particular, metric-like spaces allow for the possibility of non-zero values of $\sigma(x, x)$, unlike standard metric spaces where $\sigma(x, x)$ is always zero. This flexibility makes metric-like spaces a powerful tool for studying various types of convergence and continuity in more generalized settings.

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DEFINITION 3. *A function $\sigma : X \times X \to \mathbb{R}^+$, where X is a nonempty set, is called a* ***metric-like function*** *on X if it satisfies the following conditions for all $x, y, z \in X$:*

(σ_1) $\sigma(x, y) = 0$ implies $x = y$,

(σ_2) $\sigma(x, y) = \sigma(y, x)$ (symmetry), and

(σ_3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$ (triangle inequality).

The pair (X, σ) is referred to as a ***metric-like space****. A metric-like on X fulfills all the standard properties of a metric, with the exception that $\sigma(x, x)$ can take a positive value for $x \in X$. Every metric-like function σ on X defines a topology τ_σ on X, with the base given by the family of open σ-balls:*

$$B_\sigma(x, \varepsilon) = \{y \in X : |\sigma(x, y) - \sigma(x, x)| < \varepsilon\}, \quad \text{for all } x \in X \text{ and } \varepsilon > 0.$$

A sequence $\{x_n\}$ in the metric-like space (X, σ) converges to $x \in X$ if and only if $\lim_{n\to+\infty} \sigma(x_n, x) = \sigma(x, x)$.

Let (X, σ) and (Y, τ) be metric-like spaces, and let $\mathcal{G} : X \to Y$ be a continuous mapping. Then,

$$\lim_{n\to+\infty} x_n = x \text{ implies } \lim_{n\to+\infty} \mathcal{G}(x_n) = \mathcal{G}(x).$$

A sequence $\{x_n\}^+_{n=0}\infty \subseteq X$ is called σ-Cauchy if the limit $\lim_{m,n\to+\infty} \sigma(x_m, x_n)$ exists and is finite. The metric-like space (X, σ) is complete if every σ-Cauchy sequence $\{x_n\}^+_{n=0}\infty$ converges to some $x \in X$ such that

$$\lim_{n\to+\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{m,n\to+\infty} \sigma(x_m, x_n).$$

It is worth noting that every partial metric space is an example of a metric-like space.

EXAMPLE 1. Consider the set $X = \{0, 1\}$, and let the function $\sigma : X \times X \to \mathbb{R}^+$ be defined as:

$$\sigma(x, y) = \begin{cases} 2 & \text{if } x = y = 0, \\ 1 & \text{if } x \neq y. \end{cases}$$

In this case, the pair (X, σ) forms a metric-like space. However, since $\sigma(0, 0) \nleq \sigma(0, 1)$, we conclude that (X, σ) is not a partial metric space.

The notion of b-metric space was first introduced by Czerwik in his seminal work Czerwik (1993). This concept generalizes the traditional idea of a metric space by relaxing the triangle inequality. Specifically, in a b-metric space, the standard triangle inequality is modified by a constant factor $s \geq 1$.

DEFINITION 4. *A b-metric on a nonempty set X is a function $b : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$ and a constant $K \geq 1$, the following three conditions hold true:*

(b1) $b(x, y) = 0$ *if and only if* $x = y$,

(b2) $b(x, y) = b(y, x)$, *and*

(b3) $b(x, y) \leq s(b(x, z) + b(z, y))$.

The pair (X, b) is referred to as a ***b-metric space****.*

DEFINITION 5. *Alghamdi et al. (2013) A b-metric-like on a nonempty set X is a function $\mathcal{D} : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$ and a constant $s \geq 1$, the following three conditions hold true:*

$(\mathcal{D}1)$ $\mathcal{D}(x, y) = 0$ *implies* $x = y$,

$(\mathcal{D}2)$ $\mathcal{D}(x, y) = \mathcal{D}(y, x)$, *and*

$(\mathcal{D}3)$ $\mathcal{D}(x, y) \leq s(\mathcal{D}(x, z) + D(z, y))$.

The pair $(X, \mathcal{D})$ is referred to as a ***b-metric-like space****.*

EXAMPLE 2. Alghamdi et al. (2013); Chen et al. (2015)

1) Consider the set $X = [0, +\infty)$ and define a function $\mathcal{D} : X \times X \to [0, +\infty)$ by $\mathcal{D}(x, y) = (x + y)^2$. The pair $(X, \mathcal{D})$ forms a b-metric-like space with a constant $s = 2$. However, $(X, \mathcal{D})$ is neither a b-metric space nor a metric-like space. To verify, for any $x, y, z \in X$, we observe:

$$\mathcal{D}(x, y) = (x+y)^2 \leq (x+z+z+y)^2 = (x+z)^2+(z+y)^2+2(x+z)(z+y),$$

and since

$$(x + z)^2 + (z + y)^2 + 2(x + z)(z + y) \leq 2\big[(x + z)^2 + (z + y)^2\big],$$

it follows that

$$\mathcal{D}(x, y) \leq 2\big(\mathcal{D}(x, z) + \mathcal{D}(z, y)\big),$$

thus satisfying condition $(\mathcal{D}3)$. Additionally, conditions $(\mathcal{D}1)$ and $(\mathcal{D}2)$ are also satisfied.

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2) Let $X = [0, +\infty)$, and define the function $\mathcal{D} : X \times X \to [0, +\infty)$ by $\mathcal{D}(x, y) = \left[\max\{x, y\}\right]^2$. In this case, the pair $(X, \mathcal{D})$ forms a b-metric-like space with the constant $s = 2$. However, it is evident that $(X, \mathcal{D})$ does not qualify as either a b-metric space or a metric-like space.

3) Consider the set $X = \{0, 1, 2\}$ and define the function $\mathcal{D} : X \times X \to [0, +\infty)$ by:

$$\mathcal{D}(x, y) = \begin{cases} 2, & \text{if } x = y = 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

The pair $(X, \mathcal{D})$ forms a b-metric-like space with the constant $s = 2$.

4) Let $C_b(X) = \{\mathcal{G} : X \to \mathbb{R} : \sup_{x \in X} |\mathcal{G}(x)| < +\infty\}$. Define the function $\mathcal{D} : X \times X \to \mathbb{R}^+$ by:

$$\mathcal{D}(\mathcal{G}, \mathcal{H}) = \sqrt[3]{\sup_{x \in X} (|\mathcal{G}(x)| + |\mathcal{H}(x)|)^3}, \quad \text{for all } \mathcal{G}, \mathcal{H} \in C_b(X).$$

The function $\mathcal{D}$ satisfies the properties of a b-metric-like with the constant $s = \sqrt[3]{4}$, making $(X, \mathcal{D}, \sqrt[3]{4})$ a b-metric-like space.

To verify this, observe that for any two nonnegative real numbers e and c, the following inequalities hold:

$$(e + c)^3 \leq 4(e^3 + c^3) \quad \text{and} \quad \sqrt[3]{e + c} \leq \sqrt[3]{e} + \sqrt[3]{c}.$$

As a result, it can be deduced that:

$$\mathcal{D}(\mathcal{G}, \mathcal{H}) \leq \sqrt[3]{4} \cdot (\mathcal{D}(\mathcal{G}, \mathcal{L}) + D(\mathcal{L}, \mathcal{H})) \quad \text{for all } \mathcal{G}, \mathcal{H}, \mathcal{L} \in C_b(X).$$

DEFINITION 6. *Alghamdi et al. (2013)Let $(X, \mathcal{D})$ be a b-metric-like space, $x \in X$ and $r > 0$. The set*

$$B(x, r) = \{y \in X : |\mathcal{D}(x, y) - \mathcal{D}(x, x)| < r\}$$

is referred to as an open ball in the b-metric-like space $(X, \mathcal{D}, s)$, where x is the center and $r > 0$ represents the radius. Let $\{x_n\}$ be a sequence of points in X. The sequence $\{x_n\}$ is said to converge to a point $x \in X$ if and only if $\lim_{n \to +\infty} \mathcal{D}(x, x_n) = \mathcal{D}(x, x)$, and in this case, we denote this as $x_n \to x$ as $n \to +\infty$. A sequence $\{x_n\}$ is called Cauchy if and only if $\lim_{m,n \to +\infty} \mathcal{D}(x_n, x_m)$ exists and is finite. Furthermore, the b-metric-like *space $(X, \mathcal{D})$ is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$, satisfying:*

$$\lim_{m,n\to+\infty} \mathcal{D}(x_n, x_m) = \mathcal{D}(x, x) = \lim_{n\to+\infty} \mathcal{D}(x_n, x).$$

In 1962, Edelstein Edelstein (1962) introduced a notable modification of the Banach contraction principle. Subsequently, in 2009, Suzuki Suzuki (2009) expanded and refined the contributions of Banach and Edelstein, as highlighted in related works Salimi & karapinar (2013); Suzuki (2008). In recent years, the study of cyclic contractions and cyclic contractive mappings has gained significant traction in various mathematical investigations (Păcurar & Rus, 2010; Sintunavarat & Kumam, 2012; Petrusel, 2005; Radenović, 2016a,b). Building on these advancements, Alghamdi et al. Alghamdi et al. (2013) established fixed point theorems for cyclic Edelstein-Suzuki contractions, thereby generalizing the results of Edelstein Edelstein (1962), Suzuki Suzuki (2009), and Kirk et al. Fréchet (1906) within the framework of b-metric-like spaces.

THEOREM 1. *Alghamdi et al. (2013) Let $(X, \mathcal{D}, s)$ be a complete b-metric-like space, and let $\{A_j\}_{j=1}^m$ denote a family of nonempty closed subsets of X, where $Y = \bigcup_{j=1}^m A_j$. Suppose $\mathcal{G} : Y \to Y$ is a mapping that satisfies the condition*

$$\mathcal{G}(A_j) \subseteq A_{j+1}, \quad j = 1, 2, \ldots, m, \text{ with } A_{m+1} = A_1.$$

Furthermore, assume that for all $x \in A_i$ and $y \in A_{i+1}$, the following inequality holds:

$$\frac{1}{2s}\mathcal{D}(x, \mathcal{G}x) \le \mathcal{D}(x, y) \text{ implies}$$
$$\mathcal{D}(\mathcal{G}x, \mathcal{G}y) \le \alpha\frac{(s+1)}{s}\mathcal{D}(x, y) + \beta\big[\mathcal{D}(x, \mathcal{G}x) + \mathcal{D}(y, \mathcal{G}y)\big] + \gamma\left[\frac{\mathcal{D}(x, \mathcal{G}y) + \mathcal{D}(y, \mathcal{G}x)}{3s}\right] + \delta\left[\frac{\mathcal{D}(x, x) + \mathcal{D}(y, y)}{4s}\right]$$

where $\alpha, \beta, \gamma, \delta \ge 0$ and $\alpha + \beta + \gamma + \delta < \frac{1}{s+1}$. Under these conditions, the mapping $\mathcal{G}$ has a fixed point in $\bigcap_{j=1}^m A_j$.

Taking $\frac{\alpha(s+1)}{s} = \beta = \frac{\gamma}{3s} = \frac{\delta}{4s} = R$, in Theorem 1 leads to the derivation of the following corollary.

COROLLARY 1. *Alghamdi et al. (2013) Let $(X, \mathcal{D}, s)$ be a complete b-metric-like space, and let $\{A_j\}_{j=1}^m$ be a family of nonempty closed subsets of X with $Y = \bigcup_{j=1}^m A_j$. Let $\mathcal{G} : Y \to Y$ be a map satisfying $\mathcal{G}(A_j) \subseteq A_{j+1}$, $j = 1, 2, \ldots, m$, where $A_{m+1} = A_1$. Assume that*

$$\frac{1}{2s}\mathcal{D}(x, \mathcal{G}x) \leq \mathcal{D}(x, y)$$
$$\text{implies } \mathcal{D}(\mathcal{G}x, \mathcal{G}y) \leq R\big[\mathcal{D}(x, y) + \mathcal{D}(x, \mathcal{G}x) + \mathcal{D}(y, \mathcal{G}y) + \mathcal{D}(x, \mathcal{G}y) + \mathcal{D}(y, \mathcal{G}x) + \mathcal{D}(x, x) + \mathcal{D}(y, y)\big],$$

for all $x \in A_i$ and $y \in A_{i+1}$, where $0 \leq R < \frac{1}{(s+1)(7s+1)+s}$. Then $\mathcal{G}$ has a fixed point in $\bigcap_{j=1}^m A_j$.

In their work Chen et al. (2015), a set Φ is defined as the collection of functions $\phi : [0, +\infty) \to [0, +\infty)$ that satisfy two conditions: continuity and non-decreasing nature, along with $\phi(t) = 0$ if and only if $t = 0$. They then proceed to present the main results of the study.

THEOREM 2. *[Chen et al. Chen et al. (2015)] Consider $(X, \mathcal{D})$ as a complete b-metric-like space with a constant $s \geq 1$, and let $\mathcal{G} : X \to X$ be a map that satisfies the inequality*

$$\mathcal{D}(\mathcal{G}x, \mathcal{G}y) \leq \frac{\mathcal{D}(x, y)}{s} - \varphi(\mathcal{D}(x, y))$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then, $\mathcal{G}$ has a unique fixed point.

In Theorem 2, they showed that by choosing $\varphi(t) = \frac{t}{s} - \lambda t$ with $0 < \lambda < \frac{1}{s}$, the following corollary can be derived.

COROLLARY 2. *Chen et al. (2015) Let $(X, \mathcal{D})$ be a complete b-metric-like space with constant $s \geq 1$, and let $\mathcal{G} : X \to X$ be a mapping such that for all $x, y \in X$, the inequality*

$$\mathcal{D}(\mathcal{G}x, \mathcal{G}y) \leq \lambda \mathcal{D}(x, y) \quad \text{where} \quad 0 < \lambda < \frac{1}{s}$$

holds. Then, $\mathcal{G}$ has a unique fixed point in X.

They pointed out that by choosing $s = 1$ in Theorem 2, one can derive Theorem 2.7 from Amini-Harandi (2012). As an application of their result, the authors studied the existence of a solution to the integral equation

$$x(t) = \int_0^T K(t, r, x(r))\, dr, \tag{1}$$

where $K : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$. They proved the existence of a solution by considering $X = C[0, T]$, the space of continuous real functions on $[0, T]$, equipped with a b-metric-like structure defined by $\mathcal{D}(u, v) = \max_{t\in[0,T]} (|u(t)| + |v(t)|)^p$, with $p > 1$. The authors show that $(X, \mathcal{D})$ is a complete b-metric-like space with $s = 2^{p-1}$, and the existence of a fixed point for the mapping $f(x(t)) = \int_0^T K(t, r, x(r))\, dr$ ensures the existence of a solution to the equation.

THEOREM 3. *Chen et al. (2015) Let $K : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume there exists a continuous $\xi : [0, T] \times [0, T] \to \mathbb{R}$ such that for all $t, r \in [0, T]$:*

$$|K(t, r, x(r))| + |K(t, r, y(r))| \leq \lambda^{\frac{1}{p}} \xi(t, r) \left(|x(r)| + |y(r)|\right),$$

with $0 < \lambda < \frac{1}{s}$, and

$$\sup_{t\in[0,T]} \int_0^T \xi(t, r)\, dr \leq 1.$$

Then the integral equation (1) has a unique solution $x \in X$.

In order to present a new contractive condition, Jain et al. Jain & Kaur (2021) introduced the following new class of functions Ξ_m for any $m \in \mathbb{N}$ as the set of functions $\xi : [0, +\infty)^m \to [0, +\infty)$ that satisfy the conditions:

- $(\xi 1)$ $\xi(t_1, t_2, \ldots, t_m) < \max\{t_1, t_2, \ldots, t_m\}$ if $(t_1, t_2, \ldots, t_m) \neq (0, 0, \ldots, 0)$;
- $(\xi 2)$ If $\{t_i^{(n)}\}_{n\in\mathbb{N}}$, for $1 \leq i \leq m$, are sequences in $[0, +\infty)$ such that $\limsup_{n\to+\infty} t_i^{(n)} = t_i < +\infty$ for all $i = 1, \ldots, m$, then

$$\liminf_{n\to+\infty} \xi(t_1^{(n)}, t_2^{(n)}, \ldots, t_m^{(n)}) \leq \xi(t_1, t_2, \ldots, t_m).$$

Jain et al. Jain & Kaur (2021) proved the following result in a b-metric-like space.

THEOREM 4. *Let* $(X, \mathcal{D}, s \geq 1)$ *be a complete b-metric-like space, and let* $\mathcal{G} : X \to X$ *be a mapping such that there exists* $\xi \in \Xi_4$ *and*

$$\mathcal{D}(\mathcal{G}x, \mathcal{G}y) \leq \frac{1}{s}\xi\left(\mathcal{D}(x,y), \mathcal{D}(x,\mathcal{G}x), \mathcal{D}(y,\mathcal{G}y), \frac{\mathcal{D}(x,\mathcal{G}y) + \mathcal{D}(\mathcal{G}x,y) - \mathcal{D}(y,y)}{2s}\right)$$

for all $x, y \in X$ *with* $\mathcal{D}(x,\mathcal{G}y) + \mathcal{D}(\mathcal{G}x,y) \geq \mathcal{D}(y,y)$. *Then,* $\mathcal{G}$ *has a unique fixed point.*

Nawab et al. Hussain et al. (2014) extended the fixed point theory to partially ordered b-metric-like spaces. Let $(X, \preceq)$ be a partially ordered set, and $(X, \mathcal{D})$ a b-metric-like space, which we refer to as a partially ordered b-metric-like space.

THEOREM 5 (Nawab et al. Hussain et al. (2014)). *Consider a nondecreasing map* $\mathcal{G} : X \to X$ *such that for all* $x, y \in X$ *with* $x \preceq y$,

$$\mathcal{D}(\mathcal{G}(x), \mathcal{G}(y)) \leq \alpha M(x,y),$$

where $\alpha \in [0, \frac{1}{2s^2})$ *and*

$$M(x,y) = \max\left\{\mathcal{D}(x,y), \mathcal{D}(x,\mathcal{G}(x)), \mathcal{D}(y,\mathcal{G}(y)), \mathcal{D}(x,\mathcal{G}(y)), \mathcal{D}(y,\mathcal{G}(x))\right\}.$$

Under these conditions, the fixed point set $F(\mathcal{G})$ *is nonempty if there exists* $x_0 \in (LF)_\mathcal{G}$, *where* $(LF)_\mathcal{G} = \{x \in X : x \preceq \mathcal{G}(x)\}$, *and one of the following holds: (a)* $\mathcal{G}$ *is continuous, or (b)* $(X, \preceq, \mathcal{D})$ *satisfies the sequential limit comparison property. Additionally,* $\mathcal{G}$ *has a unique fixed point if and only if every pair of fixed points is comparable.*

Nawab et al. Hussain et al. (2014) studied fixed point results for nondecreasing mappings in the framework of partially ordered b-metric-like spaces. Prior to presenting their main theorem, they introduced the concept of the *sequential limit comparison property*:

An ordered b-metric-like space $(X, \preceq, \mathcal{D})$ is said to satisfy the sequential limit comparison property if, for any nondecreasing (or nonincreasing) sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, the convergence $x_n \to x$ implies $x_n \preceq x$ (or $x \preceq x_n$) for all $n \in \mathbb{N}$.

Their main result is stated as follows:

THEOREM 6. *Hussain et al. (2014) Let* $(X, \preceq, \mathcal{D})$ *be a partially ordered complete* b*-metric-like space with a coefficient* $s > 1$. *If a nondecreasing map*

$\mathcal{G} : X \to X$ satisfies

$$s\mathcal{D}(\mathcal{G}(x), \mathcal{G}(y)) \leq \mathcal{M}(x, y) - \varphi(m(x, y)), \tag{2}$$

for all $x, y \in X$ with $x \preceq y$, where

$$\mathcal{M}(x, y) = \frac{\mathcal{D}(x, \mathcal{G}(y)) + \mathcal{D}(y, \mathcal{G}(x))}{4s},$$

$$m(x, y) = \min\Big\{\mathcal{D}(x, \mathcal{G}(y)), \mathcal{D}(y, \mathcal{G}(x))\Big\},$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$, then the fixed point set $F(\mathcal{G})$ is nonempty, provided there exists $x_0 \in (LF)_{\mathcal{G}}$ and one of the following conditions holds:

(i) *$\mathcal{G}$ is continuous.*

(ii) *$(X, \preceq, \mathcal{D})$ satisfies the sequential limit comparison property.*

Furthermore, $\mathcal{G}$ has a unique fixed point if and only if the fixed points of $\mathcal{G}$ are comparable.

In their work, Zoto et al. Zoto et al. (2018) introduced the concept of generalized (s, p, α)-contractions and obtained fixed point theorems for this class of contractions in the framework of b-metric-like spaces. Specifically, they define a self-mapping $\mathcal{G} : X \to X$ on a complete b-metric-like space $(X, \mathcal{D})$ with parameter $s \geq 1$ as an (s, α)-Banach contraction if it satisfies the condition

$$s\mathcal{D}(\mathcal{G}x, \mathcal{G}y) \leq \alpha\mathcal{D}(x, y)$$

for some $\alpha \in [0, 1)$ and all $x, y \in X$.

Based on Círic's quasi-contractions, Zoto et al. Zoto et al. (2018) introduced the concept of generalized (ψ, s, p, α)-contractions in b-metric-like spaces. Let Ψ and Φ denote the families of altering distance functions satisfying the following conditions, respectively:

Ψ : $[0, +\infty) \to [0, +\infty)$ is an increasing and continuous function, and $\Psi(t) = 0$ if and only if $t = 0$.

Φ : $[0, +\infty) \to [0, +\infty)$ is a lower semicontinuous function, and $\Phi(t) = 0$ if and only if $t = 0$.

Using these families, they defined (ψ, s, p, α)-Ćirić-type quasi-contractions as follows:

DEFINITION 7. *Zoto et al. (2018) Let $(X, \mathcal{D})$ be a b-metric-like space with the parameter $s \geq 1$. Consider $\psi \in \Psi$, and let α and p be constants such that $0 \leq \alpha < 1$ and $p \geq 2$. A mapping $\mathcal{G} : X \to X$ is called a (ψ, s, p, α)-Ćirić-type quasi-contraction if, for all $x, y \in X$, the following inequality holds:*

$$\psi^{2sp}(\mathcal{D}(\mathcal{G}x, \mathcal{G}y)) \leq \alpha\psi\Big(\max\Big\{\mathcal{D}(x,y), \mathcal{D}(x,\mathcal{G}x), \mathcal{D}(y,\mathcal{G}y), \mathcal{D}(x,\mathcal{G}y), \mathcal{D}(y,\mathcal{G}x)\Big\}\Big).$$

REMARK 1. The authors also illustrated that these generalized contractions extend and unify several well-known results in the fixed point theory of metric-like spaces.

1. By taking $\psi(t) = \frac{1}{2}t$ (or the identity mapping $\psi(t) = t$), the notion of a (ψ, s, p, α)-Ćirić-type quasi-contraction reduces to an (s, p, α)-quasi-contraction.

2. For $\psi(t) = \frac{1}{2}t$ and $p = 2$, the definition simplifies to the (s, α)-quasi-contraction presented in Ćirić (1974).

3. Setting $s = 1$ corresponds to the special case of metric-like spaces.

THEOREM 7. *[Kastriot Zoto al. Zoto et al. (2018)] Let $(X, \mathcal{D})$ be a complete b-metric-like space with the parameter $s \geq 1$, and let $f : X \to X$ be a given self-mapping. If f is a (ψ, s, p, α)-quasicontraction, then f has a unique fixed point.*

The following corollary, derived by Zoto et al. Zoto et al. (2018), provides a version of the Hardy-Rogers result in Gordji & Rogers (1973).

COROLLARY 3. *Let $(X, \mathcal{D})$ be a complete b-metric-like space with the parameter $s \geq 1$. If $\mathcal{G} : X \to X$ is a self-mapping and there exist $p \geq 2$ and constants $a_i \geq 0$, $i = 1, \ldots, 5$, such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and*

$$s^p\mathcal{D}(\mathcal{G}x, \mathcal{G}y) \leq \alpha_1\mathcal{D}(x,y)+\alpha_2\mathcal{D}(x,\mathcal{G}x)+\alpha_3\mathcal{D}(y,\mathcal{G}y)+\alpha_4\mathcal{D}(x,\mathcal{G}y)+\alpha_5\mathcal{D}(y,\mathcal{G}x),$$

for all $x, y \in X$, then $\mathcal{G}$ has a unique fixed point in X.

In addition to the results already discussed, Zoto et al. also derived several corollaries as immediate consequences of Theorem 7. These corollaries deal with self-maps that satisfy contractive conditions expressed through rational expressions, utilizing the functions $\psi \in \Psi$ and $\phi \in \Phi$.

COROLLARY 4. *Let $(X, \mathcal{D})$ be a complete b-metric-like space with parameter $s \geq 1$, and let $\mathcal{G} : X \to X$ be a self-map. If there exist $\psi \in \Psi$, $0 \leq \alpha < \frac{1}{2}$, and $p \geq 2$, such that the condition*

$$\psi\left(2sp\,\mathcal{D}(\mathcal{G}x, \mathcal{G}y)\right) \leq \alpha \frac{\psi(M(x,y))}{1+\psi(M(x,y))}$$

is satisfied for all $x, y \in X$, where $M(x,y)$ is defined as in (2), then $\mathcal{G}$ has a unique fixed point in X.

Fabiano et al. Fabiano et al. (2020) studied cyclic (s, q)-Dass-Gupta-Jaggi type contractions in b-metric-like spaces, proving that a Picard sequence is Cauchy in this setting, thus extending and refining the existing results. They further established the equivalence between the cyclic results of Kirk et al. and the standard fixed point results for such contractions.

Before stating the main result, let us recall the notions of η-admissibility and η-continuity. A mapping $\mathcal{G} : X \to X$ is called η-admissible if $\eta(x,y) \geq 1$ implies $\eta(\mathcal{G}(x), \mathcal{G}(y)) \geq 1$ for all $x, y \in X$. Moreover, $\mathcal{G}$ is said to be η-continuous on a b-metric-like space $(X, \mathcal{D})$ if, for any sequence $\{z_n\}$ in X satisfying $\lim\limits_{n\to+\infty} z_n = z$ and $\eta(z_n, z_{n+1}) \geq 1$ for all n, it follows that $\lim\limits_{n\to+\infty} \mathcal{G}(z_n) = \mathcal{G}(z)$.

THEOREM 8. *Fabiano et al. (2020) Let $(X, \mathcal{D})$ be a $\mathcal{D}$-complete b-metric-like space, and let $\eta : X \times X \to [0, +\infty)$ and $\mathcal{G} : X \to X$ be the given mappings. Suppose that for all $x, y \in X$ with $\eta(x, \mathcal{G}x)\eta(y, \mathcal{G}y) \geq 1$ implies*

$$2s^q\mathcal{D}(\mathcal{G}x, \mathcal{G}y) \leq \phi(N(x,y))$$

where $q > 1$, $\phi : [0, +\infty) \to [0, +\infty)$ is a is non-decreasing and continuous function satisfying $\lim_{n\to+\infty} \phi^n(t) = 0$, and

$$N(x,y) = \max\left\{\mathcal{D}(x,y), \mathcal{D}(y,\mathcal{G}x), \frac{\mathcal{D}(x,y)\mathcal{D}(y,\mathcal{G}y)}{1+\mathcal{D}(x,\mathcal{G}x)}, \frac{\mathcal{D}(y,\mathcal{G}y)[1+\mathcal{D}(x,\mathcal{G}x)]}{1+\mathcal{D}(x,y)}, \frac{\mathcal{D}(x,\mathcal{G}y)+\mathcal{D}(y,\mathcal{G}x)}{4s}\right\}.$$

Assume that the mapping $\mathcal{G} : X \to X$ satisfies the following conditions:

1. *$\mathcal{G}$ is η-admissible.*
2. *There exists $z_0 \in X$ such that $\eta(z_0, \mathcal{G}(z_0)) \geq 1$.*
3. *Either $\mathcal{G}$ is η-continuous, or for any sequence $\{z_n\}$ in X such that for all $n \geq 0$, $\eta(z_n, z_{n+1}) \geq 1$ and $\lim_{n\to+\infty} z_n = z$, we have $\eta(z, \mathcal{G}(z)) \geq 1$.*

Then $\mathcal{G}$ has a fixed point $z \in X$. Furthermore, if:

4. *For all $z \in F(\mathcal{G}) = \{a \in X : \mathcal{G}(a) = a\}$, we have $\eta(z, z) \geq 1$,*

then $\mathcal{G}$ has a unique fixed point in X.

An example is given below to support Theorem 8.

EXAMPLE 3. Fabiano et al. (2020) Consider $X = [0, 1]$ equipped with the b-metric-like function $\mathcal{D}(x, y) = (x + y)^2$ for all $x, y \in X$, with the parameters $s = 2$ and $q > 1$. Let $\mathcal{G} : X \to X$ and $\eta : X \times X \to [0, +\infty)$ be defined as follows:

$$\mathcal{G}x = \frac{x}{6}\ ,\ \eta(x, y) = \begin{cases} 1, & \text{if } x = y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Additionally, let $\phi : [0, +\infty) \to [0, +\infty)$ be given by $\phi(t) = t^2$. For the pair $(x, y) = (0, 0)$, we observe that $\eta(x, \mathcal{G}x)\eta(y, \mathcal{G}y) \geq 1$ holds only when $x = y = 0$. Now, for this specific pair, we calculate $2s^q\mathcal{D}(\mathcal{G}(0), \mathcal{G}(0)) = 0$, while the expression for φ yields

$$\begin{aligned}
&\phi\Bigg(\max\Bigg\{\mathcal{D}(0,0), \mathcal{D}(0,\mathcal{G}0), \frac{\mathcal{D}(0,0)\cdot\mathcal{D}(0,\mathcal{G}0)}{1+\mathcal{D}(0,\mathcal{G}0)}, \\
&\frac{\mathcal{D}(0,\mathcal{G}0)[1+\mathcal{D}(0,\mathcal{G}0)]}{1+\mathcal{D}(0,0)}, \frac{\mathcal{D}(0,\mathcal{G}0)+\mathcal{D}(0,\mathcal{G}0)}{4\cdot 2}\Bigg\}\Bigg) \\
&= \frac{1}{2}\max\Bigg\{\mathcal{D}(0,0), \mathcal{D}(0,\mathcal{G}0), \frac{\mathcal{D}(0,0)\cdot\mathcal{D}(0,\mathcal{G}0)}{1+\mathcal{D}(0,\mathcal{G}0)}, \\
&\frac{\mathcal{D}(0,\mathcal{G}0)[1+\mathcal{D}(0,\mathcal{G}0)]}{1+\mathcal{D}(0,0)}, \frac{\mathcal{D}(0,\mathcal{G}0)+\mathcal{D}(0,\mathcal{G}0)}{8}\Bigg\} \\
&= \frac{1}{2}\max\left\{0, 0, \frac{0\cdot 0}{1+0}, \frac{0\cdot(1+0)}{1+0}, \frac{0+0}{8}\right\} \\
&= \frac{1}{2}\times 0 = 0.
\end{aligned}$$

Thus, this example demonstrates the applicability of Theorem 8; in particular, it confirms that $z = 0$ is the unique fixed point of $\mathcal{G}$.

Mirkov et al. Mirkov et al. (1994) proposed a simplified and concise proof of Banach's contraction principle using Palais's method within the generalized framework of b-metric-like spaces. Their result extends existing findings in the literature by demonstrating that the Lipschitz constant in Banach's contraction principle can belong to the entire interval $[0, 1)$ across six types of spaces: metric spaces, b-metric spaces, partial metric spaces, partial b-metric spaces, metric-like spaces, and the broader b-metric-like spaces.

First, using Palais's method, they obtained the following result.

LEMMA 1 (Mirkov et al. Mirkov et al. (1994)). *Let $(X, \mathcal{D})$ be a b-metric-like space with $s \geq 1$ and $\{x_n\}$ be a sequence in X such that*

$$\mathcal{D}(x_m, x_n) \leq k\mathcal{D}(x_{m-1}, x_{n-1}),$$

for all $m, n \in \mathbb{N}$, where $k \in [0, 1)$. Then the following inequality holds:

$$\mathcal{D}(x_m, x_n) \leq s\,(k^m + k^n)\,\mathcal{D}(x_0, x_{n_0})/\,(1 - k^{n_0}s)\,,$$

where $n_0 \in \mathbb{N}$ such that $k^{n_0}s < 1$.

THEOREM 9. *Mirkov et al. (1994) Consider a complete b-metric-like space $(X, \mathcal{D})$ with $s \geq 1$. If a mapping $\mathcal{G} : X \to X$ satisfies the condition*

$$\mathcal{D}(\mathcal{G}(x), \mathcal{G}(y)) \leq k\mathcal{D}(x, y),$$

for all $x, y \in X$, where $k \in [0, 1)$, then $\mathcal{G}$ admits a unique fixed point in X.

In recent years, extensive research has been conducted on the generalization of Geraghty contractions Karapinar (2014, et al.); Aydi et al. (2017). In 2017, Fulga et al. Fluga & Proca (2017) introduced the notion of φ_E-Geraghty contractions and established a corresponding fixed point theorem in complete metric spaces. Subsequently, this concept was explored in metric-like spaces Aydi et al. (2017), leading to the following result:

THEOREM 10. *Aydi et al. (2017) Let (X, σ) be a complete metric-like space, and let $\mathcal{G} : X \to X$ be a self-mapping. Suppose there exists $\beta \in \mathbb{B}$ such* *that*

$$\sigma(\mathcal{G}x, \mathcal{G}y) \leq \beta(F(x,y))F(x,y), \quad \forall x, y \in X,$$

where

$$F(x,y) = \sigma(x,y) + |\sigma(x, \mathcal{G}x) - \sigma(y, \mathcal{G}y)|.$$

Then, $\mathcal{G}$ has a unique fixed point.

Note that; $\mathbb{B}$ consists of all functions $\beta : [0, +\infty) \to [0, 1)$ such that:

$$\lim_{n\to+\infty} \beta(t_n) = 1 \quad \text{implies} \quad \lim_{n\to+\infty} t_n = 0.$$

Dakun Yu et al. Yu et al. (2018) initiated the concept of $(T, g)_F$-contractions of Geraghty type and studied common fixed point theorems for these contractions in b-metric-like spaces. We recall that, according to Abbas et al. Abbas & Jungck (2008), the following definition is given: Let X be a nonempty set, and let f and g be self-mappings on X. The set of common points of f and g is defined as $C(f, g) = \{x \in X : fx = gx\}$. The pair (f, g) is said to be weakly compatible if $fgx = gfx$ for all $x \in C(f, g)$. Moreover, if there exists $x \in X$ such that $fx = gx = w$, then x is called a coincidence point of f and g, and w is referred to as a ***point of coincidence*** of f and g.

Definition 8. *Yu et al. (2018) Let $(X, \mathcal{D})$ be a b-metric-like space with the coefficient $s \geq 1$, and let $T, g : X \to X$ be two mappings. The pair (T, g) is said to be a $(T, g)_F$-contraction of Geraghty type if there exists $\beta \in \mathbb{B}$ such that*

$$\mathcal{D}(Tx, Ty) \leq \beta(F_g(x,y))F_g(x,y)$$

for all $x, y \in X$, where $F_g(x,y) = \frac{1}{s^2}\left[\mathcal{D}(gx, gy) + |\mathcal{D}(gx, Tx) - \mathcal{D}(gy, Ty)|\right]$.

Theorem 11. *Yu et al. (2018) Let $(X, \mathcal{D})$ be a b-metric-like space with the coefficient $s \geq 1$, and let $T, g : X \times X \to X$ be two mappings such that $T(X) \subseteq g(X)$ and $g(X)$ is complete. If the pair (T, g) is a $(T, g)_F$-contraction of Geraghty type, then T and g have a unique point of coincidence. Furthermore, if T and g are weakly compatible, then they have a unique common fixed point.*

By specializing Theorem 11 to the case $g = I_X$ (the identity mapping), they obtained the following result.

COROLLARY 5. *Yu et al. (2018) Let $(X, \mathcal{D})$ be a complete b-metric-like space with the coefficient $s \geq 1$, and let $T : X \to X$ be a mapping. If there exists $\beta \in \mathbb{B}$ such that*

$$\mathcal{D}(Tx, Ty) \leq \beta(F(x, y))F(x, y)$$

for all $x, y \in X$, where $F(x, y) = \frac{1}{s^2}\left[\mathcal{D}(x, y) + |\mathcal{D}(x, Tx) - \mathcal{D}(y, Ty)|\right]$, then T has a unique fixed point.

As an application of the previous corollary, Dakun Yu et al. Yu et al. (2018) investigated the integral equation $(E) : x(t) = \int_0^1 K(t, r, x(r))\, dr$, where $K : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$. They considered the function space $X = C[0, 1]$ with the b-metric-like: $\mathcal{D}(u, v) = \max_{t \in [0,1]}(|u(t)| + |v(t)|)^p, \quad p \geq 1$, which forms a complete b-metric-like space with $s = 2^{p-1}$. By defining $f(x)(t) = \int_0^1 K(t, r, x(r))\, dr$, they established the existence of a solution as a direct consequence of the corollary.

THEOREM 12. *Yu et al. (2018) Let $K : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose there exists a continuous function $\xi : [0, 1] \times [0, 1] \to [0, +\infty)$ such that*

$$|K(t, r, x(r))| \leq \xi(t, r)|x(r)|.$$

If there exists $\beta \in \mathbb{B}$ satisfying

$$\sup_{t \in [0,1]} \int_0^1 \xi(t, r)dr \leq \sqrt[p]{\frac{1}{2^{2p-2}}\beta\left(\frac{1}{2^{2p-2}}(\|x - y\|_\infty + |\|x - fx\|_\infty - \|y - fy\|_\infty|)\right)}$$

then (E) has a unique solution $x \in X$.

Best proximity theory provides a framework for approximating solutions in situations where exact fixed points do not exist. Suppose U and V are two closed, nonempty subsets of a space X such that $U \cap V = \emptyset$, and let $\mathcal{G} : U \to V$ be a mapping. Since the equation $\mathcal{G}(x) = x$ has no solution, one instead seeks a point $x \in U$ that is closest to $\mathcal{G}(x) \in V$. This point is known as a *best proximity point* of $\mathcal{G}$.

The study of best proximity points initially focused on classical metric spaces before being extended to more generalized settings (see, for instance, Gardašević et al. (2023); Fallahi & Eivani (2022); Abkar & Gabeleh (2013); Aydi et al. (2020); Choudhury et al. (2015); Mongkolkeha et al. (2013)). A notable development in this area was introduced by Gordji et al. Gordji & Habibi (2017); Gordji et al. (2017), who explored the role of orthogonal sets and introduced essential concepts such as O-sequences, O-Cauchy sequences, O-continuity, O-contractions, O-preserving properties, and the P-property. Their work established best proximity point results under more generalized contractive conditions. Subsequent research has built upon these ideas, particularly in the context of orthogonality Fallahi & Eivani (2022); Mongkolkeha et al. (2013); Sawangsup & Sintunavara (2020); Garakoti et al. (2014); Yang et al. (2020), leading to significant progress in this field.

Milanka et al. Gardašević et al. (in press) addressed the open question raised in Gardašević et al. (2023) by introducing the assumption that the set X is $\perp$-transitive. Before delving into their result, we will first recall some definitions pertinent to the context of orthogonal metrics and best proximity theorems.

DEFINITION 9. *Gordji et al. (2017) Let X be a nonempty set and $\perp \subset X \times X$ be a binary relation. If there exists an element $x_0 \in X$ such that for all $y \in X$, either $y \perp x_0$ or $x_0 \perp y$, then the pair $(X, \perp)$ is called an orthogonal set (or O-set), and x_0 is referred to as the orthogonal element.*

If X is equipped with a metric d, then the pair $(X, \perp, d)$ is referred to as an orthogonal metric space. Orthogonal partial metric spaces, orthogonal b-metric spaces, and other generalized orthogonal spaces are similarly defined. For additional examples of these concepts, refer to Fallahi & Eivani (2022); Gardašević et al. (2023); Gordji & Habibi (2017); Gordji et al. (2017); Javed (et al.); Yamaod & Sintunavarat (2018).

The next example provides insight into the practical use of an orthogonal set.

EXAMPLE 4. Gordji et al. (2017) Let X be the set of all people in the world. We define $x \perp y$ if x can give blood to y. According to the following table, if x_0 is a person with blood type O$^-$, then there exists $x_0 \perp y$ for all $y \in X$.

This means that $(X, \perp)$ is an O^{-}-set, and x_0 is not unique.

Blood type	Can donate to	Can receive from
$A+$	$A+ \quad AB+$	$A+ \quad A- \quad O+ \quad O-$
$O+$	$O+ \quad A+ \quad B+ \quad AB+$	$O+ \quad O-$
$B+$	$B+ \quad AB+$	$B+ \quad B- \quad O+ \quad O-$
$AB+$	$AB+$	$everyone$
$A-$	$A+ \quad A- \quad AB+ \quad AB-$	$A- \quad O-$
$O-$	$everyone$	$O-$
$B-$	$B+ \quad B- \quad AB+ \quad AB-$	$B- \quad O-$
$AB-$	$AB+ \quad AB-$	$AB- \quad B- \quad O- \quad A-$

DEFINITION 10. *Gordji et al. (2017) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an orthogonal sequence (or O-sequence) if for all $n \in \mathbb{N}$, the following condition holds: $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$.*

DEFINITION 11. *Gordji & Habibi (2017) A function $\mathcal{G} : X \to X$ is said to be orthogonal-preserving (or $\perp$-preserving) if for all $x, y \in X$ such that $x \perp y$, it follows that $\mathcal{G}x \perp \mathcal{G}y$.*

DEFINITION 12. *Gardašević et al. (in press) An orthogonal set X is said to be $\perp$-transitive if the following condition holds: for all $x, y, z \in X$, if $(y \perp x$ or $x \perp y)$ and $(y \perp z$ or $z \perp y)$, then it follows that $(x \perp z$ or $z \perp x)$.*

DEFINITION 13. *Gordji & Habibi (2017) Let $(X, \perp, d)$ be an orthogonal metric space, which means $(X, \perp)$ is an O-set and (X, d) is a metric space. A function $\mathcal{G} : X \to X$ is called orthogonal continuous (or $\perp$-continuous) at a point $x \in X$ if, for every O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in X satisfying $x_n \to x$ as $n \to +\infty$, it follows that $\mathcal{G}(x_n) \to \mathcal{G}(x)$ as $n \to +\infty$.*

Note that, every continuous mapping is a $\perp$-continuous mapping; however, the converse does not necessarily hold. An example illustrating this can be found in Gordji & Habibi (2017).

DEFINITION 14. *Gordji et al. (2017) Let $(X, \perp, \mathcal{D})$ be an orthogonal metric space, and let $\lambda \in [0, 1)$. A function $\mathcal{G} : X \to X$ is called an orthogonal contraction (or $\perp$-contraction) with the Lipschitz constant λ if, for all $x, y \in X$ satisfying $x \perp y$, the following inequality holds:*

$$\mathcal{D}(\mathcal{G}x, \mathcal{G}y) \leq \lambda \mathcal{D}(x, y).$$

REMARK 2. Gordji et al. (2017) Every contraction mapping is a $\perp$-contraction mapping, but the converse is not true. For an example, see Gordji et al. (2017).

Let (X, d) be a metric space, and let $U, V \subset X$ be two nonempty subsets. The distance between U and V is defined as

$$D(U,V) = \inf\{d(u,v) \mid u \in U, v \in V\}.$$

A point $u \in U$ is called a best proximity point of a non-self mapping $\mathcal{G} : U \to V$ if it satisfies

$$d(u, \mathcal{G}u) = D(U,V).$$

For arbitrary sets $U, V \subset X$ such that $U, V \neq \emptyset$, we define

$$U_0 = \{u \in U \mid d(u,v) = D(U,V) \text{ for some } v \in V\},$$

$$V_0 = \{v \in V \mid d(u,v) = D(U,V) \text{ for some } u \in U\}.$$

Let (X, d) be a metric space, and let $U, V \subset X$ be nonempty subsets such that $U_0 \neq \emptyset$. The pair (U, V) is said to have the P-property if and only if

$$\begin{cases} d(u_1, v_1) = D(U,V) \\ d(u_2, v_2) = D(U,V) \end{cases} \text{ implies } d(u_1, u_2) = d(v_1, v_2),$$

for all $u_1, u_2 \in U_0$ and $v_1, v_2 \in V_0$.

Considering that $d(x, \mathcal{G}x) \geq D(U,V)$ for all $x \in U$, it follows that the global minimum of the function $x \mapsto d(x, \mathcal{G}x)$ is attained at a best proximity point. Furthermore, it is straightforward to observe that a best proximity point coincides with a fixed point when the mapping $\mathcal{G}$ is a self-mapping.

THEOREM 13. *Gardašević et al. (in press) Let $(X, \perp, \mathcal{D}, s)$ be an orthogonal 0-$\mathcal{D}$-complete b-metric-like space, and let (U, V) be a pair of two non-empty closed subsets of X with an empty intersection, satisfying the P-property. Assume that X is $\perp$-transitive. Let $\mathcal{G} : U \to V$ be a $\perp$-continuous, $\perp$-preserving, and $\perp$-contractive mapping with the Lipschitz constant $k \in (0,1)$. Suppose that $\mathcal{G}$ satisfies the following conditions:*

(i) *$\mathcal{G}(U_0) \subset V_0$;*

(ii) *There exist $x_0, x_1 \in U_0$ such that $x_0 \perp x_1$ and $\mathcal{D}(x_1, \mathcal{G}x_0) = D(U,V)$, where $D(U,V) = \inf\{\mathcal{D}(x,y) \mid x \in U, y \in V\}$.*

Then $\mathcal{G}$ has a unique best proximity point $x' \in U$, i.e., $\mathcal{D}(x', \mathcal{G}x') = D(U, V)$.

In papers [36] and [37], the issues of best proximity results in orthogonal 0-complete b-metric-like spaces were considered. For proving the results in these works, a sequence $\{x_n\}$, where $n = 0, 1, 2, \ldots$, is constructed such that the following relation holds:

$$\mathcal{D}(x_n, x_{n+1}) \leq \lambda \mathcal{D}(x_{n-1}, x_n) \quad \text{for} \quad n = 1, 2, \ldots$$

where λ belongs to the interval $[0, 1/s)$ with $s \geq 1$. Based on this relation, it is routinely proved that the constructed sequence $\{x_n\}$ is Cauchy. The open problem raised in these papers was whether the obtained results are still valid if λ belongs to the interval $[0, 1)$.

By utilizing the following result, we provide an affirmative answer to the question posed in the paper. Gardašević et al. (in press, 2023).

LEMMA 2. *Let $(X, D, s > 1)$ be a b-metric-like space, and let $\{x_n\}$ be a sequence in X satisfying the following condition: There exists $\lambda \in [0, 1)$ such that*

$$\mathcal{D}(x_n, x_{n+1}) \leq \lambda \mathcal{D}(x_{n-1}, x_n), \quad \textit{for all } n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ is Cauchy.

Proof. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, the proof is complete. Otherwise, assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. We consider two cases:

Case 1: $\lambda \in \left[0, \frac{1}{s}\right)$.

Case 2: $\lambda \in \left[\frac{1}{s}, 1\right)$.

The proof for the first case follows similarly to the arguments presented in Gardašević et al. (in press, 2023). Now, consider the second case. Since $\lambda^n \to 0$ as $n \to +\infty$, there exists $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{s}$. Define the mapping $\mathcal{G}$ on the sequence $\{x_n\}, n \in \mathbb{N}$, by

$$\mathcal{G}(x_n) = x_{n+1}, \quad \text{for all } n = 0, 1, 2, \ldots$$

It follows that $\mathcal{G}^k(x_n) = x_{k+n}$ for all $n \in \mathbb{N}$ and

$$\mathcal{D}(\mathcal{G}^k x_{n-1}, \mathcal{G}^k x_n) \leq \lambda^k \mathcal{D}(x_{n-1}, x_n).$$

Proceeding analogously as in Gardašević et al. (in press, 2023) for the first case, we establish that the sequence $\{x_{k+n}\}$ is Cauchy. Specifically, if

$n < m$, it is shown that

$$\mathcal{D}(x_{k+n}, x_{k+m}) \to 0 \quad \text{as } n, m \to +\infty.$$

Since $\{x_n\} = \{x_0, x_1, \ldots, x_{k-1}\} \cup \{x_k, x_{k+1}, \ldots, x_{k+n}, \ldots\}$, it follows that the sequence $\{x_n\}$, $n = 0, 1, 2, \ldots$, is Cauchy. □

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Espacios b-métricos: Una revisión de conceptos y aplicaciones

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CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo de revisión

Este artículo ofrece una revisión exhaustiva de los resultados de puntos fijos en espacios métricos-like, con un enfoque particular en los espacios b-métrico-like. La noción de espacios b-métrico-like generaliza varios conceptos importantes, incluidos los espacios métricos parciales, los espacios métricos-like y los espacios b-métricos. El artículo destaca contribuciones significativas desde las primeras etapas de la investigación hasta el presente, ofreciendo una compilación extensa de resultados fundamentales y recientes. Finalmente, proporcionamos respuestas afirmativas a algunos problemas abiertos derivados de los resultados de proximidad óptima en espacios b-métrico-like ortogonales 0-completos, avanzando aún más en la comprensión de la teoría de puntos fijos en estos entornos generalizados. Esta revisión tiene como objetivo proporcionar a los investigadores valiosas ideas y referencias, facilitando la exploración y el desarrollo futuros en el estudio de los espacios métricos-like y sus generalizaciones.

Espacios b-métrico-like, Espacios métricos, Espacios métricos parciales, Espacios b-métricos, Espacios métricos-like, Teoría de puntos fijos.

Обзор концепций и приложений пространств b-метрик

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РУБРИКА ГРНТИ: 36.00.00 ГЕОДЕЗИЯ. КАРТОГРАФИЯ;
36.29.00 Топография. Фототопография
36.29.33 Топографические и специализированные карты и планы. Цифровые модели местности.

ВИД СТАТЬИ: обзорная статья

В этой статье представлен подробный обзор результатов существования фиксированных точек в пространствах, подобных метрическим, с особым акцентом на b-метрические пространства. Понятие b-метрических пространств обобщает несколько важных концепций, включая частичные метрические пространства, метрические пространства и b-метрические пространства. Статья освещает значительные достижения в этой области, начиная с ранних исследований и до настоящего времени, предлагая обширную компиляцию фундаментальных и современных результатов. Кроме того, в статье даются положительные ответы на некоторые открытые вопросы, связанные с лучшими результатами близости в ортогональных 0-замкнутых b-метрических пространствах, что способствует дальнейшему развитию теории фиксированных точек в этих обобщенных структурах. Цель этого обзора — предоставить исследователям ценные идеи и ссылки, способствуя дальнейшему изучению и развитию теории пространств, подобных метрическим, и их обобщений.

Ключевые слова: b-метрические пространства, метрические пространства, частичные метрические пространства, b-метрические пространства, пространства, подобные метрическим, теория фиксированных точек.

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b-Metrijski Prostori: Pregled Koncepata i Primena

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ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: прегледни рад

У овом раду представљен је свеобухватан преглед резултата фиксних тачака у просторима сличним метричким, са посебним акцентом на b-метричке просторе. Појам b-

метричких простора обухвата неколико важних концепата, укључујући парцијалне метричке просторе, метричке просторе и b-метричке просторе. Рад истиче значајне доприносе од раних истраживања до данас, нудећи обимну компилацију фундаменталних и савремених резултата. Коначно, дајемо позитивне одговоре на неке отворене проблеме који произилазе из најбољих резултата близине у ортогоналним 0-комплетним b-метричким просторима, чиме даље унапређујемо разумевање теорије фиксних тачака у овим генерализованим окружењима. Овај преглед има за циљ да истраживачима пружи драгоцене увиде и референце, олакшавајући даље истраживање и развој у проучавању простора сличних метричким и њиховим генерализацијама.

Кључне речи: b-метрички простори, метрички простори, парцијални метрички простори, b-метрички простори, простори слични метричким, теорија фиксних тачака.

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