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On modified enriched versions of the Browder-Göhde-Kirk fixed point theorem

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Abstract:

Introduction/Purpose: In this paper, a modified enriched version of the classical Browder-Göhde-Kirk fixed point theorem in the setting of uniformly convex Banach spaces was proposed. This work aimed to extend fundamental fixed-point results to broader classes of mappings, contributing to the ongoing development of fixed-point theory and its applications in nonlinear analysis.

Methods: A modified enriched asymptotically nonexpansive mappings was introduced and analytical techniques from functional analysis and metric fixed-point theory were employed. This study leverages the geometric properties of uniformly convex Banach spaces to establish new existence and convergence results.

Results: Several key theorems extending the Goebel–Kirk fixed point theorem for modified enriched asymptotically nonexpansive mappings were proved. The results demonstrated the existence of fixed points under weaker assumptions, generalizing classical outcomes in this framework.

Conclusions: The findings provided a significant advancement in fixed-point theory, particularly for enriched mappings in uniformly convex Banach spaces. These results have potential applications in nonlinear analysis.

Key words: uniformly convex Banach spaces, nonexpansive map, convex set, fixed point

Introduction

As a fundamental principle of nonlinear analysis, fixed point theory finds remarkable applicability in mathematics and its applications, including optimization, game theory, and differential equations. A classical result in this topic is the Browder-Göhde-Kirk fixed point theorem which guarantees existence of fixed points of nonexpansive mappings on uniformly convex Banach spaces. This theorem has been improved and extended by weakening its assumptions or generalizing its range of applicability to larger classes of mappings over the years.

Several researchers have expanded upon the idea that a continuous mapping on a convex set that fulfills a restricted Lipschitz condition must also be Lipschitz continuous (see (Matkowski, 2007)). Many generalizations of the Browder-Göhde-Kirk theorem have been established on the basis of this observation (Browder: (Browder, 1965), Göhde: (Göhde, 1965), Kirk: (Kirk, 1965), and see also (Granas & Dugundji, 1982), (Goebel & Reich, 1984), (Reich, 1976), (Reich, 1980)).

Two extensions of the Browder-Göhde-Kirk theorem were shown in (Matkowski, 2022). The first one says that if a mapping T satisfies a nonlinear Lipschitz-type inequality, there is a fixed point. The second result claims that if T is continuous and there exists a positive sequence (t_n) of real numbers converging to zero and such that this implication holds for every $n \in \mathbb{N}$ and $u, v \in C$, then there exists a fixed point of T. This work helps to highlight the link between contractive mappings and fixed point theorems.

Recently, Berinde (Berinde, 2019, 2020) has extended the literature related to Banach space by introducing enrichments to mappings of contractive types. Enriched contractions (Berinde & Pacurar, 2020) refer to self-mappings T on the structure U of a normed linear space $(U, \| \cdot \|)$. These mappings adhere to a symmetric contraction condition, expressed as $\| b(u-v)+Tu-Tv \| \leq \theta \| u-v \|$, where $\theta \in [0, b+1)$ and $b \in [0, \infty)$, for each $u, v \in U$. Undoubtedly, the category of enriched contractions is more extensive, encompassing not only the conventional Banach contractions (where $b=0$) but also incorporating Lipschitz-type and nonexpansive mappings. The broader scope of the enriched contraction, which is an extension of Banach contractions, reinforces the assertion that within the Banach space context, a fixed point x^* is guaranteed to exist, and the Krasnoselskij iteration offers an approach to approximate the fixed point. This assertion has been substantiated by Berinde and Păcurar (Berinde & Pacurar, 2020). Recent years have seen an increasing interest in expanding the theory of enriched type contractions (Rawat et al., 2023).

Recently, Anjum and Abbas (Anjum & Abbas, 2024) analysed it and concluded that the idea of enriched nonexpansive mappings needs further reconsideration, because it coincides with the concept of nonexpansive mappings. Working in this direction, they defined a modified class of enriched nonexpansive mappings, and gave some fixed point results.

These developments emphasize how important it is to expand the scope of classical fixed point theorems to include more extensive contractive conditions. Our study expands on these discoveries by putting forth a brand-new enriched contractive framework that consolidates and generalizes a number of earlier findings, strengthening our knowledge of fixed point theory and its uses.

In this paper, we will generalize the Browder-Göhde-Kirk theorem by providing an enriched version of the contractive condition. Our generalization increases the usefulness of the theorem without overhauling its core structure. In particular, fixed point results are obtained under enriched Lipschitz-type conditions, which allows us to provide a much broader setting to examine nonexpansive and related classes of operators. This refinement is motivated by recent progress in the theory of metric fixed points, where more general and powerful results have been obtained by relaxing strict contractive assumptions.

Preliminaries

In this section, we give some preliminary results for subsequent use.

DEFINITION 1. *Let $(U, \|.\|)$ be a real normed vector space. It is called uniformly convex, if there is some $\delta > 0$ for every $\epsilon \in (0, 2]$, such that $\|x - y\| \geq \epsilon$ implies that $\|\frac{u+v}{2}\| \leq 1 - \delta$ for any two vectors $u, v \in U$ with $\|u\| = \|v\| = 1$.*

The fixed point theorem given by Browder, Göhde and Kirk is stated as follows.

THEOREM 1. *Let U be a uniformly convex Banach space and $T : C \to C$ be a nonexpansive mapping, where C is a non-empty, bounded, convex and closed subset of U. Then T has at least one fixed point in C.*

In (Berinde, 2019), Berinde generalized the scope of nonexpansive mappings by introducing and analysing a new set of mappings called the set of enriched nonexpansive mappings.

DEFINITION 2. *Let $(U, \|.\|)$ be a normed linear space and $T : U \to U$ be a mapping. If there exists $b \in [0, \infty)$ so that*

$$\|b(u - v) + Tu - Tv\| \leq (b + 1)\|u - v\|, \forall\, u, v \in U, \tag{1}$$

then T is known as an enriched nonexpansive mapping.

The set of enriched nonexpansive mappings is strictly bigger than the set of nonexpansive mappings, according to (Berinde, 2019). In fact, the concept of a nonexpansive mapping is obtained when $b = 0$ is specified in (1).

EXAMPLE 1. Consider a self-mapping T on $\mathbb{R}$ such that

$$T(u) = \frac{u}{2} + \frac{1}{2}.$$

For $b = 1$, we get

$$\|b(u - v) + Tu - Tv\| = \|u - v + \frac{u}{2} - \frac{v}{2}\| = \frac{3}{2}\|u - v\|.$$

Thus, T satisfies (1) with $b = 1$ and is an enriched nonexpansive mapping, but not necessarily nonexpansive.

REMARK 1. (Berinde & Pacurar, 2020) Let U be a normed space space and T be a self-mapping on U, then an averaged mapping T_λ for any $\lambda \in (0, 1)$

is obtained by

$$T_\lambda u = (1-\lambda)u + \lambda Tu, \forall u \in U. \quad (2)$$

Moreover,

$$Fix(T_\lambda) = \{u \in U : T_\lambda u = u\} = \{u \in U : Tu = u\} = Fix(T). \quad (3)$$

The b-enriched nonexpansive condition can be reformulated as a nonexpansive mapping condition, as demonstrated by the authors in (Anjum & Abbas, 2024). Thus, the writers made the following changes to the definition.

DEFINITION 3. *Consider a normed space $(U, \|.\|)$ and a mapping $T : U \to U$. If for some $b \in [0, +\infty)$ the following condition holds*

$$\|b(u-v) + Tu - Tv\| \leq \|u - v\|, \forall\, u, v \in U,$$

then T is called a b-modified enriched nonexpansive or modified enriched nonexpansive mapping.

This refined condition ensures a stronger form of contraction, leading to the next fixed point theorem.

THEOREM 2. *Let $T : U \to U$ be a modified enriched nonexpansive mapping with $b \neq 0$, where $(U, \|.\|)$ is a Banach space. Then*

1. $Fix(T) = \{u^*\}$.
2. *$\lambda \in (0, 1)$ exists such that, for any initial guess $u_0 \in U$, the Krasnoselskii iteration related to T, namely the sequence $\{u_n\}_{n=0}^{\infty}$, provided by $u_{n+1} = (1-\lambda)u_n + \lambda Tu_n, n \geq 0$, converges to u^*.*

DEFINITION 4. *Let U be a Banach space and $C \subset U$. A transformation $T : C \to C$ is known as an asymptotically nonexpansive if for every $u, v \in C$,*

$$\|T^i u - T^i v\| \leq L_i \|u - v\|, \quad u, v \in C,$$

where $\{L_i\}$ is a sequence of real numbers satisfying $\lim\limits_{i\to\infty} L_i = 1$.

In (Kirk, 1965), Goebel and Kirk proved the following result.

THEOREM 3. *Let U be a uniformly convex Banach space and T be an asymptotically nonexpansive self-mapping on C, where $C \subset U$ is a nonempty, convex, bounded and closed set. Then T has a fixed point.*

Main results

LEMMA 1. *Let U, V be normed spaces, $T : C \to V$ be a mapping where $C \subset U$ is a convex set, $b \in [0, \infty)$ and a real function $\beta : (0, \infty) \to [0, \infty)$ such that*

$$\|b(u-v) + Tu - Tv\| \leq \beta(\|u-v\|), \tag{4}$$

for all $u, v \in C$ and $u \neq v$. If

$$\limsup_{t \to 0^+} \frac{\beta(t)}{t} < +\infty, \tag{5}$$

then

$$\|b(u-v) + Tu - Tv\| \leq L\|u-v\|, \; u, v \in C, \tag{6}$$

where $\liminf\limits_{t \to 0^+} \frac{\beta(t)}{t} = L$.

Proof. Taking $\lambda = \frac{1}{b+1}$, we get $b = \frac{1-\lambda}{\lambda}$, using which (4) reduces to

$$\|(1-\lambda)(u-v) + \lambda Tu - \lambda Tv\| \leq \lambda\beta(\|u-v\|),$$

that is,

$$\|T_\lambda u - T_\lambda v\| \leq \lambda\beta(\|u-v\|), \tag{7}$$

where $T_\lambda u = (1-\lambda)u + \lambda Tu$.

Clearly, T_λ is continuous from equations (5) and (7). Also, from equation (5), for fixed $x, y \in C$ with $x \neq y$ and $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$\frac{\beta(t)}{t} \leq L + \epsilon, \text{ for all } t \in (0, t_\epsilon] \tag{3}$$

and define $n(\epsilon) = \left\lfloor \frac{\|u-v\|}{t_\epsilon} \right\rfloor$.

Now, define the sequence $\{z_k\}$ as follows:

$$z_k = \left(1 - \frac{k}{n(\epsilon)}\right)u + \frac{k}{n(\epsilon)}v, \quad k = 0, 1, \ldots, n(\epsilon).$$

Since C is convex, there exists $z_k \in C$ for all k. Moreover,

$$\|z_{k+1} - z_k\| = \frac{1}{n(\epsilon)}\|u-v\| \in (0, t_\epsilon], \quad \text{for all } k = 0, \ldots, n(\epsilon) - 1.$$

Also $z_0 = u, z_{n(\epsilon)} = v$, and

$$\|u - v\| = \sum_{k=0}^{n(\epsilon)-1} \|z_{k+1} - z_k\|.$$

Using the triangle inequality and condition (7), one obtains

$$\begin{aligned}
\|T_\lambda u - T_\lambda v\| &= \left\| \sum_{k=0}^{n(\epsilon)-1} (T_\lambda z_k - T_\lambda z_{k+1}) \right\| \\
&\leq \sum_{k=0}^{n(\epsilon)-1} \|T_\lambda z_k - T_\lambda z_{k+1}\| \\
&\leq \sum_{k=0}^{n(\epsilon)-1} \beta(\|z_k - z_{k+1}\|) \\
&\leq \lambda \sum_{k=0}^{n(\epsilon)-1} \beta\left(\frac{\|u - v\|}{n(\epsilon)}\right) \\
&\leq \lambda n(\epsilon)\beta\left(\frac{\|u - v\|}{n(\epsilon)}\right) \\
&\leq \lambda n(\epsilon)(L + \epsilon)\left(\frac{\|u - v\|}{n(\epsilon)}\right) \\
&= \lambda(L + \epsilon)\|u - v\|.
\end{aligned}$$

Taking $\epsilon \to 0$ in the inequality above, one obtains

$$\|T_\lambda u - T_\lambda v\| \leq \lambda L\|u - v\|.$$

This further implies

$$\begin{aligned}
\|(1 - \lambda)(u - v) + \lambda T u - \lambda T v\| &\leq \lambda L\|u - v\| \\
\|b(u - v) + T u - T v\| &\leq L\|u - v\|.
\end{aligned}$$

This concludes the proof. □

LEMMA 2. *Let U, V be real normed spaces and $T : C \to V$ be a continuous mapping, where $C \subset U$ is a bounded convex set. If there exists some*

$b \in [0, \infty)$, a non-negative real L and two positive sequences $(c_n), (t_n)$ satisfying

$$\lim_{n\to\infty} c_n = L \text{ and } \lim_{n\to\infty} t_n = 0,$$

so that for all $u, v \in C$ and $n \in \mathbb{N}$,

$$\|u - v\| \le t_n \implies \|b(u - v) + T(u) - T(v)\| \le c_n t_n, \tag{8}$$

then T satisfies

$$\|b(u - v) + Tu - Tv\| \le L\|u - v\|, u, v \in C.$$

Proof. Taking $\lambda = \frac{1}{b+1}$, we get $b = \frac{1-\lambda}{\lambda}$, using which (8) reduces to

$$\|(1 - \lambda)(u - v) + \lambda Tu - \lambda Tv\| \le \lambda c_n t_n,$$

that is,

$$\|u - v\| \le t_n \implies \|T_\lambda u - T_\lambda v\| \le \lambda c_n t_n, \tag{9}$$

where $T_\lambda u = (1 - \lambda)u + \lambda Tu$.

Let $u, v \in C$ such that $u \neq v$. Since C is convex and bounded, for any $n \in \mathbb{N}$, define an integer $k_n \in \mathbb{N}$ such that

$$(k_n - 1)t_n < \|u - v\| \le k_n t_n.$$

Now define a partition of the segment from u to v into k_n equal parts

$$z_k = (1 - \frac{k}{k_n})u + \frac{k}{k_n}v, \text{ for } k = 0, 1, 2, ..., k_n.$$

Then each $z_k \in C$ by convexity, and $\|z_{k+1} - z_k\| = \frac{1}{k_n}\|u - v\| \le t_n$, equation (9) implies that $\|T_\lambda z_{k+1} - T_\lambda z_k\| \le \lambda c_n t_n$.
Also $z_0 = u, z_{k_n} = v$, and

$$\begin{aligned}\|z_{k_n-1} - v\| &= \left\|\left(1 - \frac{k_n - 1}{k_n}\right)u + \frac{k_n - 1}{k_n}v - v\right\| \\ &= \frac{1}{k_n}\|u - v\| < t_n. \end{aligned}\tag{10}$$

Using the triangle inequality and condition (9), one obtains

$$\|T_\lambda u - T_\lambda v\| = \left\|\sum_{k=0}^{k_n-2}(T_\lambda z_k - T_\lambda z_{k+1}) + (T_\lambda z_{k_n-1} - T_\lambda v)\right\|$$

$$\begin{aligned}
&\leq \sum_{k=0}^{k_n-2} \|T_\lambda z_k - T_\lambda z_{k+1}\| + \|T_\lambda z_{k_n-1} - T_\lambda v\| \\
&\leq \sum_{k=0}^{k_n-2} \lambda c_n t_n + \|T_\lambda z_{k_n-1} - T_\lambda v\| \\
&= \lambda c_n t_n (k_n - 1) + \|T_\lambda z_{k_n-1} - T_\lambda v\|. \qquad (11)
\end{aligned}$$

Now, from (10), we obtain $\|z_{k_n-1} - v\| < t_n$. Therefore, we have $\lim\limits_{n\to\infty} \|z_{k_n-1} - v\| = 0$ and, in view of the continuity of T_λ, $\lim\limits_{n\to\infty} \|T_\lambda z_{k_n-1} - T_\lambda v\| = 0$. Also, taking into account that $(k_n - 1)t_n \leq \|u - v\|$, from (11) on taking $n \to \infty$, one obtains

$$\|T_\lambda u - T_\lambda v\| \leq \lambda L \|u - v\|.$$

This further implies

$$\begin{aligned}
\|(1-\lambda)(u-v) + \lambda Tu - \lambda Tv\| &\leq \lambda L \|u - v\| \\
\|b(u-v) + Tu - Tv\| &\leq L\|u - v\|.
\end{aligned}$$

This concludes the proof. □

LEMMA 3. *Let U, V be real normed spaces and $T : C \to V$ be a continuous mapping, where $C \subset U$ is a bounded convex set. If there exist a non-negative real L, some $b \in [0, \infty)$, a function $\beta : (0, \infty) \to [0, \infty)$, and a positive sequence (t_n) of real numbers with $\lim\limits_{n\to\infty} t_n = 0$ and satisfying*

$$\lim_{n\to\infty} \frac{\beta(t_n)}{t_n} = L,$$

so that for all $u, v \in C$ and $n \in \mathbb{N}$,

$$\|u - v\| = t_n \implies \|b(u-v) + Tu - Tv\| \leq \beta(\|u - v\|),$$

then T satisfies

$$\|b(u-v) + Tu - Tv\| \leq L\|u - v\|, u, v \in C.$$

Proof. Taking $c_n = \frac{\beta(t_n)}{t_n}$, we obtain $\lim\limits_{n\to\infty} c_n = L$. As for every $n \in \mathbb{N}$ and $u, v \in C$, $\|b(u-v) + Tu - Tv\| \leq \beta(t_n) = c_n t_n$ if $\|u - v\| = t_n$, so the result follows from Lemma 2. □

THEOREM 4. *Let U be a uniformly convex Banach space and T be a self-mapping on C, where $C \subset U$ is a nonempty, convex, bounded and closed set. If there exists $b \in [0,\infty)$ and a function $\beta : (0,\infty) \to [0,\infty)$ for which*

$$\|b(u-v)+Tu-Tv\| \le \beta(\|u-v\|), u,v \in C, u \neq v,$$

and

$$\limsup_{t\to 0^+} \frac{\beta(t)}{t} < +\infty, \ \liminf_{t\to 0^+} \frac{\beta(t)}{t} = 1,$$

then,

1. *$Fix(T) \neq \emptyset$.*
2. *For $b \in (0,\infty)$ the fixed point is uniquely determined. Moreover, $\lambda \in (0,1)$ exists such that, for any initial guess $u_0 \in U$, the Krasnoselskii iteration related to T, namely the sequence $\{u_n\}_{n=0}^{\infty}$, provided by*

$$u_{n+1} = (1-\lambda)u_n + \lambda T u_n, n \ge 0, \tag{12}$$

converges to u^*.

Proof. Applying Lemma 1 with $L = 1$, we get

$$\|b(u-v)+Tu-Tv\| \le \|u-v\|, \quad u,v \in C.$$

Setting $\lambda = \frac{1}{b+1}$, we derive $b = \frac{1-\lambda}{\lambda}$, which leads to

$$\|T_\lambda u - T_\lambda y\| \le \lambda\|u-v\|, \quad u,v \in C.$$

If $b \in (0,\infty)$, i.e. $\lambda < 1$, then this indicates that the averaged operator T_λ satisfies the Banach contraction principle. The Krasnoselskij iterative sequence $\{u_n\}_{n=0}^{\infty}$, defined by (12), coincides with the Picard iteration for T_λ, i.e.,

$$u_{n+1} = T_\lambda u_n, \quad n \ge 0. \tag{13}$$

Since T_λ is a Banach contraction mapping on U, where U is a Banach space, the Banach fixed-point theorem guarantees that T_λ has a unique fixed point, denoted by $u^* \in U$, and the Picard iteration related with T_λ, as defined in (13), converges to u^*. Also, if $\lambda = 1$, then T_λ is a nonexpansive mapping, so from the Browder-Göhde-Kirk fixed point theorem T_λ has a fixed point. Furthermore, from equation (3), we conclude that T also has a fixed point. □

EXAMPLE 2. Let the sets $U = V = \mathbb{R}$ with the norm $\|.\|$, $C = [0,1]$ and $b = \frac{1}{2}$. Define a function $T : C \to C$ as follows

$$T(u) = \frac{u}{2},$$

where $u \in C$. Now, for any two $u, v \in C$, $T(u) = \frac{u}{2}$ and $T(v) = \frac{v}{2}$. So,

$$\begin{aligned}\|b(u-v) + Tu - Tv\| &= \left\| \frac{1}{2}(u-v) + \frac{u}{2} - \frac{v}{2} \right\| \\ &= \|(u-v)\|\end{aligned}$$

Now, on defining $\beta(t) = t$, we obtain

$$\|b(u-v) + T(u) - T(v)\| \leq \beta(\|u-v\|).$$

Also, $\limsup\limits_{t\to 0+} \frac{\beta(t)}{t} = 1$, and $\liminf\limits_{t\to 0+} \frac{\beta(t)}{t} = 1$. Clearly, all the assumptions of the theorem are satisfied and therefore, $u = 0$ is a unique fixed point of T.

THEOREM 5. *Let U be a uniformly convex Banach space and T be a continuous self-mapping on C, where $C \subset U$ is a nonempty, convex, bounded and closed set. If there exists some $b \in [0,\infty)$, a function $\beta : (0,\infty) \to [0,\infty)$ and a sequence of positive real (t_n), $\lim\limits_{n\to\infty} t_n = 0$ satisfying the condition*

$$\lim_{n\to\infty} \frac{\beta(t_n)}{t_n} = 1,$$

so that for all $u, v \in C$ and for every $n \in \mathbb{N}$,

$$\|u-v\| = t_n \implies \|b(u-v) + T(u) - T(v)\| \leq \beta(\|u-v\|),$$

then,

1. *$Fix(T) \neq \emptyset$.*
2. *for $b \in (0,\infty)$ the fixed point is uniquely determined. Moreover, $\lambda \in (0,1)$ exists such that, for any initial guess $u_0 \in U$, the Krasnoselskii iteration related to T, namely the sequence $\{u_n\}_{n=0}^{\infty}$, provided by*

$$u_{n+1} = (1-\lambda)u_n + \lambda T u_n, n \geq 0, \tag{14}$$

converges to u^*.

Proof. Applying Lemma 2 with $L = 1$, we get

$$\|b(u - v) + Tu - Tv\| \leq \|u - v\|, \quad u, v \in C.$$

Setting $\lambda = \frac{1}{b+1}$, we derive $b = \frac{1-\lambda}{\lambda}$, which leads to

$$\|T_\lambda u - T_\lambda v\| \leq \lambda\|u - v\|, \quad u, v \in C.$$

If $b \in (0, \infty)$, i.e. $\lambda < 1$ it indicates that the averaged operator T_λ satisfies the Banach contraction principle. If $\lambda = 1$, then T_λ is a nonexpansive mapping. The rest of the reasoning is same as of Theorem 4. □

Now, we define modified enriched asymptotically nonexpansive mapping in a Banach space U.

DEFINITION 5. *Let C be a subset of a Banach space U. A transformation $T : C \to C$ is said to be modified enriched asymptotically nonexpansive if for some $b \in [0, \infty)$ and for each $u, v \in C$,*

$$\|b(u - v) + T^i u - T^i v\| \leq L_i\|u - v\|, \quad u, v \in C,$$

where $\{L_i\}$ is a sequence of real numbers satisfying $\lim_{i\to\infty} L_i = 1$.

THEOREM 6. *Let C be a nonempty, convex, closed and bounded subset of a uniformly convex Banach space U, and $T : C \to C$ be a modified enriched asymptotically nonexpansive. Then*

1. *$Fix(T) \neq \emptyset$.*
2. *for $b \in (0, \infty)$ the fixed point is uniquely determined. Moreover, $\lambda \in (0, 1)$ exists such that, for any initial guess $u_0 \in U$, the Krasnoselskii iteration related to T, namely the sequence $\{u_n\}_{n=0}^{\infty}$, provided by*

$$u_{n+1} = (1 - \lambda)u_n + \lambda T u_n, n \geq 0, \tag{15}$$

converges to u^*.

Proof. Setting $\lambda = \frac{1}{b+1}$, we derive $b = \frac{1-\lambda}{\lambda}$. So the modified enriched asymptotically nonexpansive condition reduces to

$$\|(1 - \lambda)(u - v) + \lambda T^i u - \lambda T^i v\| \leq \lambda L_i\|u - v\|,$$

i.e.,

$$\|T_\lambda^i u - T_\lambda^i v\| \leq \lambda L_i\|u - v\|, \quad u, v \in C.$$

Now, since $\lim_{i\to\infty} L_i = 1$, therefore $\lim_{i\to\infty} \lambda L_i = \lambda \leq 1$. If $b \in (0,\infty)$, then $\lambda < 1$ which indicates that for i large enough, the averaged operator T_λ^i satisfies the Banach contraction principle. If $b = 0$, then $\lambda = 1$ which indicates that for i large enough, the averaged operator T_λ^i satisfies the Browder-Göhde-Kirk fixed point theorem. The rest of the reasoning is the same as of Theorem 4. □

REMARK 2. Also note that if $\lim_{i\to\infty} L_i < 1$, then also T_λ^i satisfies the Banach contraction principle, i.e. T_λ has a unique fixed point. Furthermore, from (3), we conclude that T also has a unique fixed point.

THEOREM 7. *Let U be a uniformly convex Banach space and T be a self-mapping on C, where $C \subset U$ is a nonempty, convex, bounded and closed set. Assume that T is a modified enriched nonlinear asymptotically nonexpansive mapping, i.e., for all $i \in \mathbb{N}$, a function $\beta_i : (0,\infty) \to (0,\infty)$ exists, such that*

$$\|b(u-v) + T^i u - T^i v\| \leq \beta_i(\|u-v\|), u, v \in C, u \neq v,$$

and

$$\limsup_{t\to 0^+} \frac{\beta_i(t)}{t} < +\infty \text{ and } \liminf_{t\to 0^+} \frac{\beta_i(t)}{t} = L_i, i \in \mathbb{N},$$

where the sequence (L_i) converges with $\lim_{i\to\infty} L_i \leq 1$. Then

1. *$Fix(T) \neq \emptyset$.*
2. *for $b \in (0,\infty)$ the fixed point is uniquely determined. Moreover, $\lambda \in (0,1)$ exists such that, for any initial guess $u_0 \in U$, the Krasnoselskii iteration related to T, namely the sequence $\{u_n\}_{n=0}^\infty$, provided by*

$$u_{n+1} = (1-\lambda)u_n + \lambda T u_n, n \geq 0, \tag{16}$$

converges to u^*.

Proof. Using Lemma 1 with T^i in place of T, and L_i in place of L, for every $i \in \mathbb{N}$, we get

$$\|b(u-v) + T^i u - T^i v\| \leq L_i\|u-v\|,\ u, v \in C. \tag{17}$$

Now if $\lim_{i\to\infty} L_i = 1$, then clearly T is a modified enriched asymptotically nonexpansive mapping as given in Definition 5. So, from Theorem 6 the results follow. If $\lim_{i\to\infty} L_i < 1$ then from Remark 3.7, the results follow. □

THEOREM 8. *Let U be a uniformly convex Banach space and T be a continuous self-mapping on C, where $C \subset U$ is a nonempty, convex, bounded and closed set. Suppose, for all $i \in \mathbb{N}$ a function $\beta_i : (0, \infty) \to (0, \infty)$ and a sequence $(t_{i,n} : n \in \mathbb{N})$ with $\lim_{n\to\infty} t_{i,n} = 0$ exists, satisfying the condition*

$$\lim_{n\to\infty} \frac{\beta(t_{i,n})}{t_{i,n}} = L_i,$$

so that for all $x, y \in C$ and for every $n \in \mathbb{N}$,

$$\|u - v\| = t_{i,n} \implies \|b(u - v) + T^i(u) - T^i(v)\| \leq \beta_i(\|u - v\|),\ i \in \mathbb{N}.$$

If $L = \lim_{i\to\infty} L_i \leq 1$, then

1. *$Fix(T) \neq \emptyset$.*
2. *for $b \in (0, \infty)$ the fixed point is uniquely determined. Moreover, $\lambda \in (0, 1)$ exists such that, for any initial guess $u_0 \in U$, the Krasnoselskii iteration related to T, namely the sequence $\{u_n\}_{n=0}^{\infty}$, provided by*

$$u_{n+1} = (1 - \lambda)u_n + \lambda T u_n, n \geq 0, \tag{18}$$

converges to u^*.

Proof. Based on Lemma 3, the mapping T^i satisfies

$$\|b(u - v) + T^i u - T^i v\| \leq L_i\|u - v\|, \text{ for all } i \in \mathbb{N} \text{ and } u, v \in C. \tag{19}$$

Now if $\lim_{i\to\infty} L_i = 1$, then clearly T is a modified enriched asymptotically nonexpansive mapping as given in Definition 5. So, from Theorem 6 the results follow. If $\lim_{i\to\infty} L_i < 1$ then from Remark 3.7, the results follow. ☐

References

Anjum, R., & Abbas, M. 2024. Remarks on b-enriched nonexpansive mappings, arXiv preprint arXiv:2405.07999. Available at: https://doi.org/10.48550/arXiv.2405.07999

Berinde, V. 2019. Approximating fixed points of enriched nonexpansive mappings by Krasnoselskii iteration in Hilbert spaces, *Carpathian Journal of Mathematics*, 35(3), pp.293-304. Available at: https://www.jstor.org/stable/26905206

Berinde, V., & Pacurar, M. 2020. Approximating fixed points of enriched contractions in Banach spaces, *Journal of Fixed Point Theory and Applications*, 22, 10. Available at: https://doi.org/10.1007/s11784-020-0769-9

Berinde, V. 2020. Approximating fixed points of enriched nonexpansive mappings in Banach spaces by using a retraction-displacement condition, *Carpathian Journal of Mathematics*, 36, pp.27-34. Available at: https://www.jstor.org/stable/26898779

Browder, F.E. 1965. Nonexpansive nonlinear operators in a Banach space, *Proceedings of the National Academy of Sciences of the USA*, 54, pp.1041-1044. Available at: https://doi.org/10.1073/pnas.54.4.104

Goebel, K., & Kirk, W. A. 1972. A fixed point theorem for asymptotically nonexpansive mappings, *Proceedings of the American Mathematical Society*, 35(1), pp.171-174. Available at: https://doi.org/10.1090/S0002-9939-1972-0298500-3

Goebel, K., & Reich, S. 1984. Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings, *Series of Monographs and Textbooks in Pure and Applied Mathematics*, Dekker, New York. ISBN: 0824772237.

Göhde, D. 1965. Zum Prinzip der kontraktiven Abbildung, *Mathematische Nachrichten*, 28, pp.251-258. Available at: https://doi.org/10.1002/mana.19650300312

Granas, A., & Dugundji, J. 1982. Fixed Point Theory, *Monografie Matematyczne, Polska Akademia Nauk, War-szawa*, 1. ISBN: 0387001735, 9780387001739. Available at: https://archive.org/details/fixed-point-theory/mode/2up

Kirk, W. A. 1965. A fixed point theorem for mappings which do not increase distances, *American Mathematical Monthly*, 72, pp.1004-1006. Available at: https://doi.org/10.2307/2313345

Matkowski, J. 2007. Remarks on Lipschitzian mappings and some fixed point theorems, *Banach Journal of Mathematical Analysis*, 2, pp.237-244. Available at: 10.15352/bjma/1240336222

Matkowski, J. 2022. A refinement of the Browder–Göhde–Kirk fixed point theorem and some applications, *Journal of Fixed Point Theory and Applications*, 24, 70. Available at: https://doi.org/10.1007/s11784-022-00985-2

Rawat, S., Bartwal, A. & Dimri, R.C. 2023. Approximation and existence of fixed points via interpolative enriched contractions, *Filomat*, 37(16), pp.5455-5467. Available at: https://doiserbia.nb.rs/img/doi/0354-5180/2023/0354-51802316455R.pdf

Reich, S. 1976. The fixed point property for nonexpansive mappings, I, *American Mathematical Monthly*, 83(4), pp.266-268. Available at: https://www.tandfonline.com/doi/pdf/10.1080/00029890.1976.11994096

Reich, S. 1980. The fixed point property for nonexpansive mappings, II, *American Mathematical Monthly*, 87, pp.292-294. Available at: https://www.tandfonline.com/doi/pdf/10.1080/00029890.1980.11995019

О модификованим обогаћеним верзијама Browder-Göhde-Kirk теореме непокретне тачке

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Сажетак:

Увод/циљ: У раду се предлаже модификована обогаћена верзија класичне Browder-Göhde-Kirk теореме непокретне тачке у оквиру униформно конвексних Банахових простора. Циљ рада је проширење основних резултата о непокретним тачкама на шире класе пресликавања чиме се доприноси континуираном развоју теорије непокретне тачке и њеним применама у нелинарној анализи.

Методе: Уведена су модификована обогаћена асимптотски неекспанзивна пресликавања, а коришћене су аналитичке технике функционалне анализе и метричке теорије непокретне тачке. Ово истраживање користи геометријске особине униформно конвексних Банахових простора за успостављање нових резултата о постојању и конвергенцији.

Резултати: Доказано је неколико кључних теорема које проширују Goebel–Kirk теорему непокретне тачке за модификована обогаћена асимптотски неекспанзивна пресликавања. Резултати показују постојање непокрентних тачака под слабијим претпоставкама, генерализујући класичне исходе у овом оквиру.

Закључци: Добијени резултати представљају значајан напредак у теорији непокретне тачке, нарочито када је реч о обогаћеним пресликавањима у униформно конвексним Банаховим просторима. Ови резултати могу се применити у нелинеарној анализи.

Кључне речи: униформно конвексни Банахови простори, нееkспанзивно пресликавање, конвексни скуп, непокретна тачка

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