Topological indices and structural properties of ideal-based unit graphs in commutative rings

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Abstract:

Introduction/Purpose: This study introduced the concept of a prime ideal-based unit graph associated with a commutative ring R. In this graph, the vertices consisted of units of R that were not contained in a chosen prime ideal I, and two such vertices were considered adjacent if their difference belonged to I. The aim was to investigate the structural, algebraic, and topological properties of this graph and examine the algebraic implications of various graph-theoretic invariants.

Methods: The construction of ideal-based unit graphs was carried out using the ring $\mathbb{Z}_n$, where units excluded from the chosen prime ideal formed the vertex set. The adjacency between two vertices was determined by whether their difference lay in the ideal. The analysis involved computing several topological indices including the Zagreb indices, Wiener index, Arithmetic-Geometric index, Harmonic index, Estrada index, and graph energy. Adjacency matrices and graphical visualizations were employed to understand structural complexity and connectivity.

Results: It was observed that the structure of the resulting graph depended significantly on both the modulus n and the nature of the selected ideal. Smaller ideals produced graphs with higher connectivity, while larger ideals led to sparser or disconnected graphs. The calculated indices reflected patterns in symmetry, degree distribution, and distances, revealing deeper algebraic characteristics.

Conclusions: Prime ideal-based unit graphs provided a novel approach to studying the interaction between ring-theoretic and graph-theoretic *concepts. The findings contributed to potential applications in mathematical chemistry, secure communications, and theoretical computer science.*

Keywords: units, ideals, topological indices, commutative ring.

Introduction

Graph theory serves as a powerful tool to model and analyse relationships between algebraic structures, providing a visual and structural framework that bridges abstract mathematics with concrete representations. Among its numerous applications, the study of graph representations of commutative rings has garnered significant interest. These representations including zero-divisor graphs introduced by Anderson and Livingston (Anderson et al., 2011, pp. 23-45) and unit graphs studied by Sharma and Bhatwadekar (Sharma & Bhatwadekar, 2009, pp. 124 -127). Since then, various graphs such as annihilating-ideal graphs (Behboodi and Rakeei, 2011, pp.741-753), Cayley graphs (Abdollahi, 2008; Akhtar et al., 2009) total graphs (Akbari et al., 2009, pp.2224-2228; Asir & Chelvam, 2013, pp. 3820-3835), and unit graphs (Ashrafi, 2010, pp.2851-2871; Ramaswamy & Veena, 2009, pp.N24-N24) have been extensively studied.

Building on this foundation, this paper focuses on a novel graph construction known as the ideal-based unit graph, denoted as $G_I(R)$. This graph is built using the set of units of a commutative ring R and a chosen ideal I. Unlike the zero-divisor graph which emphasizes the multiplicative annihilation of elements, or the unit graph which focuses on the additive properties of all units, the ideal-based unit graph incorporates the influence of ideals to define adjacency. This approach opens new avenues for exploring the interplay between ideals and the unit structure of a ring.

Motivation and context

The ideal-based unit graph $G_I(R)$ captures the interplay between the additive and multiplicative structures of a ring through a chosen ideal I, highlighting how units interact modulo I. This framework enables the application of graph-theoretic invariants for computational analysis of ring-theoretic properties.

Objectives of the study

The ideal-based unit graph $G_I(R)$ is rigorously defined by its vertex set and adjacency relation, reflecting how elements of a ring interact modulo an ideal I. Its structural features such as connectivity, diameter, and girth are analyzed, along with topological indices to quantify the graph's properties numerically.

The study of $G_I(R)$ incorporates the computation and analysis of topological indices, numerical invariants that reflect the graph's structural characteristics, specifically indices such as the Zagreb indices introduced by Gutman and Trinajstić in (Gutman & Trinajstić,1972, pp.535-538) the Wiener index extensively researched within chemical graph theory (Wiener, 1947, pp.17-20) and the Estrada index developed by Estrada (Estrada, 2000, pp.713-718). Through this analysis, quantitative acumens into the graph's complexity, symmetry, and connectivity are obtained.

In the subsequent sections, $G_I(R)$ is rigorously defined, its properties are explored, and meaningful topological indices are computed, thereby showcasing the intricate interplay between algebraic and graph-theoretic concepts.

Materials and methods

The ideal-based unit graph $G_I(R)$ uses the units of R as its vertices but excludes those lying within the ideal I. Two vertices are considered adjacent if their difference lies in the ideal I. This definition captures the interaction of units with respect to I, yielding a graph that is sensitive to the algebraic properties of R and the structural role of I. To interpret this definition and its implications, a formal construction and an illustrative example using the ring Z_n and the integers modulo n is provided. The ring Z_n serves as a particularly instructive example due to its finite nature and a well-defined unit group.

DEFINITION 1. Let R be a commutative ring with unity, and let I be an ideal of R. The ideal-based unit graph is a graph constructed using the algebraic structure of R filtered through its unit group $U(R)$ and the chosen ideal I. This graph provides a novel way to study the interplay between the ring's unit structure and its ideal. The vertices of $\mathrm{G_I(R)}$ are defined as the units of R that do not lie in the ideal I.

That is, $V(G_I(R)) = \{u \in U(R) \mid u \notin I\}$, where $U(R)$ represents the set of all units (invertible elements) in R. This restriction ensures that the graph reflects the relationship between units under the influence of I, excluding any units directly contained within I.

Two distinct vertices $u, v \in V(G_I(R))$ are adjacent if and only if their difference belongs to the ideal I. That is, $\{u, v\} \in E(G_I(R)) \iff u - v \in I$. This adjacency condition establishes a connection between the units based on the additive structure of R as mediated by I.

Theoretical significance

The choice of an ideal I is vital because ideals have unique properties that influence the graph structure. An ideal restricts the differences $u - v$ to a subset of R, providing a combinatorial perspective on the ring's additive relationships (Lambek, 2009; Stanley, 2007). Since I is , it avoids trivial containment of non-units, ensuring that the graph captures meaningful connections between distinct units. This construction connects the algebraic properties of R with graph-theoretic structures, creating a bridge between commutative algebra and combinatorics (Yap, 2000).

Construction for Z_n

To illustrate the concept of $G_I(R)$, consider the specific case where $R = Z_n$, the ring of integers modulo n, and I is an ideal of Z_n. The units $U(Z_n)$ are the integers $a \in Z_n$ that are co- to n. These elements satisfy $gcd(a, n) = 1$ and have multiplicative inverses modulo n. For Z_n, an ideal $I = (d)$ is generated by a divisor d of n. If d is, $I = (d) = \{0, d, 2d, \dots, (n/d - 1)d\}$.

The vertices are the units of Z_n that are not in I. That is $V(G_I(Z_n)) = \{u \in U(Zn) \mid u \notin I\}$. Two distinct units $u, v \in V(G_I(Z_n))$ are adjacent if $u - v \in I$. This means $u - v$ is a multiple of d. For example, the graphs of $G_2(Z_{90})$ and $G_6(Z_{128})$ are depicted in the following figures, Figure 1 and Figure 2, respectively.

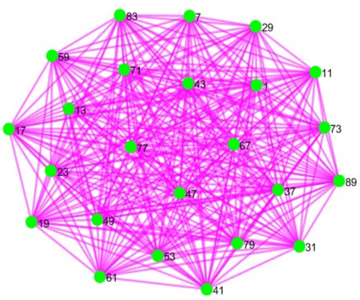

Figure 1 – Graph for $G_2(Z_{90})$

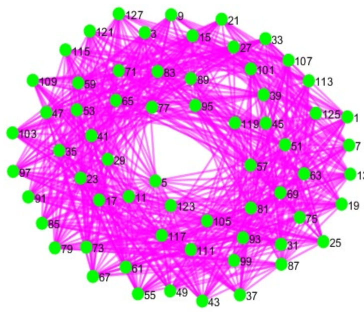

Figure 2 – Graph for $G_6(Z_{128})$

Main results

In this section, the key structural and graph-theoretic properties of the ideal-based unit graph $G_I(R)$, including its finiteness, connectivity, diameter, and girth, are investigated.

PROPOSITION 1. If R/I is finite, then the graph $G_I(R)$ has a finite number of vertices. Specifically, the size of the vertex set is given by

$$\mid V\big(G_I(R)\big) \mid=\mid U(R) \mid -\mid U(R) \cap I \mid.$$

Proof. To establish this result, recall that the vertex set $V\big(G_I(R)\big)$consists of all units of R that do not belong to the ideal I. That is, $V(G_I(R)) = \{u \in U(R) \mid u \notin I\}$.

Here, $U(R)$ represents the set of all units in R, and $U(R) \cap I$ represents the subset of these units that also belong to the ideal I. Since R/I is assumed to be finite, the set $U(R)$ must also be finite, as it is a subset of the ring R. Consequently, $U(R) \cap I$ is finite as well.

The number of vertices in the graph is simply the total number of units in R minus the number of units that belong to I. This gives the result,

$|V(G_I(R)) \mid=\mid U(R) \mid -\mid U(R) \cap I \mid$.

Thus, the finiteness of R/I ensures the finiteness of $G_I(R)$, completing the proof.

EXAMPLE 1. Consider the ring $Z_6 = \{0,1,2,3,4,5\}$ units of Z_6, which are the elements with multiplicative inverses, are $\{1,5\}$. The ideal $I = (2)$ consists of the multiples of 2 in Z_6, which are $\{0,2,4\}$. Notably, there are no units within the ideal I. Using the above proposition 3.1, the size of the vertex set of the graph $G_2(Z_6)$ as $2 - 0 = 2$.

THEOREM 1.The graph $G_I(R)$ is connected if and only if, for any two vertices $u, v \in V(G_I(R))$, there exists a finite sequence of vertices $u = u_0, u_1, \dots, u_k = v$ such that $u_i - u_{i+1} \in I$ for all i.

Proof. To prove this result, it is necessary to show that the existence of such a sequence is sufficient for connectivity. By the definition of adjacency in $G_I(R)$, two vertices u and v are directly connected by an edge if $u - v \in I$. For the vertices u, v that are not directly adjacent, the existence of a sequence $u = u_0, u_1, \dots, u_k = v$, where $u_i - u_{i+1} \in I$ for all i, ensures a path connecting u and v. Therefore, if such sequences exist for all pairs of vertices, the graph is connected.

Conversely, assume that $G_I(R)$ is connected. This implies that for any $u, v \in V(G_I(R))$, there must exist a path between them. A path is a sequence of vertices $u = u_0, u_1, \dots, u_k = v$ such that u_i is adjacent to $u_i +$ 1 for all i. By the definition of adjacency, this implies $u_i - u_{i+1} \in I$ for all i.

Hence, the existence of such a sequence is a necessary condition for connectivity.

Thus, $G_I(R)$ is connected if and only if every pair of vertices can be connected by a sequence satisfying the stated condition, completing the proof.

EXAMPLE 2. Consider the ring $Z_{20} = \{0,1,2,\dots,19\}$ The units of Z_{20} which are the elements coprime to 20, are $\{1,3,7,9,11,13,17,19\}$.The ideal $I = (5)$ consists of the multiples of 5, specifically $\{0,5,10,15\}$. Thus, the vertex set of the graph $G_5(Z_{20})$ is $V(G_5(Z_{20})) = \{1,3,7,9,11,13,17,19\}$, as these are the units not in the ideal I. To determine connectivity, consider the vertices $u = 1$ and $v = 19$. A possible sequence connecting them is $(1,11,19)$ where the differences satisfy $1 - 11 = -10 \in I$, but $11 - 19 = -8 \notin I$. Since no complete path exists connecting $u = 1$ and $v = 19$ such that all differences belong to I, the graph $G_5(Z_{20})$ is not connected. This lack of connectivity extends to other pairs of vertices, as similar interruptions occur in potential paths. This example illustrates how the structure of $G_I(R)$ depends on the interplay between the units and the ideal in higher-order rings.

THEOREM 2. If $I \neq (0)$, the diameter of $G_I(R)$, denoted $diam(G_I(R))$ satisfies the inequality $diam(G_I(R)) \leq 3$.

Proof. The diameter of a graph is the maximum distance between any two vertices, where the distance $d(u,v)$ is the length of the shortest path connecting u and v. In $G_I(R)$, adjacency is defined by the condition $u - v \in I$.

If $u, v \in V(G_I(R))$ are directly adjacent (i.e., $u - v \in I$), then $d(u,v) = 1$. For the vertices u, v that are not directly adjacent, there exists a vertex $w \in V(G_I(R))$ such that $u - w \in I$. In this case, the path $u \to w \to v$ has length 2, implying $d(u,v) = 2$.

In certain configurations, a third vertex x may be required to connect u and v, resulting in a path $u \to x \to w \to v$ of length 3. Hence, the maximum distance between any two vertices in $G_I(R)$ is at most 3. This establishes the result that $diam(G_I(R)) \leq 3$.

EXAMPLE 3. In the ring $Z_{36} = \{0,1,2,\dots,35\}$ the units are $\{1,5,7,11,13,17,19,23,25,29,31,35\}$, as these elements are co-prime to 36. The ideal $I = (4)$ consists of multiples of 4. The graph $G_4(Z_{36})$ has the vertex set $V(G_4(Z_{36})) = \{1,5,7,11,13,17,19,23,25,29,31,35\}$, which excludes the units contained in I. To verify the diameter, consider the vertices 1 and 29. These are not directly connected, but a path can be formed through intermediate vertices. For example, $1 \to 7 \to 19 \to 29$ is a valid path where the differences $1-7=-6\in I, 7-19=-12\in I$, and $19-29=-10\in I$. This path has a length of 3, and since no shorter path exists between 1 and 29, the graph's diameter is $diam(G_4(Z_{36}))=3$. It can be seen in Figure 3.

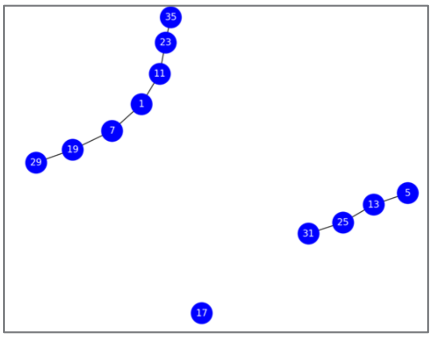

Figure 3 – *A path of length 3 in* $G_4(Z_{36})$

THEOREM 3. The girth of $G_I(R)$, defined as the length of its shortest cycle, is at least 3. If $G_I(R)$contains no cycles, its girth is infinite.

Proof. A cycle in $G_I(R)$ is a sequence of vertices $u_1, u_2, \ldots, u_k$ such that $u_1 = u_k$ and $u_i - u_i + 1 \in I$ for $i = 1,2, \ldots, k-1$. The length of the cycle is k. By definition, a cycle must involve at least three distinct vertices, as a two-vertex cycle would violate the condition that the vertices are distinct.

If $G_I(R)$ contains no cycles, then by convention, its girth is infinite. Otherwise, the shortest cycle must have a length of at least 3. This establishes the result.

EXAMPLE 4. Consider the ring $Z_8 = \{0,1,2,3,4,5,6,7\}$ where the units are $\{1,3,5,7\}$ with $I = (2)$. The vertex set of $V(\mathrm{G_I}(Z_8)) = \{1,3,5,7\}$. A cycle of length 3 is formed with the vertices $1 \to 3 \to 5 \to 1$, where each difference between connected vertices belongs to I. For instance, $1-3=-2\in I, 3-5=-2\in I$, and $5-1=4\in I$. This confirms the girth of the graph is 3.

THEOREM 4. The vertex set of $G_I(Z_n)$ is a subset of the unit group $U(Z_n)$. That is, $V(G_I(Z_n)) \subseteq U(Z_n)$.

Proof. The unit group $U(Z_n)$ consists of all integers in $\{1,2, \ldots, n-1\}$ that are co- to n. The vertices of $G_I(Z_n)$ are chosen from Z_n such that their adjacency depends on membership in the ideal (d). Particularly, a vertex $u \in Z_n$ can only be included if it is a unit, as non-units cannot satisfy the adjacency condition $u - v \in (d)$. Since only units $u \in U(Z_n)$ are to be vertices in $G_I(Z_n)$, it follows the result.

EXAMPLE 5. Consider $Z_{18} = \{0,1,2,\ldots,17\}$The unit group $U(Z_{18})$ consists of all integers coprime to 18: $\{1,5,7,11,13,17\}$. The ideal $I = (2)$ includes $\{0,2,4,6,8,10,12,14,16\}$. The vertex set of $\mathrm{G}_2(Z_{18})$ is $\{1,5,7,11,13,17\}$ as all vertices must be units. Adjacency depends on the ideal I, and vertices like 1,7,13 satisfy adjacency conditions $1 - 7 = -6 \in I, 7 - 13 = -6 \in I$. Hence, the graph demonstrates that only units form the vertex set, consistent with the above theorem.

THEOREM 5. The number of vertices in $G_I(Z_n)$ is given by $\mid V(G_I(Z_n)) \mid = \phi(n) - \mid U(Z_n) \cap (d) \mid$,where $\phi(n)$ is Euler's totient function, and $\mid U(Z_n) \cap (d) \mid$ represents the number of units in Z_n that belong to the ideal (d).

Proof. The number of units in Z_n is given by $\phi(n)$. The ideal $(d) \subseteq Z_n$ contains elements $\{kd \mid k \in Z_n\}$. The intersection $U(Z_n) \cap (d)$ contains units that are also multiples of d. The vertices in $G_I(Z_n)$ are the units $u \in U(Z_n)$ that are not in $U(Z_n) \cap (d)$. This follows the result.

EXAMPLE 6. For the ring Z_{42}, the unit group is$U(Z_{42}) = \{1,5,11,13,17,19,23,25,29,31,37,41\}$, containing 12 elements. Euler's totient function gives $\phi(42)=12$. The ideal (7) in Z_{42} consists of the elements $\{0,7,14,21,28,35\}$. As none of these elements are units, the intersection $U(Z_{42}) \cap (7)$ is empty, and $\mid U(Z_{42}) \cap (7) \mid = 0$. Using the theorem, $\mid V((G_7(Z_{42})) \mid = \phi(42) - \mid U(Z_{42}) \cap (7) \mid$, the number of vertices is $12-0=12$. Consequently, the vertex set of $G_7(Z_{42})$ is $\{1,5,11,13,17,19,23,25,29,31,37,41\}$, and the graph has 12 vertices.

THEOREM 6. Two vertices $u, v \in V(G_I(Z_n))$ are adjacent if and only if $u - v \in (d)$, that is, the difference $u - v$ is divisible by d.

Proof. By the definition of the graph $G_I(Z_n)$ two vertices $u, v \in V(\, G_I(Z_n))$ are adjacent if their difference $u - v$ belongs to the ideal (d). The ideal (d) consists of all multiples of d. That is, $(d) = \{kd \mid k \in Z_n\}$. Hence, $u - v \in (d)$ implies that there exists an integer k such that $u - v = kd$. Therefore, two vertices u and v are adjacent if and only if $u - v$ is divisible by d, which establishes the adjacency condition.

EXAMPLE 7. In the ring $Z_{15} = \{0,1,2,\ldots,14\}$ and $I = (5)$ the vertex set of $G_5(Z_{15})$ is {1,2,4,7,8,11,13,14} which includes the units of Z_{15}. The ideal (5) contains the multiples of 5, specifically $\{0,5,10\}$. For adjacency verification, consider $u = 1$ and $v = 1$. Their difference is $1 - 11 = -10$, and since $-10 \in (5)$, 1 and 11 are adjacent. Similarly, for $u = 4$ and $v = 14$, the difference is $4 - 14 = -10$, and $-10 \in (5)$, so 4 and 14 are adjacent. However, u =1 and $v = 4$ are not adjacent because $1 - 4 = -3 \notin (5)$. This illustrates that adjacency depends on the divisibility of the difference by 5, aligning with the theorem.

THEOREM 7. The graph $G_I(Z_n)$ is undirected. That is, if u is adjacent to v, then v is adjacent to u.

Proof. The adjacency condition $u - v \in (d)$ implies that $u - v = kd$ for some $k \in Z_n$. If $u - v = kd$, then $v - u = -kd$. Since $-kd \in (d)$ (as (d) is closed under multiplication by integers), it follows that $v - u \in (d)$. Thus, if u is adjacent to v, then v is also adjacent to u, making $G_I(Z_n)$ an undirected graph.

EXAMPLE 8. In the ring $Z_{20} = \{0,1,2, \dots, 19\}$ the unit group is $U(Z_{20}) = \{1,3,7,9,11,13,17,19\}$. The ideal $I = (4)$ consists of the multiples of 4, $\{0,4,8,12,16\}$. For the graph $G_4(Z_{20})$, the adjacency between two vertices u and v is determined by whether their difference $u - v$ lies in (4).

Consider the vertices $u = 1$ and $v = 9$. The difference $u - v = 1 - 9 = -8$, and since $-8 \in (4)$, u is adjacent to v. To check symmetry, calculate $v - u = 9 - 1 = 8$, which also belongs to (4). Hence, v is adjacent to u, demonstrating that adjacency is mutual. This symmetry holds for all pairs of vertices in $G_I(Z_{20})$, confirming that the graph is undirected, as per the statement of the theorem.

THEOREM 8. The connectivity of the graph $G_I(Z_n)$ depends on the properties of the ideal (d) in Z_n.If $(d) = (0)$, the graph $G_I(Z_n)$ is a complete graph with $\phi(n)$ vertices.

For $d > 0$, the graph $G_I(Z_n)$ may decompose into connected components, with the structure determined by the interaction of d with the units in Z_n.

Proof. Case 1: $(d) = (0)$

When $(d) = (0)$ the ideal consists only of 0 in Z_n, meaning $u - v \in (d)$ implies $u = v$. For all pairs $u, v \in V(\,G_I(Z_n))$ the adjacency condition $u - v \in (d)$ is always satisfied, since $0 \in (d)$. This implies that every pair of the vertices u, v is connected by an edge. As a result, $G_I(Z_n)$ is a complete graph with $\phi(n)$ vertices, because the vertex set is precisely $U(Z_n)$ and $\mid U(Z_n) \mid= \phi(n)$.

Case 2: $d > 0$

When $d > 0$, the ideal (d) consists of all multiples of d. That is, $(d) = \{kd \mid k \in Z_n\}$. Two vertices $u, v \in V(\,G_I(Z_n))$ are adjacent if $u - v \in (d)$. The divisibility condition creates a partition of the vertex set into equivalence classes, where u and v are in the same equivalence class if $u - v$ is divisible by d. The number of connected components and their size depend on the interaction of d with $U(Zn)$. If d divides many differences $u-v$, the graph is more connected. If d divides fewer differences, the graph decomposes into multiple connected components. The connectivity of $G_I(Z_n)$ is directly influenced by the ideal (d). That is, $(d)=(0)$ results in a complete graph with $\phi(n)$ vertices. $d>0$ leads to a graph whose connectivity and component structure depend on the divisibility properties of d in Z_n.

EXAMPLE 9. In the ring $Z_{10}=\{0,1,2,\dots,9\}$ the unit group is $U(Z_{10})=\{1,3,7,9\}$, consisting of 4 elements. When $I=(0)$, the ideal contains only 0, meaning that for any two vertices $u,v\in V(G_4(Z_{10}))$ the adjacency condition $u-v\in(0)$ is always satisfied. Consequently, every pair of vertices is connected by an edge, and the graph becomes a complete graph. The graph has $|U(Z_{10})|=\phi(10)=4$ vertices. In this case, $G_4(Z_{10})$ is a fully connected graph where all vertices form a single component, demonstrating the scenario when the ideal (d) is trivial.

Topological indices for ideal-based unit graphs

Topological indices are numerical descriptors that capture the structural properties of graphs. They provide critical insights into graph complexity, connectivity, and topology, with applications in fields such as algebraic graph theory, chemistry, and network analysis.

DEFINITION 2. The first Zagreb index (M_1) and the second Zagreb index (M_2) provide information about vertex degrees and their interactions. These indices of a graph G are defined as follows, $M_1(G)=\sum_{v\in V(G)} deg(v)^2$ and

$M_2(G)=\sum_{\{u,v\}\in E(G)} deg(u)\,.\,deg(v)$, where $\{u,v\}$ denotes an edge in the graph G.

DEFINITION 3. The Wiener Index captures the overall closeness of vertices in the graph. It is defined as follows $W(G)=\sum_{\{u,v\}\subseteq V(G)} d(u,v)$ where $d(u,v)$ is the shortest-path distance between the vertices u and v.

DEFINITION 4. The Randić Index reflects graph branching and is defined as

$$R(G)=\sum_{\{u,v\}\subseteq E(G)}\frac{1}{\sqrt{deg(u).deg(v)}}.$$

DEFINITION 5. The Estrada Index is based on eigenvalues of the adjacency matrix $A(G)$. It is defined as

$EE(G)=\sum_{i=1}^{n} e^{\lambda_i}$, where λ_i are the eigenvalues of $A(G)$.

DEFINITION 6. (Ramaswamy & Veena, 2009, pp.N24-N24) The graph energy (E) measures graph irregularity based on eigenvalues. It is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i are the eigenvalues of the adjacency matrix.

DEFINITION 7. (Vukičević & Furtula, 2009, pp.1369-1376) The Arithmetic-Geometric Index (AG) evaluates the interactions of adjacent vertices' degrees using an arithmetic-geometric mean. It is defined as

$$AG(G) = \sum_{\{u,v\} \in E(G)} \frac{2\, deg(u).deg(v)}{deg(u)+deg(v)}.$$

DEFINITION 8. (Zhou & Trinajstić, 2009, pp.1252-1270; Deng et al.,2013, pp.2740-2744) The Harmonic Index (HI) measures connectivity using inverse degree sums. It is defined as

$$HI(G) = \sum_{\{u,v\} \in E(G)} \frac{2}{deg(u)+deg(v)}.$$

Calculated values of topological indices

The calculated values of topological index for various values of n and the corresponding ideal structures for unit graphs on Z_n are presented in Table 1. These indices include the first and second Zagreb indices, Wiener index, Randić index, Estrada index, Graph energy, Arithmetic-Geometric index, and Harmonic index.

Table 1 - Calculated values of selected topological indices for some graphs

$G_I(Z_n)$	M_1	M_2	Wiener Index	Randić Index	Estrada Index	Graph Energy	AG	HI
$G_2(Z_6)$	8	4	1	0.5	3.086	2	2	0.5
$G_2(Z_8)$	18	12	2	0.5	6.502	4	4.5	1.0
$G_2(Z_{10})$	32	22	4	0.4	12.248	6	6.2	1.25
$G_3(Z_{12})$	50	36	6	0.4	19.292	8	9	1.5
$G_3(Z_{15})$	72	50	10	0.35	24.532	10	12	1.75
$G_4(Z_{16})$	96	72	12	0.3	30.456	12	16	2.0
$G_3(Z_{18})$	128	90	15	0.28	35.127	14	20	2.25
$G_5(Z_{20})$	160	110	20	0.25	41.052	16	25	2.5
$G_4(Z_{24})$	288	204	30	0.2	58.728	18	40	3.0
$G_5(Z_{25})$	400	280	40	0.18	63.456	20	50	3.25
$G_6(Z_{30})$	750	550	60	0.15	83.452	24	75	3.5
$G_6(Z_{36})$	1152	816	90	0.12	112.892	28	112	4.0

$G_8(Z_{40})$	1600	1120	120	0.1	125.532	30	150	4.25
$G_9(Z_{45})$	2025	1410	150	0.09	150.412	32	225	4.5
$G_{10}(Z_{50})$	2500	1750	200	0.08	175.236	34	300	4.75

The comparative graphs which provide a clear visualization of how different topological indices evolve with increasing n in the unit graphs $G_I(Z_n)$ can be seen in Figures 4 and 5. The first Zagreb Index (M_1) and the second Zagreb Index (M_2) demonstrate a steady growth, correlating with the increasing vertex degrees and their interactions, while the Wiener Index reflects the growing average distances between vertices in larger graphs. The Estrada Index, with its exponential-like rise, highlights the influence of spectral contributions as graph complexity increases. Conversely, the Randić Index shows a slight decline, indicating reduced branching as the graphs become denser. In contrast, the Harmonic Index (HI) grows gradually, signalling enhanced vertex connectivity. Together, these graphs reveal how different aspects of graph topology such as degree distribution, connectivity, and spectral characteristics respond to changes in n, offering a comprehensive perspective on structural dynamics.

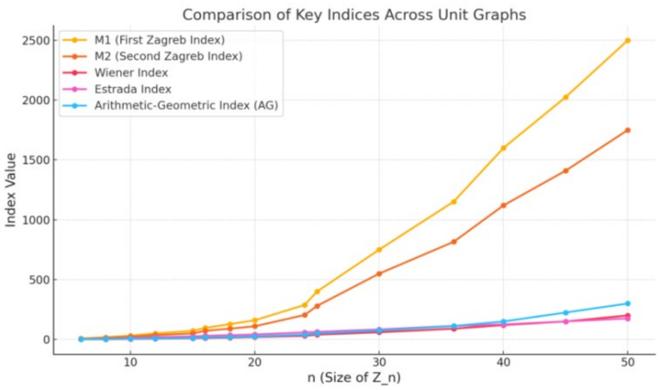

Figure 4 - Comparison of the key indices across unit graphs

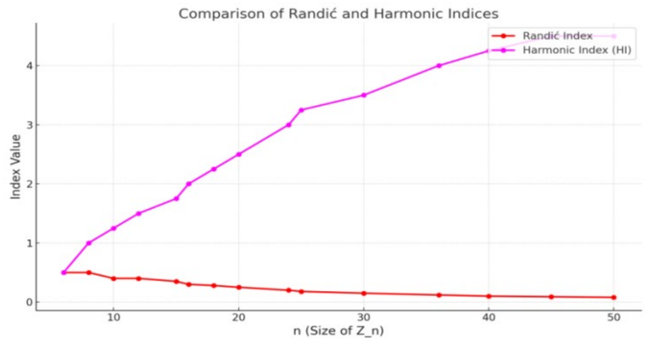

Figure 5 - Comparison of Randić and Harmonic indices

Algebraic properties and graph indices

This section explores the elaborate relationship between the algebraic properties of commutative rings Z_n and the topological indices derived from their associated unit graphs. These indices, computed for various ideals and values of n, are analyzed to uncover their dependence on the ring structure and the selected ideal. The algebraic structure of Z_n profoundly impacts the topology of its associated unit graph $G_I(Z_n)$.

The unit graph $G_I(Z_n)$ serves as a bridge between algebraic structures and graph theory. Its topology is inherently influenced by the properties of Z_n, particularly the chosen ideal $I = (d)$. The divisor d and its relationship with n shape the graph's connectivity, symmetry, and vertex interactions. The ideal $I = (d)$ is defined as the set of all multiples of d modulo n. Since d must divide n, the size of I is determined by the quotient n/d. This quotient reflects how many elements are part of the ideal, with significant implications for graph adjacency and structure. When d is small, I contains a larger number of elements, making adjacency conditions more relaxed and resulting in a denser graph. For example, if $n = 12$ and $d = 1$, the ideal contains all elements of Z_{12}, making the graph complete. Conversely, larger d values, such as $d = 6$ in Z_{12}, produce a smaller ideal, reducing adjacency and potentially causing the graph to become fragmented or disconnected.

Impact on connectivity

The divisor d determines how elements of Z_n interact within the graph. When $d = 1$, the ideal $I = (1)$ spans the entire ring, ensuring every pair of vertices is connected and forming a complete graph. However, as d increases, the connectivity diminishes because fewer differences $u - v$ fall within I. For larger values of d, the graph may split into disconnected components, corresponding to different residue classes modulo d.

The adjacency matrix of the graph provides a clear representation of the connections between vertices. For $G_I(Z_{12})$, the matrix below highlights how the adjacency condition $u - v \in \{0,3,6,9\}$ results in limited connectivity.

The size of each connected component depends on the overlap between the cosets of I and the set of units $U(Z_n)$. For example, in Z_8 with $d = 2$, the ideal $I = \{0,2,4,6\}$ splits the graph into two components, one containing the units $\{1,3,5,7\}$ and another with non-units overlapping with I. $G_3(Z_{12})$ can be see visually in Figure 6. The unit graphs of $G_2(Z_8)$ and $G_3(Z_{12})$ can be seen in Figure 7.

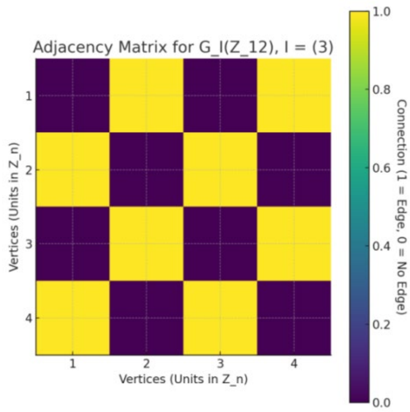

Figure 6 - Heat map of the adjacency matrix for $G_3(Z_{12})$

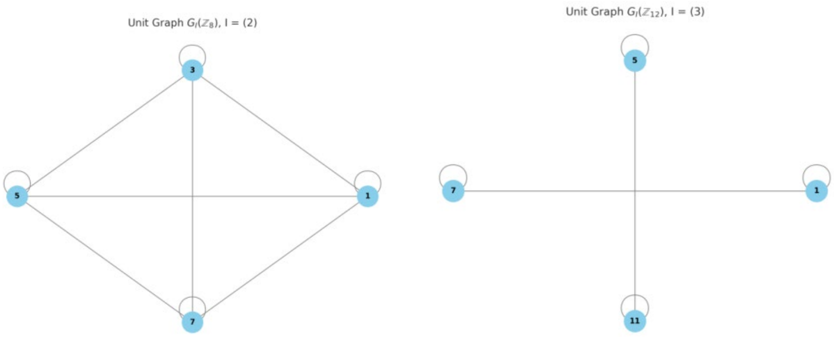

Figure 7 - Unit graphs of $G_2(Z_8)$ *and* $G_3(Z_{12})$

Cyclic structure of Z_n

The additive group of Z_n is cyclic, meaning all elements can be generated by repeated addition of a single element g. This cyclicity introduces periodicity in the graph's structure. The ideal I partitions Z_n into cosets, where each coset is of the form $a + I$ for some representative a. These cosets form disjoint subsets, and adjacency in the graph depends on whether the difference between the vertices u and v lies within I.

For smaller d, the cosets are larger, and many elements are connected, creating denser graphs. For larger d, the cosets are smaller, leading to sparser connections. This modular partitioning of Z_n ensures that the graph reflects the periodic and symmetric nature of the ring. For example, Figure 8 depicts the cyclic structure of $G_3(Z_{15})$.

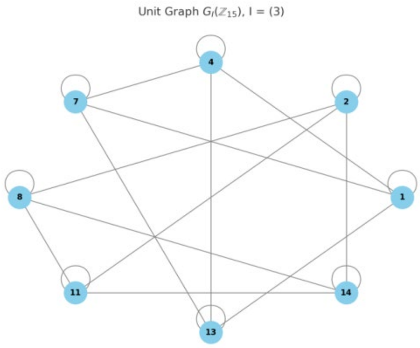

Figure 8 - Unit graph of $G_3(Z_{15})$

The ideal I forms a subgroup of Z_n under addition modulo n. Each coset of I contains n/d elements, and the graph structure depends on how these cosets interact with the units $U(Zn)$. Residue classes play a crucial role in determining the adjacency of vertices. If two vertices u and v belong to the same coset, their difference $u - v$ is in I, making them adjacent. Connections between cosets depend on their representatives. For example, in Z_{10} with $d = 5$, the ideal $I = \{0,5\}$ divides Z_{10} into two cosets $\{0,5\}$ and $\{1,2,3,4,6,7,8,9\}$. The resulting graph is sparse, with limited adjacency between these cosets.

The adjacency matrix for $G_I(Z_{10})$, where $I = \{0,5\}$, reveals the sparsity of connections caused by the ideal's partitioning into cosets. The matrix below highlights the restricted adjacency between vertices. It can be seen in Figure 9.

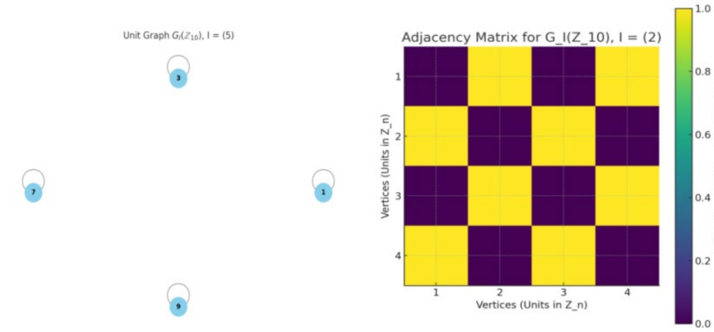

Figure 9 -Unit graph and the heat map of the adjacency matrix of $G_5(Z_{10})$

Interaction with units and degree distribution

The multiplicative group of units $U(Z_n)$ interacts with the additive structure of I. Units correspond to vertices with potentially higher degrees, as they are more likely to form edges based on the adjacency condition $u - v \in I$. Non-units, on the other hand, often contribute to sparsity, particularly when they overlap with I. Degree distributions in the graph are shaped by this interaction. In highly connected graphs (e.g., small d), most vertices have similar degrees, resulting in symmetric degree distributions. In sparse graphs (e.g., large d), the degrees vary significantly, reflecting irregular adjacency patterns.

For $G_I(Z_{15})$, the adjacency matrix below reflects how the ideal $I = \{0,3,6,9,12\}$ influences the degree distribution. Vertices within the same coset show higher connectivity, as seen in the matrix. This can be seen in Figure 10.

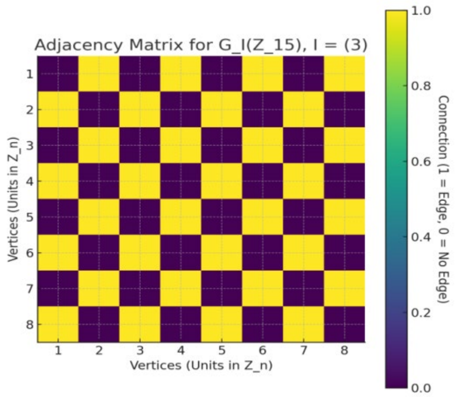

Figure 10 - Heat map of the adjacency matrix for $G_3(Z_{15})$

Graph symmetry and algebraic properties

The symmetry of Z_n is particularly evident when n is please include 15. In such cases, all non-zero elements of Z_n are units, and the graph exhibits uniformity. For composite n, the presence of zero-divisors and varying interactions between units and non-units introduce asymmetry, leading to diverse graph structures.

The algebraic properties of Z_n, including its cyclic nature, residue classes, and ideal structure, are vividly reflected in the topology of $G_I(Z_n)$. These properties influence vertex connectivity, graph symmetry, and degree distribution, highlighting the interplay between ring theory and graph topology.

Applications

The applications of ideal-based unit graphs span structural, algebraic, and comparative domains. By leveraging topological indices and their correlation with ring properties, these graphs serve as a powerful tool for analyzing the underlying algebraic structures. Their ability to unify graph-theoretic and ring-theoretic perspectives makes them a promising framework for future research in algebraic graph theory and related fields.

Structural insights

The study of topological indices in ideal-based unit graphs $G_I(R)$ provides valuable structural insights into graph complexity and connectivity. Indices such as the First Zagreb Index (M_1) and the Second Zagreb Index (M_2) quantify vertex degree interactions, offering a measure of graph density and edge distribution. Higher values of M_1 and M_2 correlate with dense connectivity, typically observed when the ideal $I = (d)$ has smaller d, allowing for more relaxed adjacency conditions. Conversely, sparse graphs with lower values of these indices emerge when d is larger, limiting connections.

The Wiener Index captures the average pairwise distances between vertices, providing a measure of graph compactness. Dense graphs with small d exhibit lower Wiener Index values, reflecting shorter paths between vertices, while sparse graphs with larger d show increased distances. Similarly, the Estrada Index reveals spectral characteristics, with exponential growth indicating the influence of eigenvalues on connectivity. The Randić Index and the Harmonic Index further enrich the analysis, highlighting how branching and connectivity vary with the ideal's size. These indices collectively allow researchers to classify graphs, identify patterns, and predict structural properties based on the choice of R and I.

Ring-theoretic implications

The topology of $G_I(R)$ directly correlates with the algebraic properties of the ring R and its ideal I. The adjacency condition $u - v \in I$ reflects the additive subgroup structure of R, and the resulting graph connectivity encapsulates how the ideal I partitions R. For instance, smaller ideals encompass more differences $u - v$, leading to connected graphs, while larger ideals, containing fewer elements, can fragment the graph into disconnected components.

The interaction of the multiplicative group of units $U(R)$ with the additive structure of I further highlights ring-theoretic nuances. For example, the distribution of units and non-units within R determines vertex degrees and edge formation in $G_I(R)$. The analysis of these graphs provides insights into ring symmetry, residue class structures, and the relationship between ring elements and their cosets modulo I. This bridge between graph topology and ring theory can uncover properties such as the presence of zero divisors, the nature of subgroups, and the behaviour of R under different ideal selections.

Comparisons

The ideal-based unit graph $G_I(R)$ offers a new perspective for comparing standard unit graphs and zero-divisor graphs (Ashrafi et al.,2010, pp.2851-2871; Anderson et al.,2011, pp.23-45). Unlike standard unit graphs, where adjacency is based solely on the sum of units being invertible, $G_I(R)$ introduces an additional layer of complexity by restricting adjacency via the ideal I. This leads to a richer variety of graph structures, ranging from dense to sparse, based on the ideal's size and position within R.

In contrast to zero-divisor graphs, which focus on the multiplicative behaviour of non-units, $G_I(R)$ emphasizes additive properties. While zero-divisor graphs reveal information about ring annihilators and zero-divisors, $G_I(R)$ highlights the distribution of units and their differences modulo I. Comparing these graphs provides a comprehensive view of the interplay between additive and multiplicative structures within R. Such comparisons can guide applications in algebraic graph theory, where understanding the balance between addition and multiplication in ring structures is critical.

Properties of ideal-based unit graphs

The study of graph-theoretical representations of algebraic structures has gained significant attention in recent years, particularly through the exploration of ideal-based zero-divisor graphs. Mallika, Kala, and Selvakumar developed foundational properties of zero-divisor graphs where the adjacency of vertices is influenced by an ideal of the ring (Mallika et al., 2017, pp.177-187). These graphs have proven to be powerful tools for understanding the interplay between ring-theoretic properties and graph invariants, such as chromatic number, clique number, and girth. Inspired by their approach, this section extends similar concepts to ideal-based unit graphs which focus on the additive difference relationship among the units of a ring, constrained by an ideal. By adapting and generalizing the results from zero-divisor graphs, this section aims to provide a deeper structural understanding of ideal-based unit graphs and their chromatic, connectivity, and regularity properties.

THEOREM 9. The girth of the graph $G_I(R)$, denoted by $girth\ (G_I(R))$, satisfies the following properties:

If $G_I(R)$ contains cycles, then $girth\ (G_I(R)) \geq 3$ as the cycles in $G_I(R)$ must involve at least three distinct vertices due to the additive adjacency condition.

If $G_I(R)$ is acyclic, then $girth(G_I(R)) = \infty$.

Proof. The graph $G_I(R)$ is constructed with vertices as $u, v, w, \dots \in U(R) \setminus I$, the set of units of R excluding those in the ideal I. Two vertices u and v are adjacent if $u - v \in I$. To determine the girth of $G_I(R)$, the cycle structure of the graph is analysed.

For a cycle to exist in $G_I(R)$, there must be a sequence of vertices $u_1, u_2, \dots, u_k$ such that $u_i - u_{i+1} \in I$ for $1 \leq i < k$, and $u_k - u_1 \in I$. This sequence forms a closed loop where each adjacent pair satisfies the additive adjacency rule $u - v \in I$. The girth of $G_I(R)$, defined as the length of the shortest cycle, is determined by the minimum k for which such a sequence exists. Due to the additive nature of adjacency and the exclusion of self-loops ($u - u = 0$) and two-vertex cycles, the minimum cycle length is $k \geq 3$ if cycles exist.

If $G_I(R)$ does not contain any cycles, it is considered acyclic. In this case, the girth of the graph is defined as ∞, reflecting the absence of closed paths. This situation arises when no sequence of vertices $u_1, u_2, \dots, u_k$ satisfies the adjacency rule for a complete cycle.

The proof assurances that if cycles exist in $G_I(R)$, the girth is at least 3, as cycles must involve at least three vertices due to the exclusion of self-loops and two-vertex cycles. Otherwise, the graph is acyclic, and the girth is ∞. This completes the proof.

EXAMPLE 10. Ring: $R = Z_{18}$, Ideal: $I = (3) = \{0,3,6,9,12,15\}$, the vertices set is $U(R)\backslash I = \{1,5,7,11,13,17\}$. Edges : the vertices u, v are adjacent if $u - v \in I$. There exists the cycle 1→7→11→1. Graph Properties: A single large cycle involving all 8 vertices.

THEOREM 10. Let R be a commutative ring and I an ideal of R. The graph $G_I(R)$ contains a cycle if and only if there exist distinct $u, v, w \in U(R) \setminus I$ such that $(u - v), (v - w), (w - u) \in I$.

Proof. To prove the theorem, both the necessity and sufficiency of the stated condition for the presence of a cycle in $G_I(R)$ are established.

Necessity. A cycle in $G_I(R)$ implies the existence of vertices $u, v, w \in U(R)$ such that $u \to v \to w \to u$. By the definition of adjacency in G $G_I(R)$, two vertices $x, y \in U(R) \setminus I$ are adjacent if and only if their difference $x - y \in I$. For the cycle $u \to v \to w \to u$ in $G_I(R)$, the adjacency conditions are satisfied because $u - v \in I$, and $w - u \in I$. These adjacency relations demonstrate that the differences between the consecutive vertices u, v, w are contained in the ideal I. Hence, the existence of a cycle in $G_I(R)$ necessitates the presence of distinct vertices $u, v, w \in U(R) \setminus I$ such that $(u - v), (v - w), (w - u) \in I$.

Sufficiency. Conversely, assume there exist distinct vertices $u, v, w \in U(R) \setminus I$ such that $u - v \in I, v - w \in I$, and $w - u \in I$. These conditions ensure that u and v are adjacent, v and w are adjacent, and w and u are adjacent in $G_I(R)$. As a result, the edges $u \to v \to w \to u$ form a cycle in $G_I(R)$. Thus, the presence of such u, v, w is sufficient to guarantee a cycle in the graph.

EXAMPLE 11. Let $R = Z_9 = \{0,1,2,3,4,5,6,7,8\}$ and $I = (3) = \{0,3,6\}$. The units $U(R) = \{1,2,4,5,7,8\}$. The vertices of $G_I(R)$ are $U(R) \setminus I = \{1,2,4,5,7,8\}$, as I contains no units. Two vertices $u, v \in U(R) \setminus I$ are adjacent if $u - v \in I$. The adjacency relations, $1 - 4 = -3 \equiv 6 \; mod \; 9 \in I, 4 - 7 = -3 \equiv 6 \; mod \; 9 \in I, 7 - 1 = -6 \equiv 3 \; mod \; 9 \in I$. This forms a cycle $1 \to 4 \to 7 \to 1$.

THEOREM 11. Let R be a commutative ring and I an ideal. If $G(R/I)$ is disconnected, $G_I(R)$ is regular only if all connected components of $G(R/I)$ are regular, and cosets contribute uniformly to $G_I(R)$.

Proof. To establish the regularity conditions for $G_I(R)$, the relationships between $G_I(R)$ and $G(R/I)$ and the uniform contribution of cosets has to be analysed. If $G(R/I)$ is regular with the degree k, each vertex in $G(R/I)$ has exactly k adjacent vertices. This regularity arises from the structure of R/I, where vertices (cosets of I) are connected if their difference lies in I. The adjacency relations in $G_I(R)$ are inherited from $G(R/I)$, as the vertices of $G_I(R)$ (units in $R \setminus I$) are distributed among the cosets $a + I$. For the regularity to propagate from $G(R/I)$ to $G_I(R)$, it is necessary that each coset $a + I$ contributes uniformly to the vertex set of $G_I(R)$.

Uniform contribution means that all cosets $a + I$ in R/I contribute an equal number of vertices to $G_I(R)$. Specifically, the number of vertices contributed by each coset $a + I$ is $|U(R) \setminus I|/|R/I|$, where $|U(R) \setminus I|$ represents the total number of units not in I and $|R/I|$ is the number of cosets in R/I. If this contribution is uniform, the adjacency relations between the vertices of $G_I(R)$ mirror the regular structure of $G(R/I)$, ensuring consistent vertex degrees in $G_I(R)$.

The graph $G_I(R)$ is regular if $G(R/I)$ is regular and all cosets contribute uniformly to the vertex set of $G_I(R)$. The uniform contribution ensures that the vertex degrees in $G_I(R)$ remain consistent across all vertices, thereby propagating the regularity of $G(R/I)$ to $G_I(R)$. This concludes the proof.

EXAMPLE 12. The ring Z_{24} consists of the elements $\{0,1,2,\ldots,23\}$. The ideal $I = (3)$ includes $\{0,3,6,9,12,15,18,21\}$. The units are $U(Z_{24}) = \{1,5,7,11,13,17,19,23\}$. The vertex set of $G_I(R)$ is $U(Z_{24})\setminus I = \{1,5,7,11,13,17,19,23\}$.

Adjacency relations as follows, 1−5 = −4 ≡ 20 mod 24 ∈ I, 5−7 = −2 ≡22 mod 24 ∈ I, 7−11 = −4 ≡ 20 mod 24 ∈ I, 11−1 = −10 ≡ 14 mod 24 ∈ I (forming cycle 1). Similarly, 13→17→19→23→13 forms cycle 2.

Comparision of unit graphs and chemical graphs

Graphs are powerful tools for representing both algebraic and chemical structures. Unit graphs $G_I(R)$ from algebraic ring theory and molecular graphs from chemistry share deep structural similarities. This section explores these similarities, focusing on regularity, symmetry, and cyclic properties. By comparing ideal-based unit graphs of commutative rings with well-known chemical graphs, the study bridges abstract algebra with real-world molecular systems.

DEFINITION 9. For the unit graph $G_I(R)$, the vertices are units of R, $U(R)$ excluding those in the ideal I. The edges are defined as two vertices u and v which are adjacent if $u - v \in I$.Topology depends on the structure of the ring R and the ideal I.

For the chemical graphs, the vertices are atoms in the molecule (e.g., carbon, hydrogen, oxygen). The edges are covalent bonds (single, double, triple) connecting atoms. Topology reflects the molecular structure and bond distribution.

Case studies and comparisons

EXAMPLE 13. Benzene (C_6H_6) vs. $G_I(Z_9), I = (3)$. Unit graph $G_I(Z_9), I = (3)$; vertices: $\{1,2,4,5,7,8\}$; edges: two disjoint triangles: 1→4→7→1 ; 2→5→8→2. It is 2-regular (each vertex connects to 2 neighbours). Chemical graph (benzene): vertices: six carbon atoms arranged cyclically. Edges: alternating single and double bonds. It is 2-regular (each carbon connects to 2 neighbours).

Observation: Both graphs exhibit cyclic symmetry, with benzene's molecular structure reflecting the disjoint triangles of $G_I(Z_9)$. Figure 12 depicts this example.

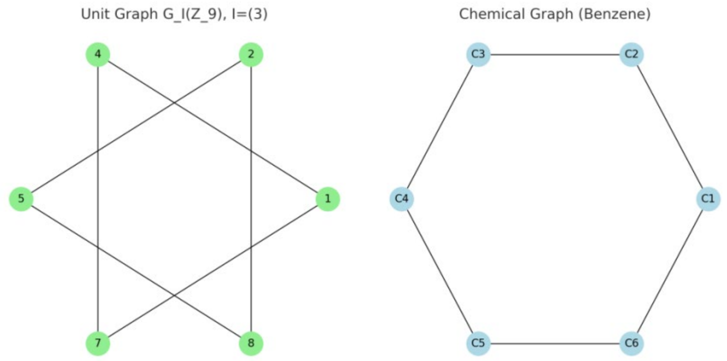

Figure 12 - Unit graph of $G_3(Z_9)$ and the chemical graph of benzene

EXAMPLE 14. Cyclooctane (C_8H_{16}) vs. $G_I(Z_{24}), I = (3)$. Unit graph $G_I(Z_{24}), I = (3)$: vertices: $\{1,5,7,11,13,17,19,23\}$. Edges: single cycle: 1→5→7→11→13→17→19→23→1.Regularity:2-regular.
Chemical graph (cyclooctane): vertices: eight carbon atoms in a cyclic structure. Edges: single bonds between consecutive atoms. Regularity: 2-regular.

Observation: The large cyclic structure of cyclooctane parallels the single cycle in $G_I(Z_{24})$. The following figure (Figure 13) depicts this example.

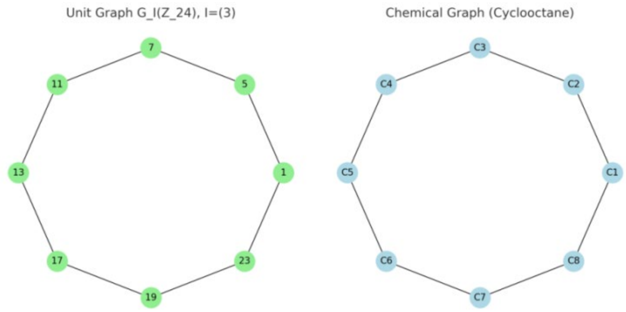

Figure 13- Unit graph of $G_3(Z_{24})$ and the chemical graph of cyclooctane

This comparative study reveals the versatility of graph theory in linking algebraic and chemical systems. The symmetry, regularity, and cyclic properties of unit graphs and chemical graphs illustrate how abstract algebra can model real-world molecular systems. Future work can explore further connections, enhancing interdisciplinary understanding and applications.

Conclusion

The study of ideal-based unit graphs $G_I(Z_n)$ provides a fascinating intersection of algebraic structures and graph theory, revealing deep connections between the properties of Z_n, the choice of ideals, and the resulting graph topology. By analysing these graphs, it is observed that the divisor d of n, which generates the ideal $I = (d)$, plays a pivotal role in determining the connectivity, symmetry, and sparsity of the graph. Smaller values of d lead to dense, highly connected graphs, often complete, while larger d values result in sparse graphs that may fragment into disconnected components. This interplay is further enriched by the cyclic structure of Z_n, where adjacency is influenced by residue classes and coset interactions. The interaction between the additive subgroup I and the multiplicative group of units $U(Z_n)$ highlights the algebraic properties shaping vertex connectivity and degree distributions.

The evaluation of topological indices on these graphs such as the Zagreb indices, Wiener Index, Estrada Index, Randić Index, and others provides a numerical lens to quantify their structural and spectral characteristics. The key insights include the steady growth of the first and second Zagreb Indices with increasing n, reflecting enhanced degree interactions, and the exponential rise of the Estrada index, indicating the growing influence of spectral contributions. Conversely, the Randić index declines slightly, showcasing reduced branching, while the Harmonic index increases, emphasizing improved connectivity in denser graphs. These indices, when considered collectively, reveal the dynamic interplay between algebraic properties and graph topology, offering a comprehensive framework for analysing unit graphs of commutative rings.

The comparison of these unit graphs with chemical graphs not only highlights structural parallels but also opens pathways for interdisciplinary applications in mathematical chemistry and cryptography. Future work could extend this framework to weighted graphs, modular systems, or higher algebraic structures, further broadening the scope of ideal-based unit graph research.

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Тополошки индекси и структурне особине јединичних графова заснованих на идеалима у комутативним прстеновима

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ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/сврха: У овом раду уводи се појам јединичних графова заснован на простом идеалу који је повезан са комутативним прстеном R. Чворови овог графа јесу јединице R које не припадају изабраном простом идеалу I, а два таква чвора сматрају се суседним ако њихова разлика припада идеалу I. Циљ је да се истраже структурна, алгебарска и тополошка својства овог графа, као и да се испитају алгебарске импликације различитих граф-теоријских инваријанти.

Методе: Јединични графови засновани на идеалима конструишу се коришћењем прстена $\mathbb{Z}_n$ где јединице искључене из изабраног простог идеала формирају скуп чворова. Суседност између два чвора одређује се по томе да ли се њихова разлика налази у идеалу. Приликом анализе израчунато је неколико тополошких индекса, укључујући Загребачке индексе, Винеров индекс, аритметичко-геометријски индекс, хармонијски индекс, Естрадинов индекс, и енергију графа. Геометријске визуализације и матрице суседства користе се за тумачење комплексности и повезаности графова.

Резултати: Резултати показују да структура добијеног графа у знатној мери зависи од модулуса n и природе изабраног идеала. Мањи идеали доводе до графова са великом повезаношћу, док већи идеали дају ређе или неповезане графове. Израчунати индекси одражавају обрасце у симетрији, расподели степена и растојању, указујући тако на суштинске алгебарске карактеристике.

Закључци: Јединични графови засновани на идеалима представљају нови оквир за проучавање интеракције између алгебарских својстава прстена и структурних особина графова. Добијени резултати доприносе развоју алгебарских алата примењљивих у математичкој хемији, безбедној комуникацији и теоријској рачунарској науци.

Кључне речи: јединице, идеали, тополошки индекси, комутативни прстен.

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