

Applications of the expanded Darbo's fixed point theorem to fractional differential equations with finite delay

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Abstract:

Introduction/purpose: The aim of this paper is to study the existence of mild solutions to the initial value problem (IVP for short) of the Darboux problem for partial hyperbolic fractional differential equations via the Caputo derivative with finite delay in Fréchet spaces.

Methods: In our analysis, Darbo's fixed point theorem is employed together with the notion of measure of noncompactness to establish the existence of solutions by reducing the research to proving the existence and uniqueness of fixed points of appropriate operators.

Results: Under suitable conditions, existence of mild solutions to the considered fractional partial hyperbolic differential equations is proved. An illustrative example demonstrates the theoretical results applicability.

Conclusion: Throughout this paper, sufficient conditions were considered to establish the existence and uniqueness of solutions to the Darboux problem by applying Darbo's fixed point theorem on unbounded interval. The study provides a rigorous methodological framework for analyzing similar classes of problems and contributes to the broader understanding of mild solutions in fractional calculus and functional analysis. The results offer potential applications for researchers investigating fractional differential equations in infinite-dimensional spaces.

Key words: differential partial hyperbolic equation, fractional order derivative, mild solution, Fréchet space.

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Introduction

For many years, not only have mathematicians been interested in fractional calculus and fractional order differential equations, but physicists and engineers have been as well. In fact, there is a wide range of applications in the fields of porous media, electrochemistry, viscoelasticity, electromagnetism and rheology (Baleanu, et al.; Hilfer, 2000; Ortigueira, 2011; Tarasov, 2010). In recent years, ordinary and partial fractional differential equations have undergone significant development. For more details, see Abbas *et al.* (Abbas et al., 2012) and Podlubny (Podlubny, 1999), Abbas and Bechohra (Abbas and Benchohra, 2009, 2012), Benchohra *et al.* (Benchohra and Hellal, 2014a,b; Benchohra et al., 2017), Helal (Helal, 2020), Kilbas and Marzan (Kilbas and Marzan, 2005), Vityuk and Golushkov (Vityuk and Golushkov, 2004) and the references therein.

In 1930, Kuratowski (Kuratowski, 1930) first proposed the notion of a measure of noncompactness. This concept is a powerful tool in functional analysis, particularly in operator equation theory in Banach spaces and metric fixed point theory. The study of the existence of solutions for integral and integro-differential, ordinary and partial differential equations also makes use of this notion. We refer the reader to (Banaei, 2018; Banás et al., 2017; Banás and Mursaleen, 2014) for further information.

In this paper, we examine the existence of solutions to fractional order initial value problem (*IVP* for short). More precisely, we will consider the following problem:

$$({}^c D_0^r u)(t, x) = f(t, x, u_{(t,x)}), \text{ for a. a } (t, x) \in J, \quad (1)$$

$$u(t, x) = \phi(t, x), \text{ if } (t, x) \in \tilde{J}, \quad (2)$$

$$\begin{cases} u(t, 0) = \varphi(t), \\ u(0, x) = \psi(x), & (t, x) \in J, \\ \varphi(0) = \psi(0), \end{cases} \quad (3)$$

where $J := [0, \infty) \times [0, \infty)$, $\tilde{J} := [-\alpha, \infty) \times [-\beta, \infty) \setminus [0, \infty) \times [0, \infty)$, ${}^c D_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $f : J \times E \rightarrow E$ is a given function, $(E, \|\cdot\|_E)$ is a (real or complex) Banach space, $\phi : \tilde{J} \rightarrow E$ is a given continuous function with $\phi(t, 0) = \varphi(t)$, $\phi(0, x) = \psi(x)$ for each $(t, x) \in J$, $\varphi, \psi : [0, \infty) \rightarrow E$ are given absolutely continuous and $\varphi(t)$ and $\psi(x)$ are measurable for all $(t, x) \in J$.

We denote by $u_{(t,x)}$ the element of $C(\tilde{J}, E)$ defined by

$$u_{(t,x)}(s, \tau) = u(t + s, x + \tau); \quad (s, \tau) \in \tilde{J},$$

where $u_{(t,x)}(\cdot, \cdot)$ represents the history of the state u .

Inspired by other studies, we examine the existence of solutions for the problem (1)-(3). The existence of solutions for this problem will be found by using the expanded Darbo's fixed point theorem (Dudek, 2017; Dudek and Olszowy, 2015), and the notion of measure of noncompactness (Ayerbee Toledano et al., 1997) in Fréchet spaces. To the author's knowledge, there are few papers on fractional differential equations with delay in Fréchet spaces. This work aims to further continue this investigation.

The paper is organized as follows: we will go over some essential information in Section 2 that will be applied to the next sections. The main results are presented in Section 3, where we demonstrate that the problems (1)-(3) have mild solutions. In the last section, an example is given to justify the application of our results.

Background materials and preliminaries

In this section, we give some notations, essential definitions of fractional partial integrals and derivatives and preliminary facts to simplify the forthcoming analysis.

Let $J_0 = [0, n] \times [0, n]$; $n > 0$. A measurable function $u : J_0 \rightarrow E$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. For properties of the Bochner integral, see, for instance, Yosida (1980).

$C(J_0)$ is the Banach space of all continuous functions from J_0 into E with the norm

$$\|u\|_{\infty} = \sup_{(t,x) \in J_0} \|u(t, x)\|.$$

$AC(J_0)$ denotes the space of absolutely continuous functions from J_0 into E . $L^1(J_0)$ is the Banach space of measurable functions $u : J_0 \rightarrow E$ which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_0^n \int_0^n \|u(t, x)\|_E dt dx.$$

Now, we provide some basic overview of fractional calculus.

Definition 1. (Vityuk and Golushkov, 2004) Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ and $u \in L^1(J_0)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t, x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} u(s, \tau) d\tau ds.$$

In particular,

$$(I_\theta^\sigma u)(t, x) = u(t, x), \quad (I_\theta^\sigma u)(t, x) = \int_0^t \int_0^x u(s, \tau) d\tau ds;$$

for almost all

$$(t, x) \in J_0,$$

where $\sigma = (1, 1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0, \infty) \times (0, \infty)$, when $u \in L^1(J_0)$.

Note also that when $u \in C(J_0)$, then $(I_\theta^r u) \in C(J_0)$. Moreover

$$(I_\theta^r u)(t, 0) = (I_\theta^r u)(0, x) = 0; \quad (t, x) \in J_0.$$

Example 1. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}, \quad \text{for almost all } (t, x) \in J_0.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in (0, 1] \times (0, 1]$. Denoted by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2. Vityuk and Golushkov (2004) Let $r \in (0, 1] \times (0, 1]$ and $u \in L^1(J_0)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression

$$D_\theta^r u(t, x) = (D_{tx}^2 I_\theta^{1-r} u)(t, x)$$

and the Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^c D_\theta^r u)(t, x) = \left(I_\theta^{1-r} \frac{\partial^2}{\partial t \partial x} u \right) (t, x).$$

The case $\sigma = (1, 1)$ is included, thus we have

$$(D_\theta^\sigma u)(t, x) = ({}^c D_\theta^\sigma u)(t, x) = (D_{tx}^2 u)(t, x), \quad \text{for almost all } (t, x) \in J_0.$$

Example 2. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, then

$$D_{\theta}^r t^{\lambda} x^{\omega} = \frac{\Gamma(1 + \lambda)\Gamma(1 + \omega)}{\Gamma(1 + \lambda - r_1)\Gamma(1 + \omega - r_2)} t^{\lambda - r_1} x^{\omega - r_2}, \text{ for almost all } (t, x) \in J_0.$$

We recall the definition of the concept of a sequence of measures of noncompactness, as given in [Dudek \(2017\)](#); [Dudek and Olszowy \(2015\)](#), as follows:

Definition 3. Let \mathcal{M}_X be the family of all nonempty and bounded subsets of a Fréchet space X . A family of functions α where $\alpha : \mathcal{M}_X \rightarrow [0, \infty)$ is said to be a family of measures of noncompactness in the real Fréchet space X if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_X$:

- (a) α is full, that is: $\alpha(B) = 0$ if and only if B is precompact,
- (b) $\alpha(B_1) \leq \alpha(B_2)$ for $B_1 \subset B_2$,
- (c) $\alpha(\text{Conv}B) = \alpha(B)$,
- (d) If $\{B_i\}_{i=1, \dots}$ is a sequence of closed sets from \mathcal{M}_X such that $B_{i+1} \subset B_i$; $i = 1, \dots$ and if $\lim_{i \rightarrow \infty} \alpha(B_i) = 0$, then the intersection set $B_{\infty} := \bigcap_{i=1}^{\infty} B_i$ is nonempty.

Some Properties:

- (e) We call the family of measures of noncompactness α to be homogeneous if $\alpha(\lambda B) = |\lambda|\alpha(B)$, for $\lambda \in \mathbb{R}$.
- (f) If the family α satisfies the condition $\alpha(B_1 \cup B_2) \leq \alpha(B_1) + \alpha(B_2)$, it is called subadditive.
- (g) It is sublinear if both conditions (e) and (f) hold.
- (h) We say that the family of measures α has the maximum property if $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$.
- (i) The family of measures of noncompactness α is said to be regular if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

For more details on measure of noncompactness and its properties see [Appell \(1981\)](#).

Lemma 1. A nonempty subset $B \subset X$ is said to be bounded if

$$\sup_{u \in X} \|u\|_n < \infty, \quad \text{for } n \in \mathbb{N}.$$

Lemma 2. *Bothe (1998)* If Y is a bounded subset of Fréchet space X , then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^\infty \subset Y$ such that

$$\alpha(Y) \leq 2\alpha(\{y_k\}_{k=1}^\infty) + \epsilon.$$

Lemma 3. ? If $\{u_k\}_{k=1}^\infty \subset L^1(J_0)$ is uniformly integrable, then $\alpha(\{u_k\}_{k=1}^\infty)$ is measurable and for each $(t, x) \in J_0$,

$$\alpha\left(\left\{\int_0^t \int_0^x u_k(s, \tau) d\tau ds\right\}_{k=1}^\infty\right) \leq 2 \int_0^t \int_0^x \alpha(\{u_k(s, \tau)\}_{k=1}^\infty) d\tau ds.$$

Definition 4. Let Ω be a nonempty subset of a Fréchet space X , and let $A : \Omega \rightarrow X$ be a continuous operator which transforms bounded subsets. One says that A satisfies the Darbo condition with constants $(k_n)_{n \in \mathbb{N}}$ with respect to a family of measures of noncompactness α , if

$$\alpha(A(B)) \leq k_n \alpha(B),$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}$.

If $k_n < 1$, $n \in \mathbb{N}$ then A is called a contraction with respect to α .

The following expanded Darbo's fixed point theorem for Fréchet spaces will be utilized in the sequel.

Theorem 1. *Dudek (2017); Dudek and Olszowy (2015)* Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space X and let $V : \Omega \rightarrow \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness α . Then V has at least one fixed point in the set Ω .

Existence and uniqueness of mild solutions

This section is devoted to the proof of the global existence of solutions for the problem (1)-(3).

For each $n \in \mathbb{N}$ we consider the following set,

$$C_n = C([- \alpha, n] \times [- \beta, n], E)$$

and we define in $C_0 := C([- \alpha, \infty) \times [- \beta, \infty), E)$ the semi-norms by:

$$\|u\|_n = \{\sup \|u(t, x)\| : - \alpha \leq t \leq n, - \beta \leq x \leq n\}.$$

Then C_0 is a Fréchet space with the family of semi-norms $\{\|u\|_n\}$.

Let us begin by explaining what we mean by a solution of the problem (1)-(3).

Definition 5. A function $u \in C_0$ is said to be a solution of (1)-(3) if u satisfies the equations (1) and (3) on J and the condition (2) on \tilde{J} .

We shall use the following lemma for the existence of solutions for the problem (1)-(3):

Lemma 4. A function $u \in C_0$ is a solution of problem (1)-(3) if and only if u satisfies the equation

$$u(t, x) = z(t, x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} f(s, \tau, u(s, \tau)) d\tau ds,$$

for all $(t, x) \in J$ and the condition (2) on \tilde{J} , where

$$z(t, x) = \varphi(t) + \psi(x) - \varphi(0).$$

we propose the following assumptions to prove our main results:

(H1) The functions $\varphi(t)$ and $\psi(x)$ are measurable and bounded for a.e. $(t, x) \in J_0$,

(H2) The function $(t, x) \rightarrow f(t, x, u)$ is measurable on J_0 , and the function $u \rightarrow f(t, x, u)$ is continuous on E for a. e. $(t, x) \in J_0$.

(H3) There exist functions $p_1, p_2 : J_0 \rightarrow [0, \infty)$ such that for each $(t, x) \in J_0$,

$$\|f(t, x, u)\|_E \leq p_1(t, x) + p_2(t, x)\|u\|_n, \quad \text{for all } u \in E.$$

(H4) For each $(t, x) \in J_0$, and any bounded $B \subset X$,

$$\alpha(f(t, x, B)) \leq p_2(t, x)\alpha(B),$$

where α is a measure of noncompactness on the Banach space E .

(H5) There exists a positive real number R_n such that

$$z^* + \frac{[p_{1n}^* + p_{2n}^* R_n] n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \leq R_n,$$

where

$$z^* = \sup_{(t,x) \in J_0} \|z(t, x)\|_E, \quad p_{in}^* = \sup_{(t,x) \in J_0} p_i(t, x), i = 1, 2.$$

Theorem 2. Assuming that assumptions (H1)-(H5) hold. If

$$\ell_n := \frac{4p_{2n}^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

then the proposed problem (1)-(3) has at least one mild solution.

Proof. Define the operator $N : C_0 \rightarrow C_0$ by

$$(Nu)(t, x) = \begin{cases} \phi(t, x), & (t, x) \in \tilde{J}, \\ z(t, x) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} \\ \times f(s, \tau, u(s, \tau)) d\tau ds, & (t, x) \in J. \end{cases} \quad (4)$$

It is evident that the fixed points of the operator N are solutions of our problem (1)-(3).

We consider the ball $B_{R_n} = \{u \in C_0 : \|u\|_n \leq R_n\}$, then from (H3), for any $u \in B_{R_n}$, we obtain

$$\begin{aligned} & \| (Nu)(t, x) \|_E \leq \\ & \leq \| z(t, x) \|_E + \int_0^t \int_0^x \frac{(t-s)^{r_1-1} (x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \| f(s, \tau, u(s, \tau)) \|_E d\tau ds \\ & \leq \| z(t, x) \|_E + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} p_1(s, \tau) d\tau ds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} p_2(s, \tau) \| u(s, \tau) \| d\tau ds \\ & \leq z^*(w) + \frac{p_{1n}^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} d\tau ds \\ & + \frac{p_{2n}^* R_n}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} d\tau ds \\ & \leq z^* + \frac{[p_{1n}^* + p_{2n}^* R_n] n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ & \leq R_n. \end{aligned}$$

This demonstrates that N transforms the ball B_{R_n} into itself.

We shall show that the operator $N : B_{R_n} \rightarrow B_{R_n}$ satisfies all the conditions of Theorem 1.

The proof will proceed in a sequence of steps.

Step 1: N is continuous.

Let $\{u_m\}_{m \in \mathbb{N}}$ be a sequence such that $u_m \rightarrow u$ in C_0 . Then, for each $(t, x) \in J_0$, we have

$$\begin{aligned} & \| (Nu_m)(t, x) - (Nu)(t, x) \| \\ & \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1} (x-\tau)^{r_2-1} \| f(s, \tau, u_m(s, \tau)) - \\ & \quad - f(s, \tau, u(s, \tau)) \| d\tau ds. \end{aligned}$$

Since $u_m \rightarrow u$ as $m \rightarrow \infty$, the Lebesgue dominated convergence theorem implies that

$$\|Nu_m - Nu\|_n \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Step 2: $N(B_{R_n})$ is bounded.

Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.

Step 3: For each bounded subset D of B_{R_n} , we have

$$\alpha(N(D)) \leq \ell_n \alpha(D).$$

From Lemmas 2 and 3, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_m\}_{m=0}^\infty \subset D$, such that for all $(t, x) \in J_0$, we have

$$\begin{aligned} & \alpha((N(D))(t, x)) \\ & = \alpha \left(\left\{ z(t, x) + \int_0^t \int_0^x \frac{(t-s)^{r_1-1} (x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, \tau, u(s, \tau)) d\tau ds; u \in D \right\} \right) \\ & \leq \left\{ \int_0^t \int_0^x \frac{(t-s)^{r_1-1} (x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, \tau, u_m(s, \tau)) d\tau ds \right\}_{m=1}^\infty + \epsilon \\ & \leq 4 \int_0^t \int_0^x \alpha \left(\left\{ \frac{(t-s)^{r_1-1} (x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, \tau, u_m(s, \tau)) \right\}_{m=1}^\infty \right) d\tau ds + \epsilon \\ & \leq 4 \int_0^t \int_0^x \frac{(t-s)^{r_1-1} (x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha(\{f(s, \tau, u_m(s, \tau))\}_{m=1}^\infty) d\tau ds + \epsilon \\ & \leq 4 \int_0^t \int_0^x \frac{(t-s)^{r_1-1} (x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s, \tau) \alpha(\{u_m(s, \tau)\}_{m=1}^\infty) d\tau ds + \epsilon \\ & \leq \left(4 \int_0^t \int_0^x \frac{(t-s)^{r_1-1} (x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s, \tau) d\tau ds \right) \alpha(\{u_m\}_{m=1}^\infty) + \epsilon \\ & \leq \left(4 \int_0^t \int_0^x \frac{(t-s)^{r_1-1} (x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s, \tau) d\tau ds \right) \alpha(D) + \epsilon \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{4p_{2n}^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \right) \alpha(D) + \epsilon \\ &= \ell_n \alpha(D) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha(N(D)) \leq \ell_n \alpha(D).$$

As a consequence of steps 1 to 3, together with Theorem 1, we can conclude that N has at least one fixed point in B_{R_n} which is a mild solution of problem (1)-(3).

Applications

Let us examine the following example as an application of our findings.

Example 3. Let $E = \mathbb{R}$ be equipped with the usual σ -algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$.

$$({}^c D_0^r u)(t, x) = \frac{c_n}{e^{t+x+2}(1 + |u(t-1, x-2)|)}; \text{ if } (t, x) \in J := [0, \infty) \times [0, \infty), \quad (5)$$

$$u(t, x) = t + x^2, \quad (t, x) \in \tilde{J} := [-1, \infty) \times [-2, \infty) \setminus [0, \infty) \times [0, \infty), \quad (6)$$

$$\begin{cases} u(t, 0) = t, \\ u(0, x) = x^2, \end{cases} \quad (t, x) \in J, \quad (7)$$

Where

$$c_n = \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{n^{r_1+r_2}}; \quad n \in \mathbb{N}^*.$$

$$f(t, x, u(t, x)) = \frac{c_n}{e^{t+x+2}(1 + |u(t-1, x-2)|)}; \quad (t, x) \in J.$$

The function $(t, x) \mapsto f(t, x, u)$ is jointly continuous for all $u \in C(J, E)$ and hence jointly measurable for all $u \in C(J, E)$. Also the map $u \mapsto f(t, x, u)$ is continuous for all $(t, x) \in J$. Thus, the function f is Carathéodory on $J \times E$. For each $u \in E$, $(t, x) \in J$, we have

$$|f(t, x, u)| \leq 1 + \frac{c_n |u|}{e^3}.$$

Hence the condition (H3) is satisfied with $p_{1n}^* = 1$, $p_{2n}^* = c_n e^{-3}$.

Moreover, the condition (H4) is satisfied. We shall show that the condition $\ell_n < 1$ holds for each $(r_1, r_2) \in (0; 1] \times (0; 1]$ we obtain

$$\begin{aligned} \ell_n &= \frac{4p_{2n}^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &= \frac{4}{e^3} \\ &< 1. \end{aligned}$$

As a result, Theorem 2 implies that our problem (5)-(7) has a fixed point u defined on J which is a solution of our problem (5)-(7).

Conclusion

Fractional calculus is a broad topic that calls for a variety of techniques and sophisticated tools. In this paper, we have contributed to the study of various classes of Darboux problems for partial hyperbolic functional fractional differential equations via the Caputo derivative with finite delay in Fréchet spaces.

Throughout most of this paper, sufficient conditions were considered in order to obtain the existence and uniqueness of solutions for our problem by reducing the research to the existence and uniqueness of fixed points of suitable operators, using the expanded Darbo's fixed point theorem associated with the notion of measure of noncompactness. There are numerous directions in which the work presented here can be extended. To find existence results, one should examine the properties of the operators and the structure of the space. The presence of weighted continuous functions in the space, their uniqueness, the composition of the solution set, and the optimality of the conditions met by the operators are just a few of the numerous other topics and questions that can be investigated.

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Primene proširene Darboove teoreme o fiksnoj tački na frakcione diferencijalne jednačine sa konačnim kašnjenjem

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OBLAST: matematika

KATEGORIJA (TIP) ČLANKA: originalni naučni rad

Sažetak:

Uvod/cilj: Cilj rada je proučavanje postojanja blagih rešenja problema početne vrednosti Darbuovog problema za parcijalne hiperboličke frakcione diferencijalne jednačine pomoću Kaputovog izvoda sa konačnim kašnjenjem u Freševim prostorima.

Metode: U analizi se koriste Darboova teorema o fiksnoj tački sa pojmom mere nekompaktnosti kako bi se utvrdilo postojanje rešenja svođenjem istraživanja na dokazivanje postojanja i jedinstvenosti fiksnih tačaka odgovarajućih operatora.

Rezultati: Postojanje blagih rešenja posmatranih parcijalnih hiperboličkih frakcionih diferencijalnih jednačina je dokazano pod odgovarajućim uslovima. Primenljivost teorijskih rezultata je ilustrovana primerom.

Zaključak: U radu se razmatraju dovoljni uslovi za utvrđivanje postojanja i jedinstvenosti rešenja Darbuovog problema primenom Darboove teoreme o fiksnoj tački na neograničenom intervalu. Istraživanje pruža metodološki okvir za analizu sličnih problema i doprinosi širem razumevanju blagih rešenja u frakcionom računju i funkcionalnoj analizi. Dobijeni rezultati nude potencijalne primene za istraživače koji se bave frakcionim diferencijalnim jednačinama u beskonačno-dimenzionalnim prostorima.

Ključne reči: parcijalna hiperbolička diferencijalna jednačina, frakcioni izvod, blago rešenje, Frešev prostor.

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