ORIGINAL SCIENTIFIC PAPERS

On the sum of powers of vertex degrees of polycyclic graphs

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Abstract:

Introduction/purpose: The sum of powers of vertex degrees of graphs is much studied in the literature, but only for some fixed values of the power. General properties of these sums are studied to a much lesser extent. In this paper, we offer results along these lines.

Methods: Combinatorial graph theory is applied.

Results: A few lower bounds for the sum of powers of vertex degrees are established, applicable to all powers. Classes of polycyclic graphs with a minimum sum of powers of vertex degrees are characterized.

Conclusion: The paper contributes to the study of the sum of powers of vertex degrees of graphs.

Keywords: degree (of a vertex), sum of powers of vertex degrees of graphs, polycyclic graphs.

Introduction

In this paper, we consider simple graphs. Let G be such a graph, possessing n vertices and m edges. Throughout this paper, the parameters n and m are assumed to be constant. In order to avoid trivialities, we assume that the graphs considered are connected.

The cyclomatic number (or simply: the number of cycles) of a connected graph G is c=m-n+1 (Bondy & Murty, 1976). We say that a graph is polycyclic if it has two or more cycles (i.e., if $c \geq 2$).

The results reported in this paper extend and somewhat correct those from (Gutman, 2014). Therefore, we use the same notation as in (Gutman, 2014).

The degree of a vertex v of the graph G, denoted by deg(v), is the number of first neighbors of this vertex. In this paper, we will examine the sum

$$Z_p = Z_p(G) = \sum_v \deg(v)^p = \sum_{k \geq 1} k^p n_k \tag{1}$$

where p is a positive integer, and where n_k denotes the number of vertices of the graph G whose degrees are equal to k. Recall that

$$\sum_{k \geq 1} n_k = n \; . \tag{2}$$

For p=1, we have the well-known relation

$$\sum_v \deg(v) = \sum_{k \geq 1} k n_k = 2m \; . \tag{3}$$

Let it be noted that formula (3) was discovered by Leonhard Euler in 1736, within his seminal work on the Königsberg bridge problem (Biggs et al, 1976).

The sum $Z_p(G)$ was separately investigated for a large number of particular values of the exponent p. Thus, $Z_2(G)$ is the much studied first Zagreb index (Gutman & Trinajstić, 1972, Nikolić et al, 2003); $Z_3(G)$ is the forgotted index, introduced in (Furtula & Gutman, 2015); $Z_4(G)$ has been named the Yemen index (Alameri et al, 2020); $Z_5(G)$, the so-called S-index, was investigated by (Nagarajan et al, 2021); the sums $Z_p(G)$ for p=6 and p=7 were studied by (Yousef et al, 2025) under the names G-index and P-index.

For the sake of completenesss, we mention here the inverse degree index $Z_{-1}(G)$ (Das et al, 2016) and the much studied zero-th order connectivity index $Z_{-1/2}(G)$ (Kier & Hall, 1976).

The results on the general properties of $Z_p(G)$, for arbitrary positive integer values of p, seem to be missing in the literature, with the exception of (Gutman, 2014). The present work is aimed at helping to fill in this gap.

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A transformed version of Z_p

The dependence of $Z_p(G)$ on the degrees of the vertices of the graph G, based on Eq. (1) is simple, evident and straightforward. Therefore, in order to get a better insight into this dependence, we transform the sum $Z_p(G)$ by adding to it Eq. (2) multiplied by some constant α, and Eq. (3) multiplied by some constant β. By this we arrive at a transformed expression of the form

$$T_p(G) = Z_p(G) + \alpha n + 2\beta m \tag{4}$$

which in view of Eqs. (1), (2), and (3), yields

$$T_p(G) = \sum_{k \geq 1} \left(k^p + \alpha + \beta k \right) n_k = \sum_{k \geq 1} \Theta_p(k) n_k \,. \tag{5}$$

Via Eq. (5), we introduce the term

$$\Theta_p(k) = k^p + \alpha + \beta k \tag{6}$$

which is a polynomial of degree p, in the variable k.

It is purposeful to choose α=-γ and β=2γ, which results in the simplification

$$T_p(G) = Z_p(G) - 2\gamma(m-n) = Z_p(G) - 2\gamma(c-1) \tag{7}$$

where c is the cyclomatic number, and

$$\Theta_p(k) = k^p - \gamma k + 2\gamma \,. \tag{8}$$

Throughout this paper, we assume that the parameter γ is positive valued. Then, from Eq. (7) we conclude the following:

In the case of trees (c=0), the terms we added to $Z_p(G)$ are necessarily positive valued. Therefore, whatever structural feature of G decreases/increases T_p(G), it will equially decrease/increase $Z_p(G)$, i.e., the proposed transformation is not applicable to trees.

The proposed transformation is also useless in the case of unicyclic graphs (c=1), since then T_p(G) and $Z_p(G)$ coincide.

Therefore, in what follows, we will only consider graphs whose cyclomatic number is greater than 1, that is, polycyclic graphs.

The parameter γ may be (reasonably) chosen to depend on p in many different ways. We first follow the model from (Gutman, 2014), and then offer a few more.

The first model: $\gamma = \lambda^p$

In (Gutman, 1914), the special case λ=3 was considered. Instead of it, here we report to have envisaged another such suitable option, $\lambda = 2\sqrt{2}$.

For $\lambda = 2\sqrt{2}$, Eq. (8) becomes

$$\Theta_p(k) = k^p - (2\sqrt{2})^p k + 2(2\sqrt{2})^p \,. \tag{9}$$

A simple calculation reveals the following:

For *p*=2, $\theta_p(k)=(k-4)^2$. Therefore, for all positive integer values of *k*, $\theta_2(k)$ is positive valued, except for *k*=4 when it is zero.

For *p*=3, by evaluating $\theta_2(1)$, $\theta_2(2)$, $\theta_2(3)$, $\theta_2(4)$ it becomes evidents that for all positive integer values of k, $\theta_2(k)>0$.

The same is found also for *p*=4 and for higher values of *p*.

This implies:

Theorem 1. Let *G* be a connected polycyclic graph with *n* vertices and *m* edges. Let among its vertices there are n_h vertices with degree *h* for some *h* different from 4. Then, for any value of the exponent *p*,

$$Z_p(G) \geq 2(2\sqrt{2})^p(m-n) - \Theta_p(h)n_h$$

where θ_p is the polynomial defined by Eq. (9).

Equality is attained only if *p*=2 and only if all the $n-n_h$ remaining vertices of *G* are of degree 4. This equality case pertains to a class of polycyclic (*n,m*) graphs with a fixed number of vertices of degree *h*. Their Z_2 values are minimal.

Proof. Rewriting Eqs. (5) and (7), we have

$$Z_p(G) = 2\gamma(m-n) - \sum_{k\geq 1}\Theta_p(k)n_k \,.$$

Above, we have shown that the terms $\theta_p(k)$ are always positive or zero. Therefore, by deleting the terms $\theta_p(k)n_k$ for all $k \neq h$, we obtain the inequality

$$Z_p(G) \geq 2\gamma(m-n) - \Theta_p(h)n_h \,.$$

Equality will happen if all the deleted terms were equal to zero. This requires that *p*=2 and k=4, i.e., that all the remaining vertices be of degree 4.

Thus, the graphs with a given number of vertices of degree h and all other vertices of degree 4 have the minimum Z_2 value.

As it was clarified above, consideration based on the transformation $Z_p \rightarrow T_p$ requires that the underlying graphs be polycyclic. █

The analogous results for the choice λ=3 read as follows (the details are found in (Gutman, 1914)):

$\Theta_p(1)=1+3^p>0$ for any value of $p\geq 2$,

$\Theta_p(2)=2^p>0$ for any value of $p\geq 2$, and

$\Theta_p(3)=0$ for any value of $p\geq 2$.

For $p=2$, $\Theta_2(4)=-2$; for $p=3$, $\Theta_3(4)=10$; $\Theta_p(4)$ is positive valued also for $p>3$.

For $k\geq 5$, $\Theta_p(k)$ is positive valued for any $p\geq 2$.

This implies:

Theorem 2. Let G be a connected polycyclic graph with n vertices and m edges. Let among its vertices there are n_h vertices with degree h for some h different from 3. Then, for any value of the exponent $p\geq 3$,

$$Z_p(G) \geq 2\cdot 3^p(m-n) - \Theta_p(h)n_h$$

where θ_p is the polynomial defined by Eq. (10):

$$\Theta_p(k) = k^p - 3^p k + 2\cdot 3^p. \tag{10}$$

Equality is attained if and only if all the n-n_h remaining vertices of G are of degree 3. This equality case pertains to a class of polycyclic (n,m)-graphs with a fixed number of vertices of degree h. Their Z_p values are minimal for all $p\geq 3$.

Proof is analogous to the proof of Theorem 1, *mutatis mutandis*. █

In Theorem 2, it is required that the exponent p be greater than 2. For the case $p=2$, from Eq. (10), we find that $\theta_2(4)=\theta_2(5)=-2$ and $\theta_2(6)=0$. Therefore, Theorem 2 cannot be extended to this case.

The second model: $\gamma = p^p$

Within this model,

$$\Theta_p(k) = k^p - p^p k + 2\cdot p^p. \tag{11}$$

For $p=2$, $\theta_2(k)=(k-2)^2+4$, which is positive valued for all k. Therefore, we may skip this case.

For $p=3$, we calculate that $\theta_3(1)=28$, $\theta_3(2)=8$, $\theta_3(3)=0$, $\theta_3(4)=202$, and all the folowing $\theta_3(k)$ values are evidently positive.

For $p=4$, we find that $\theta_4(1)$, $\theta_4(2)$, $\theta_4(4)$, $\theta_4(5)$, ... are positive, but $\theta_4(3)$ is negative.

For p=5, we find that $\theta_5(1)$, $\theta_5(2)$, $\theta_5(6)$, $\theta_5(7)$, ... are positive, whereas $\theta_5(3)$, $\theta_5(4)$ and $\theta_5(5)$ are negative. The same kind of anomaly exists also at p>6.

Thus, the model examined in this section can be used only at p=3, and we get:

Theorem 3. Let G be a connected polycyclic graph with n vertices and m edges. Let among its vertices there are n_h vertices with degree h for some h different from 3. Then, for $p=3$,

$$Z_p(G) \geq 2p^p(m-n) - \Theta_p(h)n_h$$

where θ_p is the polynomial defined by Eq. (11).

Equality is attained if and only if all the n-n_h remaining vertices of G are of degree 3. This equality case pertains to a class of polycyclic (n,m)-graphs with a fixed number of vertices of degree h. Their Z_3 values are minimal.

Note that the equality case in Theorem 3 (necessarily) agrees with that in Theorem 2.

The third model: $\gamma = \lambda p^p$

In the previous sections, we have seen that meaningful conclusions on the property of the sum of powers of vertex degrees can be deduced in the case when the polynomial $\theta_p(k)$ is zero for some particular value of p and k, whereas it is positive valued for all other choices of p and k. Therefore, in this section, we only focus on the condition $\theta_p(k)$=0 within our third model for the multiplier γ.

In this case, $\gamma=\lambda\, p^p$, and we get

$$\Theta_p(k) = k^p - \lambda p^p k + 2\lambda \cdot p^p \,. \tag{12}$$

Consider the functions $F_1(k) = k^p$ and $F_2(k) = -\lambda p^p k + 2\lambda \cdot p^p$. Noting that $F_1(0)$=0 and $F_2(0)=2\lambda p^p > 0$, and that F_1 is monotonically increasing whereas F_2 monotonically decreases, there must exist a point (not necessarily at an integer k) where they meet. If such a k is an integer, then $\theta_p(k)$, defined via Eq. (12), would become equal to zero. Then, the condition $\theta_p(k)$=0, for any chosen value of the exponent p, would determine a particular value of the parameter λ.

In what follows, we show how λ is determined when the condition $\theta_p(p)$=0 is imposed.

If $\theta_p(p)$=0, then by Eq. (12),

$$p^p - \lambda p^p \cdot p + 2\lambda \cdot p^p = 0 \,.$$

from which it immediately follows that $\lambda=1/(p-2)$. This result also indicated that the condition $\theta_p(p)=0$ is not applicable in the case $p=2$.

Additional examinations of our third model for γ, as well as of other possible models of this kind, is left for some later work. In any case, we believe that we clearly demonstrated that the paper (Gutman, 2014), recognized only the tip of an iceberg. It may be that here we did the same.

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Sobre la suma de potencias de grados de vértices de grafos policíclicos

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CAMPO: matemáticas (clasificación de asignaturas de matemáticas: primaria 05c07, secundaria 05c09)
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Resumen:

Introducción/propósito: La suma de potencias de los grados de los vértices de grafos se estudia ampliamente , pero solo para algunos valores fijos de la potencia. Las propiedades generales de estas sumas se estudian en mucha menor medida. En este artículo, presentamos resultados en este sentido.

Métodos: Se aplica la teoría de grafos combinatorios.

Resultados: Se establecen unos límites inferiores para la suma de potencias de grados de vértice, aplicables a todas las potencias. Se caracterizan las clases de grafos policíclicos con una suma mínima de potencias de grados de vértice.

Conclusión: El artículo contribuye al estudio de la suma de potencias de grados de vértices de grafos.

Palabras clave: grado (de un vértice), suma de potencias de grados de vértices de grafos, grafos policíclicos.

О сумме степеней вершин полициклических графов

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РУБРИКА ГРНТИ: 27.29.19 Краевые задачи и задачи на собственные значения для обыкновенных дифференциальных уравнений и систем уравнений
ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: Несмотря на то что сумма степеней вершин графов широко изучена в литературе, многим значениям степени не уделено внимания. Общие свойства этих сумм *изучены в гораздо меньшей степени. В данной статье представлены новые результаты в этом направлении.*

Методы: В статье применена комбинаторная теория графов.

Результаты: Установлено несколько нижних границ для суммы степеней вершин, применимых ко всем степеням. Описаны классы графов с минимальной суммой степеней вершин.

Выводы: Результаты данной статьи вносят вклад в изучение суммы степеней вершин графов.

Ключевые слова: степень (вершины), сумма степеней вершин графов, полициклические графы.

О збиру потенцираних степена чворова полицикличних графова

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ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Збир потенцираних степена чворова графа проучаван је у литератури само за одређене вредности потенције. Опште особине ових сума недовољно су истраживане, па се у овом раду наводе нови резултати истраживања.

Методе: Примењена је комбинаторна теорија графова.

Резултати: Добијено је неколико граница за збир потенцираних степена чворова, применљивих за све потенције. Карактерисане су класе графова са минималном сумом потенцираних степена.

Закључак: Рад доприности проучавању збира потенцраних степена чворова графова.

Кључне речи: степен (чвора), збир потенцираних степена чворова графа, полициклични графови.

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